

Categorical logic meets double categories

Hayato Nasu

Kyoto Univ.

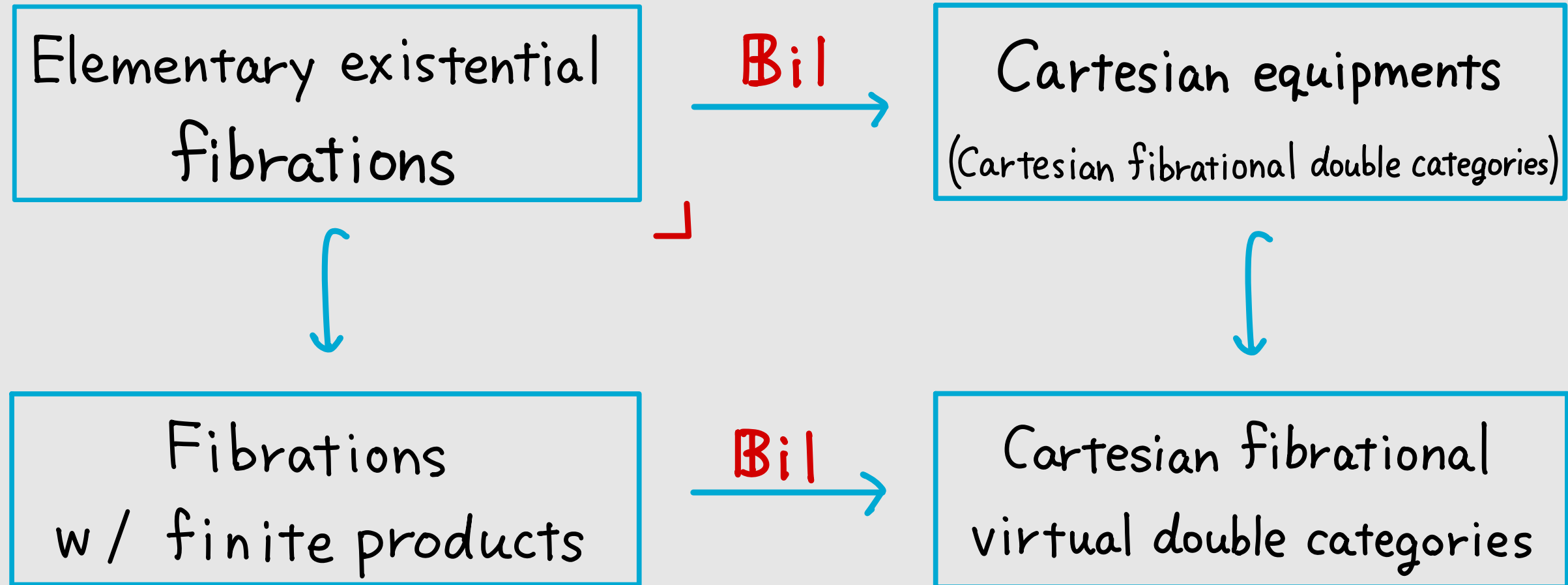
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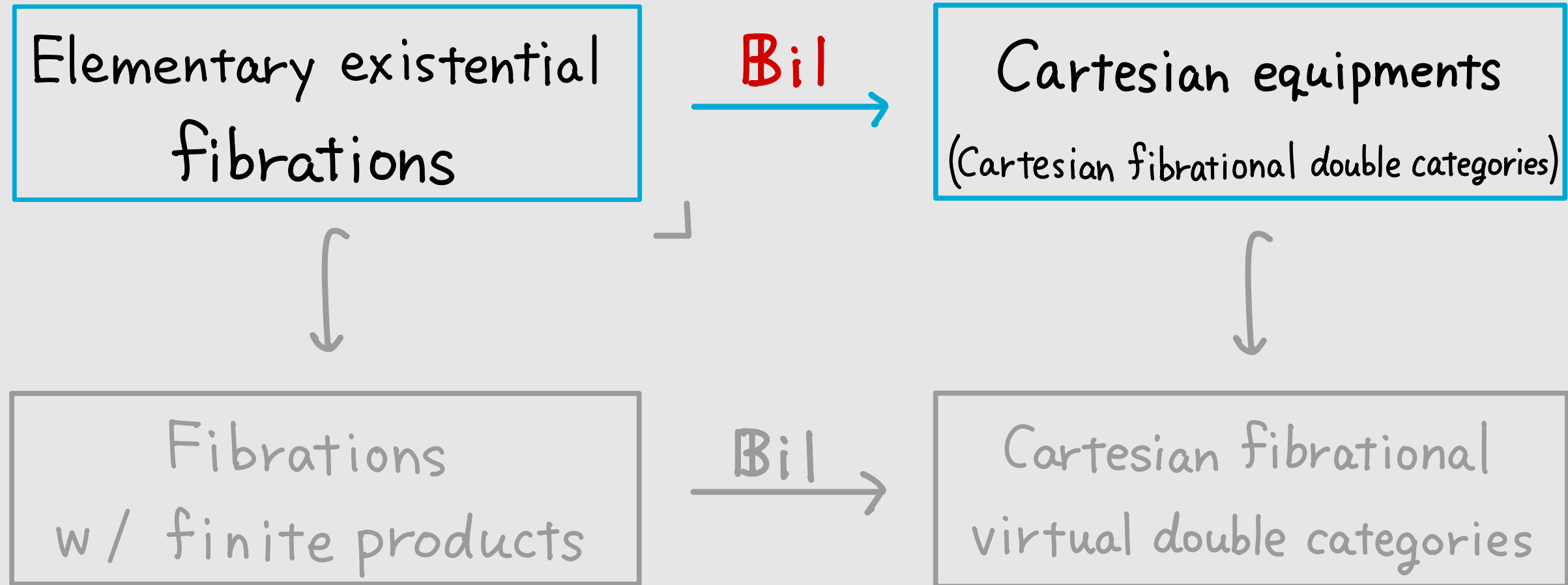
Logical Aspects of Virtual double categories , master's thesis, 2025

