Double-category of sites

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Introduction

(Grothendieck) topoi can be presented through sites.

As well, geometric morphisms can be presented either:

in a contravariant way, by:	in a covariant way, by:
morphisms of sites	**comorphisms of sites**
characterized by some *cover-preserving* property	characterized by some *cover-lifting* property

Can those two classes be mixed in a same 2-categorical structure on sites ?

Problem: morphisms and comorphisms of sites do not compose with each other !

Solution: make them the horizontal and vertical arrows of a *double-category* of sites.

Such a structure does not require them to compose with each other.

Moreover the sheaf topos construction arranges nicely into a double-functor.

Some results of topos theory will rephrase as double-categorical statements.

This talk is based on the preprint:

O. Caramello and A. Osmond. "Morphisms and comorphisms of sites I – Double categories of sites". In: *arXiv:2505.08766* (2025)

Outlay

Morphisms and comorphisms of sites

Double-category of sites and sheafification double-functor

(Co)morphisms as (co)lax morphisms of coalgebras

Morphisms and comorphisms of sites

Sieves and sites

Definition

A *sieve* on an object c in a category $\mathcal{C}$ is a subobject of the representable $S \rightarrowtail \text{よ}_c$.

A *coverage* J on $\mathcal{C}$ consists for each c of a set $J(c)$ of sieves on c such that

- for each c, the *maximal sieve*, which is よ_c, is in $J(c)$
- for each arrow $a : d \to c$ and each S in $J(c)$, the *pullback sieve* below is in $J(d)$

$$a^*S = \left\{ v : d' \to d \mid \exists u : c' \to c \in S \text{ and a factorization } \begin{array}{ccc} d' & \overset{\exists}{\dashrightarrow} & c' \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle u \in S} \\ d & \underset{a}{\longrightarrow} & c \end{array} \right\}$$

We will assume coverages to be *sifted*: if $S \leq R$ and $S \in J(c)$, then $R \in J(c)$.

A *site* is a pair $(\mathcal{C}, J)$ with J a (sifted) coverage on $\mathcal{C}$.

Any (small generated) site $(\mathcal{C}, J)$ induces a topos $\mathbf{Sh}(\mathcal{C}, J)$.

Notions of morphisms between sites

A functor between sites may behave in two relevant ways relative to the coverages:

- either by *preserving* covering sieves;
- either by *lifting* covering sieves.

Combined with flatness, cover-preservation defines a notion of *morphism of sites*.

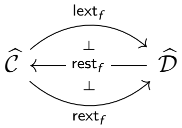
On the other hand cover-lifting defines a notion of *comorphism of sites*.

Both induce geometric morphisms between associated sheaf topoi.

Let us revisit those ideas through the formalism of *extension and restriction*.

Extension and restriction

Recall that any functor $f : \mathcal{C} \to \mathcal{D}$ induces a triple of adjoints



where lext_f (resp. rext_f) sends a presheaf $X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ to

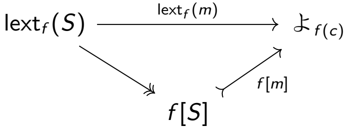
$$\mathrm{lext}_f X = \mathrm{lan}_{f^{\mathrm{op}}} X \qquad\qquad \mathrm{rext}_f X = \mathrm{ran}_{f^{\mathrm{op}}} X$$

while rest_f sends a presheaf $Y : \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$ to

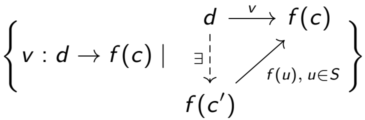
$$\mathrm{rest}_f Y = Y \circ f^{\mathrm{op}}$$

Beware that lext and rext are covariant in f, while rest is contravariant in f.

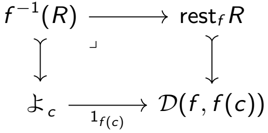
For a sieve $S \rightarrowtail \text{よ}_c$ in $\mathcal{C}$, the left extension $\mathrm{lext}_f S$ induces a sieve on $f(c)$ by taking the image



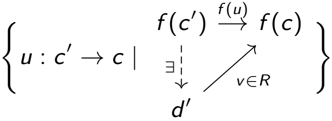
corresponding to the set of arrows



For a sieve $R \rightarrowtail \text{よ}_d$ in $\mathcal{D}$, the restriction $\mathrm{rest}_f(R)$ induces a sieve on c by taking the pullback



corresponding to the set of arrows



Morphisms and comorphisms of sites

Let $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a functor between sites; then :

Definition

f is *cover-preserving* if for any c in $\mathcal{C}$ and any $S \rightarrowtail よ_c$ in $J(c)$, the sieve $f[S]$ is in $K(f(c))$.

