Eckmann-Hilton Arguments in Weak ω-Categories

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The Eckmann-Hilton Argument

Theorem (Eckmann-Hilton)

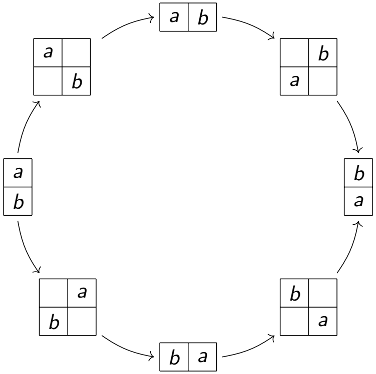
Let $(M, *, e)$ and $(M, \circ, e)$ be unital magmas on the same set sharing a unit e. Suppose the interchange law is satisfied for all $a, b, c, d \in M$:

$$(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$$

Then $* = \circ$ and both are commutative, that is $a * b = a \circ b = b * a = b \circ a$ for all $a, b \in M$

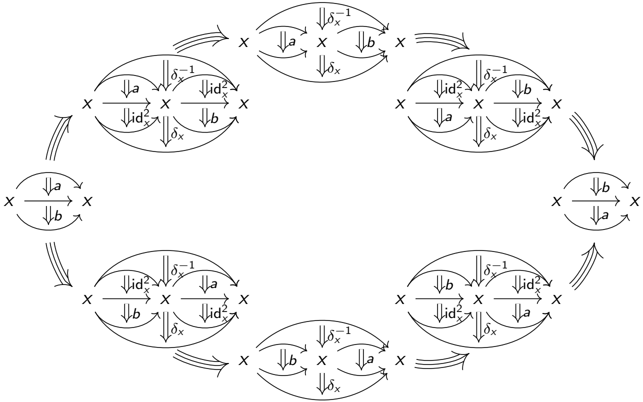
The Eckmann-Hilton Clock

We can represent this graphically, writing $\begin{array}{|c|}\hline a\\\hline b\\\hline\end{array}$ for $a * b$, $\begin{array}{|c|c|}\hline a & b\\\hline\end{array}$ for $a \circ b$, and $\square$ for e. The interchange law is automatic from this notation.



A weak Eckmann-Hilton

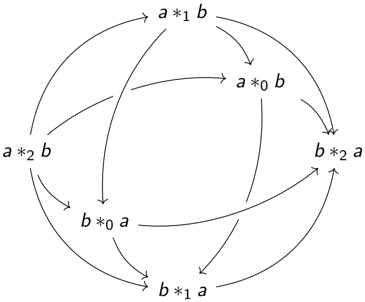
Arguably the most general setting in which Eckmann-Hilton-like arguments hold is higher category theory. Working in the setting of Batanin-Leinster ω-categories, given 2-cells $a, b : \mathrm{id}_x \to \mathrm{id}_x$, we construct equivalences:



We call the clockwise equivalence $\mathrm{EH}^2_{1,0}(a, b)$. It witnesses commutativity of the 1-composition ($a *_1 b \cong b *_1 a$) going via the 0-composite $a *_0 b$.

Higher Dimensions?

Higher-dimensional cells satisfy even more Eckmann-Hilton relations. In dimension 3, the above Eckmann-Hilton clock can be expanded to the **Eckmann-Hilton sphere:**



The additional degrees of freedom allow us to relate each of the 0, 1, and 2-composites of any 3-cells $a, b : \mathrm{id}_x^2 \to \mathrm{id}_x^2$. For instance, moving clockwise round the equator, we have $\mathrm{EH}^3_{2,0}(a, b) : a *_2 b \to b *_2 a$.

Our Contribution

Theorem (BMOSV, 2025)

Let $a, b : \mathrm{id}_x^n \to \mathrm{id}_x^n$ *be* $(n+1)$*-cells in an* ω*-category, for* $n > 0$*. Then for any distinct* $k, l \in \{0, \dots, n\}$ *we can produce equivalences:*

$$H_{k,l}^{n+1} : a *_k b \to \Theta(a *_l b)$$

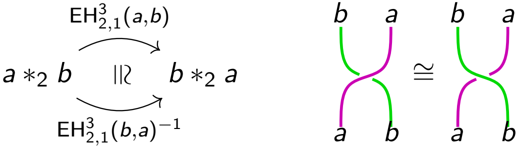
where $\Theta(-)$ *is an operation that "pads" its argument with appropriate unitors. Furthermore, using these we can build equivalences:*

$$EH_{k,l}^{n+1} : a *_k b \to b *_k a$$

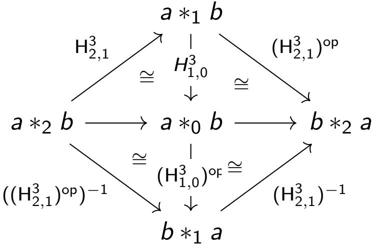
witnessing commutativity of k*-composition, going via the* l*-composite.*

What's Next?

In semi-strict settings, it is known that these higher-dimensional Eckmann-Hilton proofs satisfy further coherences themselves, for instance the **syllepsis:**



We have proved the syllepsis (and are in the process of formalising the result) in the fully weak setting, using these intermediate Eckmann-Hilton moves:



References

[1] Thibaut Benjamin et al. *Higher Eckmann-Hilton Arguments in Type Theory*. 2025. arXiv: 2501.16465 [math.CT].

[2] Eugenia Cheng and Alex Corner. *A Higher-Dimensional Eckmann–Hilton Argument*. 2024. URL: https://alex-corner.github.io/slides/corner-ct24.pdf.

[3] Beno Eckmann and Peter J. Hilton. 'Structure maps in group theory'. In: *Fundamenta Mathematicae* 50 (1961), pp. 207–221. DOI: 10.4064/fm-50-2-207-221.

[4] Eric Finster et al. 'A Type Theory for Strictly Unital ∞-Categories'. In: *Proceedings of the 37th Annual ACM / IEEE Symposium on Logic in Computer Science (LICS 2022)*. 2022, pp. 1–12. DOI: 10.1145/3531130.3533363.

[5] Kristina Sojakova and G. A. Kavvos. 'Syllepsis in Homotopy Type Theory'. In: *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2022)*. 2022. DOI: 10.1145/3531130.3533347.