arXiv : 2501.17869

Main result



Main result



The double category of relations

the double category **Rel**

- $A, B, \dots$: sets
- $f, g, \dots$: functions
- $R, S, \dots$: **binary relations**
- $\alpha, \beta, \dots$: implication

$$\begin{array}{ccccc}
 A & \xrightarrow{R} & D & \xrightarrow{S} & G \\
 f \downarrow & \alpha & \downarrow h & \gamma & \downarrow m \\
 B & \longrightarrow & E & \longrightarrow & H \\
 g \downarrow & \beta & \downarrow k & \delta & \downarrow n \\
 C & \xrightarrow{T} & F & \xrightarrow{U} & I
 \end{array}$$

Remark

$$R \odot S (a, g) \stackrel{\text{def}}{\iff} \exists d \in D \left(R(a, d) \wedge S(d, g) \right)$$

Fibrations as stages of predicate logic

Set-theoretic
semantics



Semantics
with a fibration $\begin{array}{c} \mathcal{E} \\ \downarrow \Pi \\ \mathcal{B} \end{array}$

sets $X, Y, \dots$

objects $I, J, \dots$ in $\mathcal{B}$

functions $f: X \rightarrow Y, \dots$

morphisms $f: I \rightarrow J, \dots$ in $\mathcal{B}$

subsets $P \subseteq X, Q \subseteq Y, \dots$

objects P in $\underline{\mathcal{E}_I}$, Q in $\underline{\mathcal{E}_J}$
fibers

Double categories of relations w.r.t. fibrations

the double category $\mathbf{Rel}(\mathcal{P}: \mathcal{E} \rightarrow \mathcal{B})$

- $A, B, \dots$: objects in $\mathcal{B}$
- $f, g, \dots$: morphisms in $\mathcal{B}$
- $A \xrightarrow{R} D, \dots$: objects in $\mathcal{E}_{A \times D}$
- $\alpha, \beta, \dots$: morphisms in $\mathcal{E}$

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$\mathcal{P}$ needs to have structures for $\exists$ and $=$ in order to define composition.
 = elementary existential fibrations

* In my thesis, I write this as $\mathbf{Bil}(\mathcal{P})$.

