Categorical logic meets double categories

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Logical Aspects of Virtual double categories, master's thesis, 2025

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Main result

$$\begin{tikzcd} \boxed{\text{Elementary existential fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{\begin{array}{c}\text{Cartesian equipments} \\ \text{(Cartesian fibrational double categories)}\end{array}} \arrow[d, hook] \\ \boxed{\text{Fibrations w/ finite products}} \arrow[r, "\mathbb{B}\mathrm{il}"] & \boxed{\text{Cartesian fibrational virtual double categories}} \end{tikzcd}$$

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\end{tikzcd}
$$

The double category of relations

the double category $\mathbb{R}\mathrm{el}$

- $A, B, \ldots$: sets
- $f, g, \ldots$: functions
- $R, S, \ldots$: binary relations
- $\alpha, \beta, \ldots$: implication

$$
\begin{tikzcd}
A \arrow[r, "R", "\shortmid"{marking}] \arrow[d, "f"'] \arrow[dr, phantom, "\alpha"] & D \arrow[r, "S", "\shortmid"{marking}] \arrow[d, "h"] \arrow[dr, phantom, "\gamma"] & G \arrow[d, "m"] \\
B \arrow[r, "\shortmid"{marking}] \arrow[d, "g"'] \arrow[dr, phantom, "\beta"] & E \arrow[r, "\shortmid"{marking}] \arrow[d, "k"] \arrow[dr, phantom, "\delta"] & H \arrow[d, "n"] \\
C \arrow[r, "T"', "\shortmid"{marking}] & F \arrow[r, "U"', "\shortmid"{marking}] & I
\end{tikzcd}
$$

Remark

$$R \odot S\ (a, g) \overset{\text{def}}{\iff} \exists\, d \in D\ \big(R(a,d) \wedge S(d,g) \big)$$

Fibrations as stages of predicate logic

Set-theoretic semantics	$\rightsquigarrow$	Semantics with a fibration $\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathbb{B}\end{array}$
sets $X, Y, \ldots$		objects $I, J, \ldots$ in $\mathbb{B}$
functions $f : X \to Y, \ldots$		morphisms $f : I \to J, \ldots$ in $\mathbb{B}$
subsets $P \subseteq X,\ Q \subseteq Y, \ldots$		objects P in $\mathcal{E}_I$, Q in $\mathcal{E}_J$

$\underbrace{\mathcal{E}_I \quad \mathcal{E}_J}_{\text{fibers}}$

Double categories of relations w.r.t. fibrations

the double category $\mathbb{R}\mathrm{el}(p : \mathcal{E} \to \mathcal{B})$

- $A, B, \dots$: objects in $\mathcal{B}$
- $f, g, \dots$: morphisms in $\mathcal{B}$
- $A \overset{R}{\nrightarrow} D, \dots$: objects in $\mathcal{E}_{A \times D}$
- $\alpha, \beta, \dots$: morphisms in $\mathcal{E}$

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p needs to have structures for $\exists$ and $=$ in order to define composition.
$=$ elementary existential fibrations

$*$ In my thesis, I write this as $\mathbb{B}\mathrm{il}(p)$.

Double categories of relations w.r.t. fibrations

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Proposition.
$\mathbb{R}\mathrm{el}(p)$ is a cartesian equipment.

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Main result

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\boxed{\text{Elementary existential fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{\begin{array}{c}\text{Cartesian equipments} \\ \text{(Cartesian fibrational double categories)}\end{array}} \arrow[d, hook] \\
\boxed{\begin{array}{c}\text{Fibrations} \\ \text{w/ finite products}\end{array}} \arrow[r, "\mathbb{B}\mathrm{il}"] & \boxed{\begin{array}{c}\text{Cartesian fibrational} \\ \text{virtual double categories}\end{array}}
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$\begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathbb{B} \end{array}$: a fibration with finite products $\rightsquigarrow$

✓ functions

✓ composition of functions

✓ relations

✗ composition of relations

$\begin{array}{c}\mathcal{E}\\ \downarrow p\\ \mathcal{B}\end{array}$: a fibration with finite products $\rightsquigarrow$

✓ functions

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✓ relations

✗ composition of relations

Virtual double categories!

Virtual double categories

Definition. A virtual double category (VDC) $\mathbb{D}$ consists of

- objects $I, J, \ldots$
- vertical arrows $\begin{array}{c} I \\ \downarrow u \\ J \end{array}, \ldots$
- horizontal arrows $I \overset{\alpha}{\nrightarrow} K, \ldots$
- (virtual) cells

$$\begin{tikzcd} I_0 \arrow[r, "R_1", "\shortmid"{marking}] \arrow[d, "u"'] & I_1 \arrow[r, "\shortmid"{marking}] & \cdots \arrow[r, "R_n", "\shortmid"{marking}] \arrow[d, phantom, "\nu"] & I_n \arrow[d, "v"] \\ J_0 \arrow[rrr, "S"', "\shortmid"{marking}] & & & J_1 \end{tikzcd}$$

for $n \geq 0$

with compositions of vertical arrows and cells that satisfy some laws.

Virtual double categories = Double categories w/o arbitary horizontal composition.