A *morphism of sites* is a **flat** functor that is cover-preserving.

Call $\mathbf{Sit}^\flat$ the 2-category of

- sites
- morphisms of sites
- and transformations

Definition

f is *cover-lifting* if for any c in $\mathcal{C}$ and any $R \rightarrowtail よ_{f(c)}$ in $K(f(c))$, the sieve $f^{-1}(R)$ is in $J(c)$.

A *comorphism of sites* is a functor that is cover-lifting.

Call $\mathbf{Sit}^\sharp$ the 2-category of

- sites
- comorphisms of sites
- and transformations

Geometric morphisms induced from (co)morphisms of sites

A morphism of sites

$$f : (\mathcal{C}, J) \to (\mathcal{D}, K)$$

induces a geometric morphism

$$\mathbf{Sh}(f) : \widehat{\mathcal{D}}_K \to \widehat{\mathcal{C}}_J$$

with inverse image

$$\begin{array}{ccc} \widehat{\mathcal{C}} & \xrightarrow{\mathrm{lext}_f} & \widehat{\mathcal{D}} \\ {\scriptstyle \mathfrak{i}_J}\big\uparrow & & \big\downarrow{\scriptstyle \mathfrak{a}_K} \\ \widehat{\mathcal{C}}_J & \xrightarrow[\mathbf{Sh}(f)^*]{} & \widehat{\mathcal{D}}_K \end{array}$$

This defines a 1-contravariant, 2-covariant pseudofunctor

$$(\mathbf{Sit}^\flat)^{\mathrm{op}} \xrightarrow{\mathbf{Sh}} \mathbf{Top}$$

A comorphism of sites

$$F : (\mathcal{C}, J) \to (\mathcal{D}, K)$$

induces a geometric morphism

$$C_F : \widehat{\mathcal{C}}_J \to \widehat{\mathcal{D}}_K$$

with inverse image

$$\begin{array}{ccc} \widehat{\mathcal{D}} & \xrightarrow{\mathrm{rest}_F} & \widehat{\mathcal{C}} \\ {\scriptstyle \mathfrak{i}_J}\big\uparrow & & \big\downarrow{\scriptstyle \mathfrak{a}_K} \\ \widehat{\mathcal{D}}_L & \xrightarrow[C_F^*]{} & \widehat{\mathcal{C}}_J \end{array}$$

This defines a 1-covariant, 2-contravariant pseudofunctor

$$(\mathbf{Sit}^\sharp)^{\mathrm{co}} \xrightarrow{C} \mathbf{Top}$$

Double-category of sites and sheafification double-functor

Double-category of sites

Definition

We define the double-category $\mathbf{Sit}^{\natural}$ as having

- as objects (small generated) sites,
- as horizontal arrows morphisms of sites,
- as vertical arrows comorphisms of sites,
- and as a double-cells lax squares as below, with $\begin{cases} f, h \text{ morphisms of sites} \\ G, K \text{ comorphisms of sites} \end{cases}$

$$\begin{array}{ccc} (\mathcal{A}, M) & \xrightarrow{f} & (\mathcal{B}, L) \\ {\scriptstyle G}\big\downarrow\!\!\!\!+ & \phi & +\!\!\!\!\big\downarrow{\scriptstyle K} \\ (\mathcal{C}, J) & \xrightarrow[h]{} & (\mathcal{D}, K) \end{array} \qquad \begin{array}{ccc} (\mathcal{A}, M) & \xrightarrow{f} & (\mathcal{B}, L) \\ {\scriptstyle G}\big\downarrow & \overset{\phi}{\swarrow} & \big\downarrow{\scriptstyle K} \\ (\mathcal{C}, J) & \xrightarrow[h]{} & (\mathcal{D}, K) \end{array}$$