Double categories of relations w.r.t. fibrations

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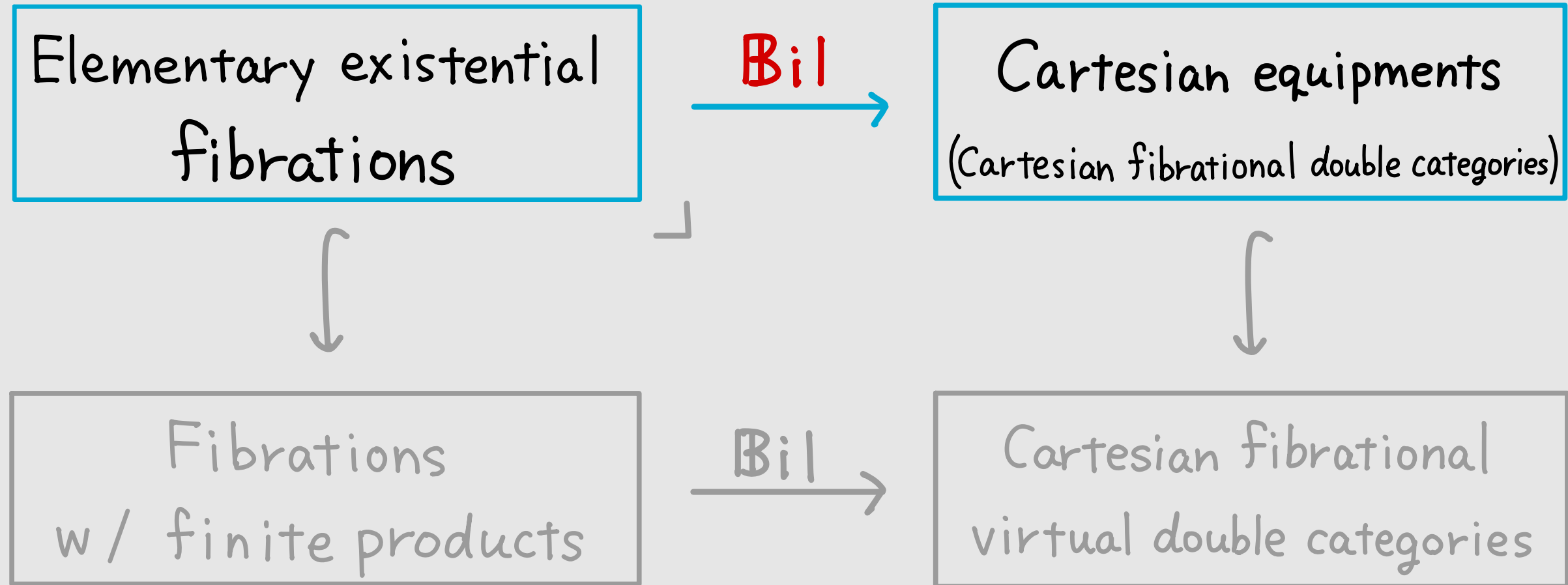
Proposition.

$\mathbf{Rel}(P)$ is
a cartesian equipment.

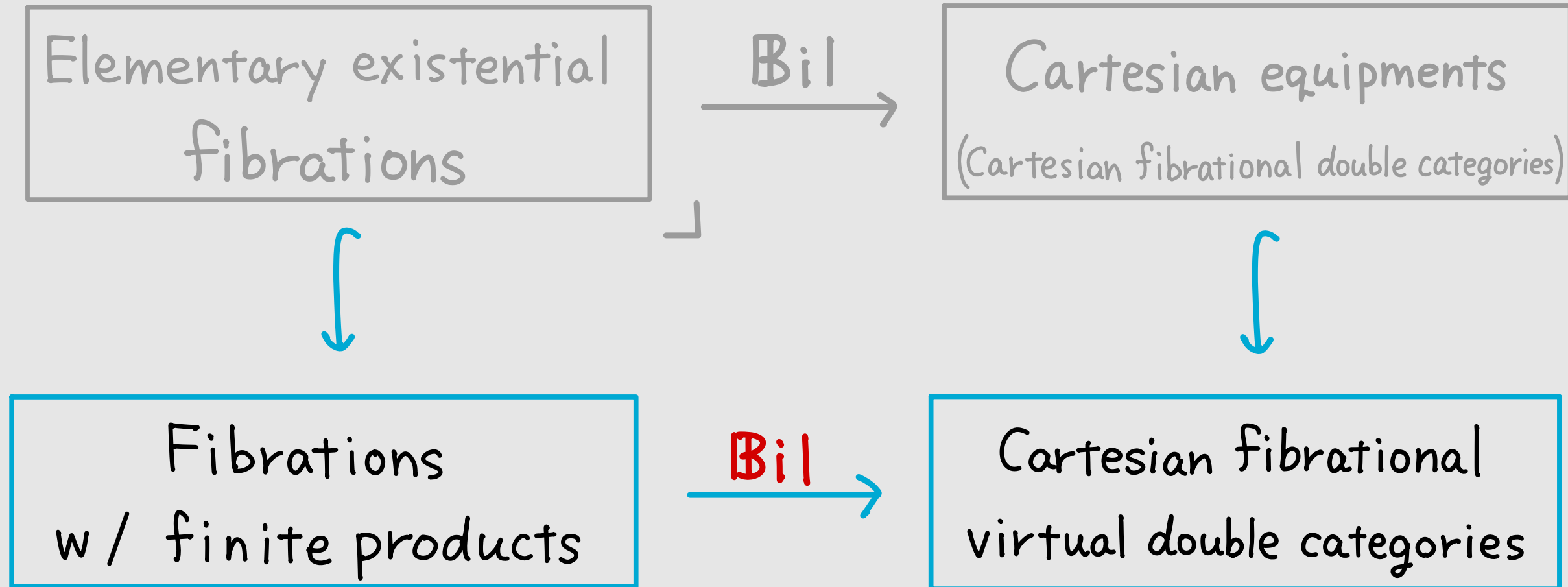
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$\mathcal{E} \downarrow \mathcal{B} : \text{a fibration}$
 $\text{with finite products}$



- ✓ functions
- ✓ composition of functions
- ✓ relations
- ✗ composition of relations

$\mathcal{E} \downarrow \mathcal{B}$: a fibration
with finite products



- ✓ functions
- ✓ composition of functions
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Virtual double categories!