$\rightsquigarrow$ Double categories $\simeq$ VDCs with horizontal composition.

Key observation

$$\begin{tikzcd} X \arrow[r, "R", "|" marking] \arrow[d, equal] & Y \arrow[r, "S", "|" marking] \arrow[d, phantom, "\cap\!\mathrm{I}"] & Z \arrow[d, equal] \\ X \arrow[rr, "T"', "|" marking] & & Z \end{tikzcd} \quad \text{in } \mathbf{Rel}$$

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X \arrow[r, "R", "|" marking] \arrow[d, equal] & Y \arrow[r, "S", "|" marking] \arrow[d, phantom, "\cap\mid"] & Z \arrow[d, equal] \\
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\end{tikzcd} \quad \text{in } \mathbb{R}\mathrm{el}$$

$$\Longleftrightarrow \quad \exists y \in Y \ \big(R(x,y) \wedge S(y,z) \big) \Rightarrow T(x,z) \quad (\forall x, z)$$

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$$\iff \quad R(x,y) \wedge S(y,z) \Rightarrow T(x,z) \qquad (\forall x, y, z)$$

$\exists$ is not necessary to define virtual cells !

Go virtual !

$$\begin{array}{c} \mathcal{E} \\ \downarrow p \\ \mathcal{B} \end{array}$$: a fibration with finite products $\rightsquigarrow$

- ✓ functions
- ✓ composition of functions
- ✓ relations
- × composition of relations
- ✓ virtual cells

Proposition [N.]

$\mathbb{R}\mathrm{el}(p)$ is a cartesian fibrational virtual double category.

Main result

$$\begin{tikzcd}
\boxed{\text{Elementary existential fibrations}} \arrow[r, "\mathbb{B}\mathrm{il}"] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \boxed{\begin{array}{c}\text{Cartesian equipments} \\ \text{(Cartesian fibrational double categories)}\end{array}} \arrow[d, hook] \\
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__Theorem.__ [N.] p : fibration with finite products

p : an e.e. fibration $\iff$ $\mathbb{B}\mathrm{il}(p)$: a cartesian equipment

What's the importance?

1. This theorem justifies the idea that composition of relations requires $\exists$ and $=$.

2. Elementary existential fibrations are assumed to satisfy $\begin{cases} \text{the Beck-Chevalley condition} \\ \text{the Frobenius reciprocity} \end{cases}$.

$\mathbb{R}\mathrm{el}(\mathbb{P})$ being a cartesian equipment is enough to imply these.

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<u>Theorem.</u>[N.] p : fibration with finite products

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Thank you!

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https://hayatonasu.github.io/hayatonasu/

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Corollaries

- $\mathcal{C}$: a category with finite limits $\rightsquigarrow$ $\mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$: a cartesian fibration.

  $\mathbb{R}\mathrm{el}(\mathcal{C}) := \mathbb{B}\mathrm{il}(\mathrm{Sub}(\mathcal{C}) \to \mathcal{C})$ is a cartesian equipment

  $\iff$ $\mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$ is an e.e. fibration $\overset{\text{[Jacobs '99]}}{\iff}$ $\mathcal{C}$: regular

- Q : a $\wedge$-semilattice $\rightsquigarrow$ $\mathrm{Fam}(Q) \xrightarrow{f_Q} \mathrm{Set}$: a cartesian fibration.

  $Q\text{-}\mathbb{R}\mathrm{el} := \mathbb{B}\mathrm{il}(f_Q)$ is a cartesian equipment

  $\iff$ f_Q is an e.e. fibration $\overset{\text{[Jacobs '99]}}{\iff}$ Q : a frame.

Definitions

A fibration p is cartesian if

- $\mathcal{B}$ has finite products, and
- all fibers $\mathcal{E}_I$ have finite products preserved by the base change functors.

p is elementary existential (e.e.) if

- it is cartesian,
- the base change functors along $I \times J \xrightarrow{\pi} I$ & $I \times J \xrightarrow{id \times \Delta} I \times J \times J$'s have left adjoints, and
- BC condition and Frobenius reciprocity hold for these adjoints.

A restriction of

$$\begin{tikzcd} A \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "\gamma"', "|" marking] & D \end{tikzcd}$$

is the universal cell

$$\begin{tikzcd} A \arrow[r, "|" marking] \arrow[d, "f"'] \arrow[dr, phantom, "\bullet"] & C \arrow[d, "g"] \\ B \arrow[r, "\gamma"', "|" marking] & D \end{tikzcd}.$$

A VDC is fibrational if it admits all restrictions.

A fibrational VDC is cartesian if the right adjoints below exist.

$$\begin{tikzcd} \mathbb{D} \arrow[r, bend left, "!"] \arrow[r, phantom, "\perp"] & \mathbb{1} \arrow[l, bend left] \end{tikzcd}, \quad \begin{tikzcd} \mathbb{D} \arrow[r, bend left, "\Delta"] \arrow[r, phantom, "\perp"] & \mathbb{D} \times \mathbb{D} \arrow[l, bend left] \end{tikzcd} \quad \text{in } \mathcal{FVDC}.$$