Constructing a sheafification double-functor

We want a double-functor $\begin{cases} \text{with horizontal component the pseudofunctor } \mathbf{Sh} \\ \text{with vertical component the pseudofunctor } C \end{cases}$

For double-cell, suppose one has a 2-cell of the following form:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle g}\downarrow & \overset{\phi}{\swarrow\!\!\!\swarrow} & \downarrow{\scriptstyle k} \\ \mathcal{C} & \xrightarrow[h]{} & \mathcal{D} \end{array}$$

Such a square induces a cross-adjoint square constructed from the composite 2-cell

$$\begin{array}{ccc} \widehat{\mathcal{A}} & \xrightarrow{\mathrm{lext}_f} & \widehat{\mathcal{B}} \\ {\scriptstyle \mathrm{rest}_g}\uparrow & \overset{\overline{\phi}}{\searrow\!\!\!\searrow} & \uparrow{\scriptstyle \mathrm{rest}_k} \\ \widehat{\mathcal{C}} & \xrightarrow[\mathrm{lext}_h]{} & \widehat{\mathcal{D}} \end{array} \qquad \begin{array}{ccc} \mathrm{lext}_f \mathrm{rest}_g & \xRightarrow{\eta_h} & \mathrm{lext}_f \mathrm{rest}_g \mathrm{rest}_h \mathrm{lext}_h \\ {\scriptstyle \overline{\phi}}\Downarrow & & \Downarrow{\scriptstyle \mathrm{rest}_\phi} \\ \mathrm{rest}_k \mathrm{lext}_h & \xLeftarrow[\epsilon_f]{} & \mathrm{lext}_f \mathrm{rest}_f \mathrm{rest}_k \mathrm{lext}_h \end{array}$$

Sheafification of double-cells

If now one has sites $(\mathcal{A}, M)$, $(\mathcal{B}, L)$, $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$, related through a lax square

$$\begin{array}{ccc} (\mathcal{A}, M) & \xrightarrow{f} & (\mathcal{B}, L) \\ {\scriptstyle G}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle K} \\ (\mathcal{C}, J) & \xrightarrow[h]{} & (\mathcal{D}, K) \end{array}$$

f, h morphisms of sites
G, K comorphisms of sites

then the sheafification functor $\mathfrak{a}_L : \widehat{\mathcal{B}} \to \widehat{\mathcal{B}}_L$ sends $\overline{\phi}$ a 2-cell $\phi^\flat$ corresponding to the inverse image part of a geometric transformation

$$\begin{array}{ccc} \widehat{\mathcal{A}}_M & \xrightarrow{\widehat{f}^*} & \widehat{\mathcal{B}}_L \\ {\scriptstyle C_G^*}\uparrow & \overset{\phi^\flat}{\searrow} & \uparrow{\scriptstyle C_K^*} \\ \widehat{\mathcal{C}}_J & \xrightarrow[\widehat{h}^*]{} & \widehat{\mathcal{D}}_K \end{array} \qquad \begin{array}{ccc} \widehat{\mathcal{A}}_M & \xleftarrow{\mathbf{Sh}(f)} & \widehat{\mathcal{B}}_L \\ {\scriptstyle C_G}\downarrow & \overset{\widehat{\phi}}{\searrow} & \downarrow{\scriptstyle C_K} \\ \widehat{\mathcal{C}}_J & \xleftarrow[\mathbf{Sh}(h)]{} & \widehat{\mathcal{D}}_K \end{array}$$

This is a 2-cell in the *lax quintet* double-category $\mathbf{Top}^\square$ of Grothendieck topoi.

Sheafification as a double-functor

Theorem

Sheafification defines a surjective on object double-functor into the lax quintet double-category of topoi, which is

- *horizontally contravariant with horizontal component* **Sh**
- *vertically covariant with vertical component C*

$$
\begin{array}{ccccc}
(\mathbf{Sit}^{\flat})^{\mathrm{op}} & \xrightarrow{h} & (\mathbf{Sit}^{\natural})^{hop}_{vco} & \xleftarrow{v} & (\mathbf{Sit}^{\sharp})^{\mathrm{co}} \\
& {\scriptstyle \mathbf{Sh}} \searrow & \downarrow & \swarrow {\scriptstyle C} & \\
& & \mathbf{Top}^{\square} & &
\end{array}
$$

Lax squares inverted by this double-functor generalize the notion of *exact squares*.