Virtual double categories

Definition. A **virtual double category (VDC)** $\mathbb{D}$ consists of

- objects $I, J, \dots$
- vertical arrows $\begin{matrix} I \\ \downarrow u \\ J \end{matrix}, \dots$
- horizontal arrows $I \xrightarrow{\alpha} K, \dots$

• **(virtual) cells**

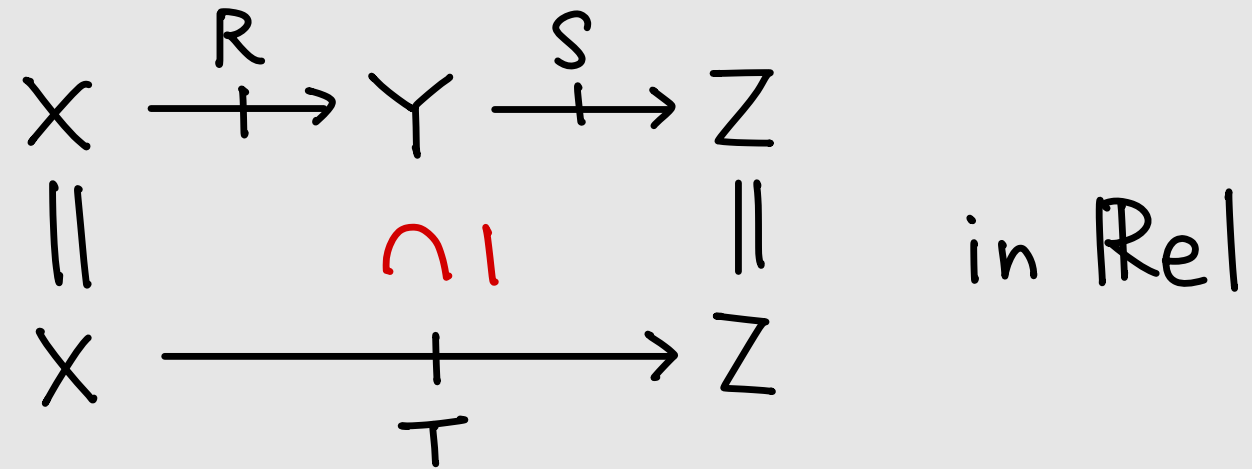
$$\begin{array}{ccccccc}
 I_0 & \xrightarrow{R_1} & I_1 & \xrightarrow{\quad} & \dots & \xrightarrow{R_n} & I_n \\
 \downarrow u & & & \searrow \wr & & & \downarrow v \\
 J_0 & \xrightarrow{\quad} & & S & \xrightarrow{\quad} & & J_1
 \end{array}
 \quad \text{for } n \geq 0$$

with compositions of vertical arrows and cells that satisfy some laws.

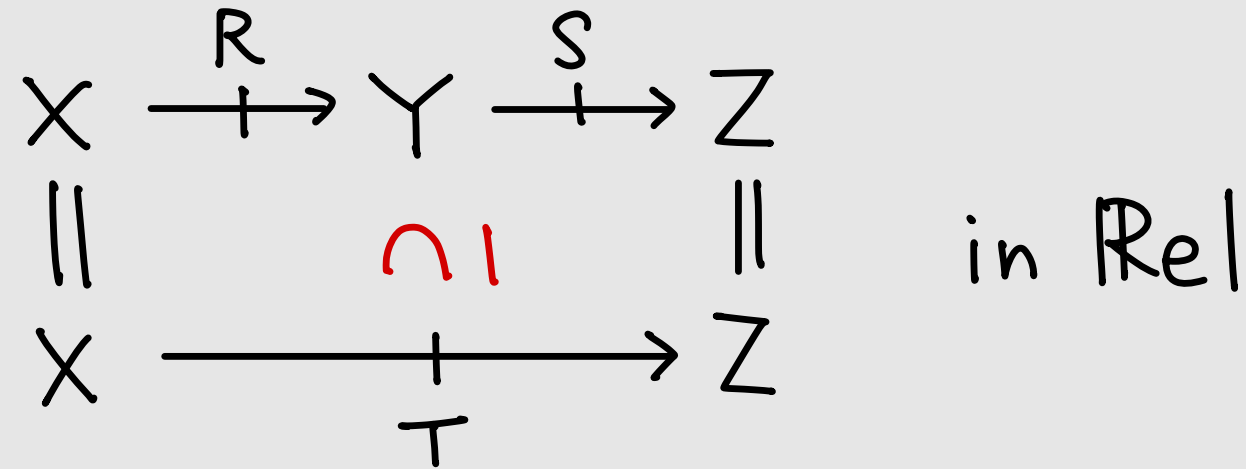
Virtual double categories = Double categories
w/o arbitrary horizontal composition.

$\rightsquigarrow$ Double categories $\simeq$ VDCs with horizontal composition.

Key observation

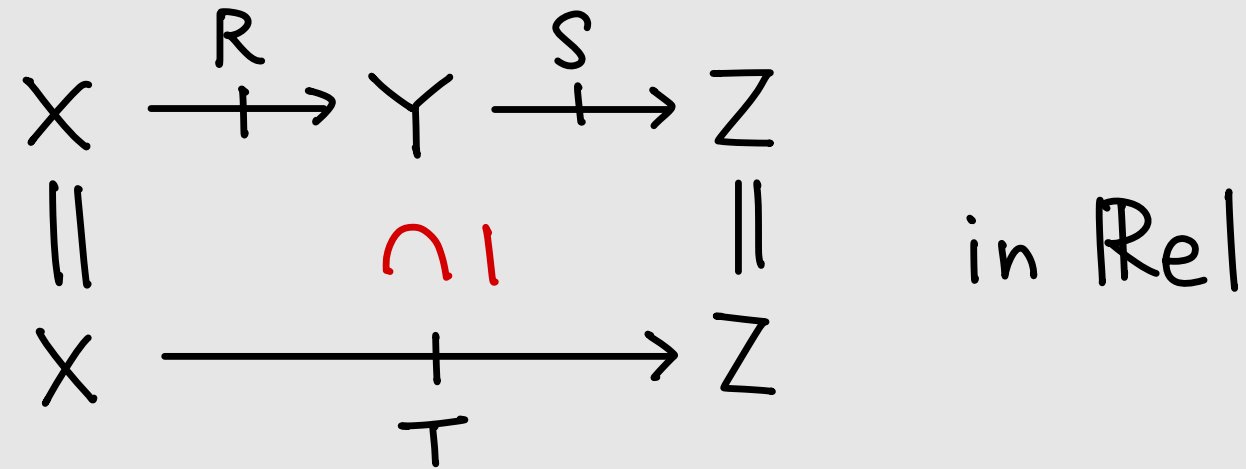


Key observation



$$\Leftrightarrow \exists y \in Y \left(R(x, y) \wedge S(y, z) \right) \Rightarrow T(x, z) \quad (\forall x, z)$$

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Key observation

$$\begin{array}{ccccc}
 X & \xrightarrow{R} & Y & \xrightarrow{S} & Z \\
 \parallel & & & & \parallel \\
 X & \xrightarrow[\quad T \quad]{} & & & Z
 \end{array}
 \quad \text{in Rel}$$

$$\Leftrightarrow \exists y \in Y \left(R(x, y) \wedge S(y, z) \right) \Rightarrow T(x, z) \quad (\forall x, z)$$

$$\Leftrightarrow R(x, y) \wedge S(y, z) \Rightarrow T(x, z) \quad (\forall x, y, z)$$

$\exists$ is not necessary to define virtual cells !

Go virtual !

$\begin{array}{c} \mathcal{E} \\ \downarrow \Pi \\ \mathcal{B} \end{array} : \begin{array}{l} \text{a fibration} \\ \text{with finite products} \end{array}$