Exact squares

A lax square in **Cat** is *exact* if the associated extension/restriction 2-cell is invertible

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ {\scriptstyle g}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle k} \\ \mathcal{C} & \xrightarrow[h]{} & \mathcal{D} \end{array} \qquad \begin{array}{ccc} \widehat{\mathcal{A}} & \xrightarrow{\mathrm{lext}_f} & \widehat{\mathcal{B}} \\ {\scriptstyle \mathrm{rest}_g}\uparrow & \underset{\simeq}{\overline{\phi}} & \uparrow{\scriptstyle \mathrm{rest}_k} \\ \widehat{\mathcal{C}} & \xrightarrow[\mathrm{lext}_h]{} & \widehat{\mathcal{D}} \end{array}$$

Definition

A lax square as below left (underlying a double-cell of $\mathbf{Sit}^\natural$) will be said *locally exact* if the corresponding transformation below right is invertible

$$\begin{array}{ccc} (\mathcal{A}, M) & \xrightarrow{f} & (\mathcal{B}, L) \\ {\scriptstyle G}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle K} \\ (\mathcal{C}, J) & \xrightarrow[h]{} & (\mathcal{D}, K) \end{array} \qquad \begin{array}{ccc} \widehat{\mathcal{A}}_M & \xrightarrow{\mathbf{Sh}(f)^*} & \widehat{\mathcal{B}}_L \\ {\scriptstyle C_G^*}\uparrow & \underset{\widetilde{\phi}^\flat}{\searrow} & \uparrow{\scriptstyle C_K^*} \\ \widehat{\mathcal{C}}_J & \xrightarrow[\mathbf{Sh}(h)^*]{} & \widehat{\mathcal{D}}_K \end{array}$$

This admits a concrete characterization in term of *relative cofinality* à la [2].

Comma and cocomma squares in $\mathbf{Sit}^\natural$

In **Cat** the ur-examples of exact squares are comma and cocomma; similarly:

Proposition

If $\begin{cases} f\colon(\mathcal{C},J)\to(\mathcal{D},K) \text{ morph.} \\ G\colon(\mathcal{B},L)\to(\mathcal{D},K) \text{ comorph.}\end{cases}$

then there is a topology $J_{G,f}$ on $G\downarrow f$ that makes the comma square a double-cell of $\mathbf{Sit}^\natural$

$$\begin{array}{ccc} (G\downarrow f, J_{G,f}) & \xrightarrow{\pi_0} & (\mathcal{B},L) \\ {\scriptstyle \pi_1}\big\downarrow & \lambda_{G,f} & \big\downarrow{\scriptstyle G} \\ (\mathcal{C},J) & \xrightarrow[f]{} & (\mathcal{D},K) \end{array}$$

Proposition

If $\begin{cases} f\colon(\mathcal{C},J)\to(\mathcal{B},L) \text{ morph.} \\ G\colon(\mathcal{C},J)\to(\mathcal{D},K) \text{ comorph.}\end{cases}$

then there is a topology $J_{f,G}$ on $f\uparrow G$ that makes the cocomma square a double-cell of $\mathbf{Sit}^\natural$

$$\begin{array}{ccc} (\mathcal{C},J) & \xrightarrow{f} & (\mathcal{B},L) \\ {\scriptstyle G}\big\downarrow & \lambda_{f,G} & \big\downarrow{\scriptstyle \iota_0} \\ (\mathcal{D},K) & \xrightarrow[\iota_1]{} & (f\uparrow G, J_{f,G}) \end{array}$$

In both cases, the underlying square in **Cat** is exact.

Some effects of the mixed variancy of Sh

Vertical and horizontal cells can be related either through:

Definition

- *conjoint squares*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \| & \epsilon & \downarrow G \\ A & = & A \end{array} \qquad \begin{array}{ccc} B & = & B \\ G\downarrow & \eta & \| \\ A & \xrightarrow[f]{} & B \end{array}$$

$$\epsilon \bullet \eta = 1_G \qquad \eta \circ \epsilon = \mathrm{id}_f$$

Definition

- *companion squares*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ G\downarrow & \phi & \| \\ B & = & B \end{array} \qquad \begin{array}{ccc} A & = & A \\ \| & \psi & \downarrow G \\ A & \xrightarrow[f]{} & B \end{array}$$

$$\phi \bullet \psi = 1_G \qquad \psi \circ \phi = \mathrm{id}_f$$

By mixed variancy, **Sh** sends $\begin{cases} \text{companions to conjoints} \\ \text{conjoints to companions} \end{cases}$

Those squares are alike those in **Cat** whose exactness captures:

- the fact of being adjoint
- the fact of being respectively fully faithful and absolutely dense.