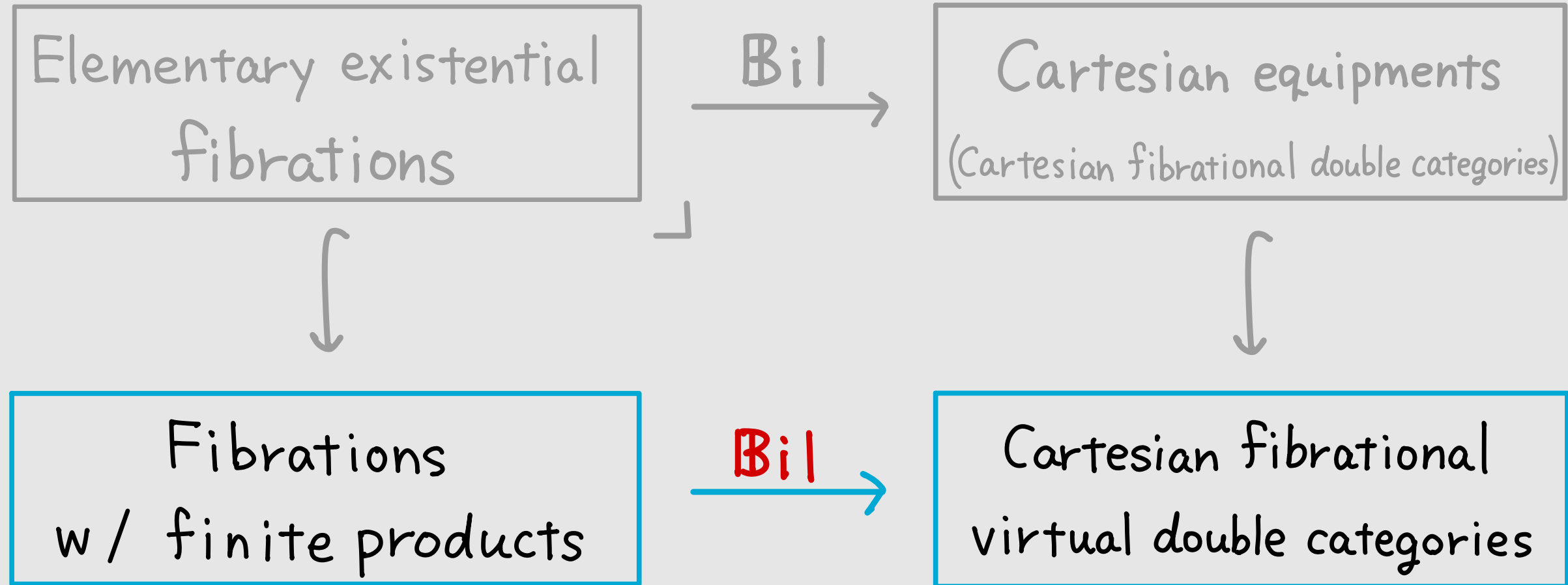


- ✓ functions
- ✓ composition of functions
- ✓ relations
- ✗ composition of relations
- ✓ virtual cells

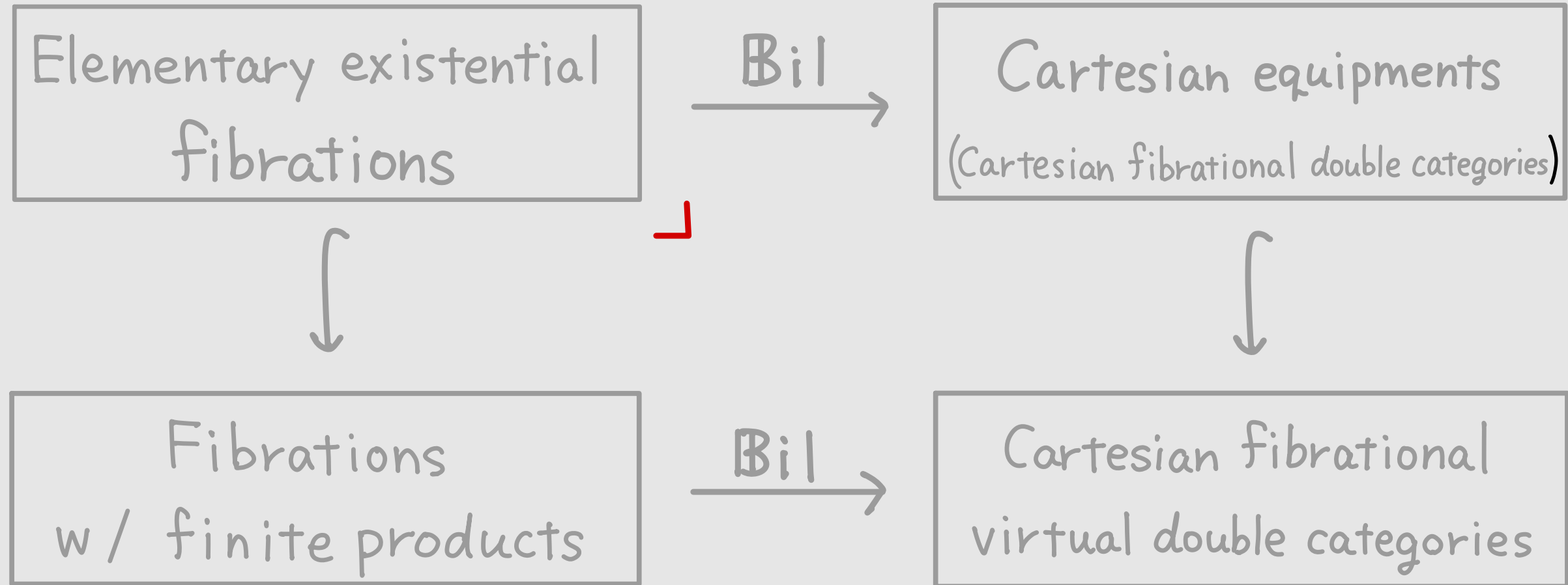
Proposition [N.]

$\text{Rel}(\Pi)$ is a cartesian fibrational virtual double category.

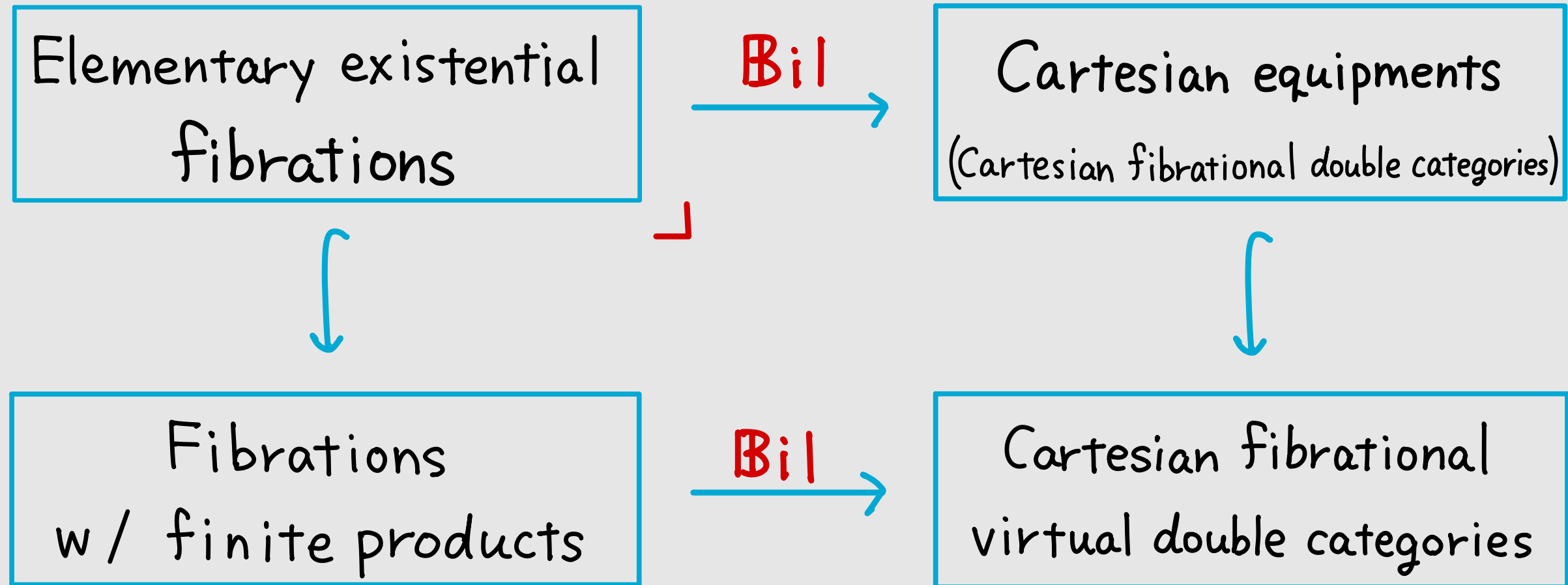
Main result



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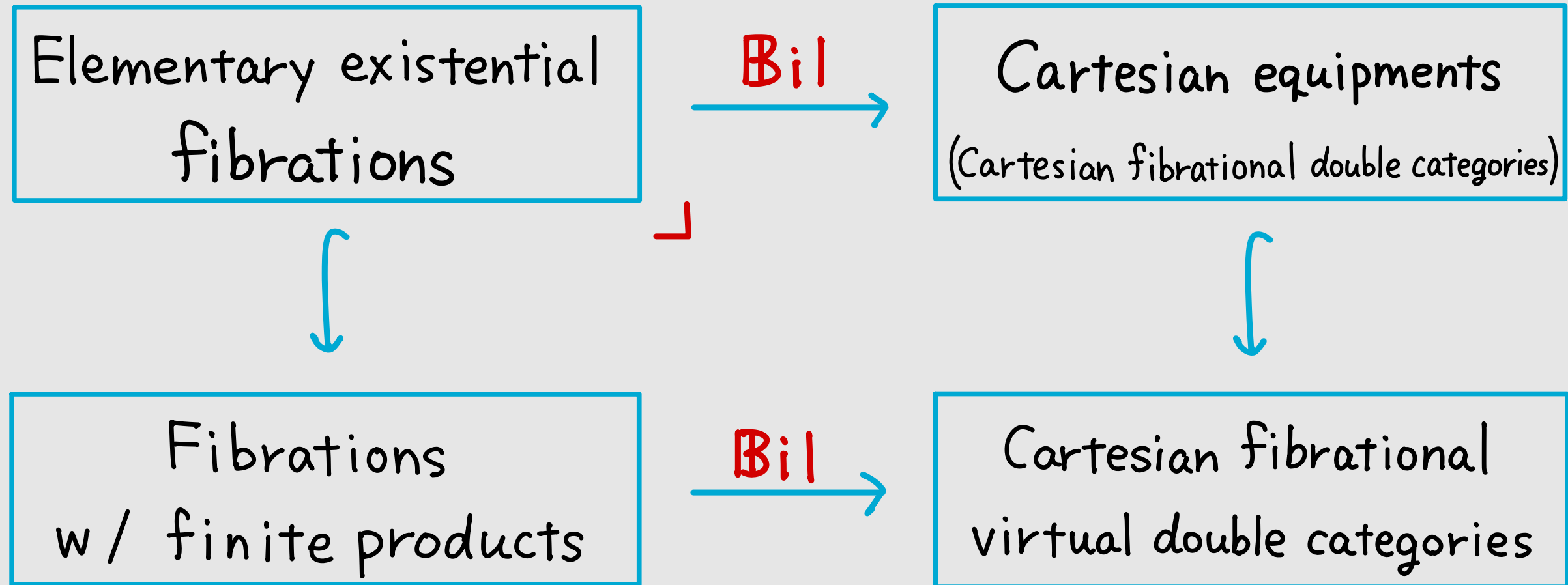
Theorem.[N.] $\mathcal{P}$: fibration with finite products

$\mathcal{P}$: an e.e. fibration $\iff \mathbb{B}il(\mathcal{P})$: a cartesian equipment

What's the importance?

1. This theorem justifies the idea that composition of relations requires $\exists$ and $=$.
 2. Elementary existential fibrations are assumed to satisfy $\begin{cases} \text{the Beck-Chevalley condition} \\ \text{the Frobenius reciprocity} \end{cases}$.
- $\mathbb{R}el(\mathcal{P})$ being a cartesian equipment is enough to imply these.

Main result



Theorem.[N.] $\mathcal{P}$: fibration with finite products

$\mathcal{P}$: an e.e. fibration $\iff \mathbb{B}il(\mathcal{P})$: a cartesian equipment

Thank you!

Hayato Nasu

Kyoto Univ

hnasu@kurims.kyoto-u.ac.jp

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<https://hayatonasu.github.io/hayatonasu/>

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Corollaries

- $\mathcal{C}$: a category with finite limits $\rightsquigarrow \text{Sub}(\mathcal{C}) \rightarrow \mathcal{C}$: a cartesian fibration.

$\text{Rel}(\mathcal{C}) := \text{Bil}(\text{Sub}(\mathcal{C}) \rightarrow \mathcal{C})$ is a cartesian equipment

$\Leftrightarrow \text{Sub}(\mathcal{C}) \rightarrow \mathcal{C}$ is an e.e. fibration $\xLeftrightarrow[\text{Jacobs '99}] \mathcal{C} : \text{regular}$

- Q : a $\wedge$ -semilattice $\rightsquigarrow \text{Fam}(Q) \xrightarrow{f_Q} \text{Set}$: a cartesian fibration.

$Q\text{-Rel} := \text{Bil}(f_Q)$ is a cartesian equipment

$\Leftrightarrow f_Q$ is an e.e. fibration $\xLeftrightarrow[\text{Jacobs '99}] Q : \text{a frame.}$

Definitions

A fibration p is **cartesian** if

- B has **finite products**, and
- all fibers E_I have **finite products** preserved by the base change functors.

p is **elementary existential (e.e.)** if

- it is cartesian,
- the base change functors along $I \times J \xrightarrow{\pi} I$ & $I \times J \xrightarrow{id \times \Delta} I \times J \times J$ have **left adjoints**, and
- BC condition and Frobenius reciprocity hold for these adjoints.

A **restriction** of $f \downarrow B \xrightarrow[\gamma]{+} D$ is

the universal cell $f \downarrow B \xrightarrow[\gamma]{+} D$ with a blue dot in the center of the square.

A VDC is **fibrational** if it admits all restrictions.

A fibrational VDC is **cartesian** if the right adjoints below exist.

$$\mathbb{D} \xleftarrow[\perp]{!} \mathbb{1}, \quad \mathbb{D} \xleftarrow[\perp]{\Delta} \mathbb{D} \times \mathbb{D} \quad \text{in } FVDC.$$