Adjunction through exactness

A functor f is right adjoint to G in **Cat** iff there is an exact square as below

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \| & \overset{\epsilon}{\Longleftarrow} & \downarrow G \\ \mathcal{C} & = & \mathcal{C} \end{array} \qquad \mathrm{lext}_f \simeq \mathrm{rest}_G$$

Definition

A morphism of site $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ will be said to be *weakly right adjoint* to a comorphism G if there exist a locally exact square as below:

$$\begin{array}{ccc} (\mathcal{C}, J) & \xrightarrow{f} & (\mathcal{D}, K) \\ \| & \overset{\epsilon}{\Longleftarrow} & \downarrow G \\ (\mathcal{C}, J) & = & (\mathcal{C}, J) \end{array}$$

Proposition

f is weakly right adjoint to G iff f and G induce the same geometric morphism

$$\mathbf{Sh}(f) \simeq C_G : \mathbf{Sh}(\mathcal{D}, K) \to \mathbf{Sh}(\mathcal{C}, J)$$

Local exactness criterion for local and totally connected morphisms

A functor $f : \mathcal{C} \to \mathcal{D}$ is fully faithful iff its identity square below is exact:

$$\begin{array}{ccc} \mathcal{C} & = & \mathcal{C} \\ \| & & \downarrow{\scriptstyle f} \\ \mathcal{C} & \xrightarrow{f} & \mathcal{D} \end{array} \qquad \mathrm{rest}_f \mathrm{lext}_f = \mathrm{id}$$

A functor $f : \mathcal{C} \to \mathcal{D}$ is *absolutely dense* iff the square below is exact:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ {\scriptstyle f}\downarrow & & \| \\ \mathcal{D} & = & \mathcal{D} \end{array} \qquad \mathrm{lext}_f \mathrm{rest}_f = 1$$

Proposition

Let f be both $\left\{ \begin{array}{l} \textit{a morphism} \\ \textit{a comorphism} \end{array} \right.$

$\mathbf{Sh}(f)$ *is a* **local geometric morphism** *iff this square is locally exact:*

$$\begin{array}{ccc} (\mathcal{C}, J) & = & (\mathcal{C}, J) \\ \| & & \downarrow{\scriptstyle f} \\ (\mathcal{C}, J) & \xrightarrow[f]{} & (\mathcal{D}, K) \end{array}$$

Proposition

Let f be both $\left\{ \begin{array}{l} \textit{a morphism} \\ \textit{a comorphism} \end{array} \right.$

C_f *is a* **totally connected geometric morphism** *iff this square is locally exact:*

$$\begin{array}{ccc} (\mathcal{C}, J) & \xrightarrow{f} & (\mathcal{D}, K) \\ {\scriptstyle f}\downarrow & & \| \\ (\mathcal{D}, K) & = & (\mathcal{D}, K) \end{array}$$

(Co)morphisms as (co)lax morphisms of coalgebras

Why a double-categorical structure ?

Why morphisms and comorphisms do arrange in a double-category ?

This double-category is an instance of a special family of double-categories.

double-categories of profunctors have a different flavour and are less symmetrical.

Here horizontal and vertical maps are two classes of functors with dual properties, rather than a class of functors and a class of relations generalizing them.

This reminds a more symmetric kind of double-categories, those that arise as double-categories of (co)algebras with lax and colax morphisms for (co)monads !

Coalgebra and (co)lax morphisms for a copointed endofunctor

Definition

Let $\mathcal{K}$ be a 2-category and $T : \mathcal{K} \to \mathcal{K}$ a copointed endo-2-functor;
A *coalgebra* for (T, ε) is a pair (C, γ) with C in $\mathcal{K}$ and α a section of the counit

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & TC \\ & \searrow^{=} & \downarrow{\scriptstyle \varepsilon_C} \\ & & C \end{array}$$

A *lax* (resp. *colax*) *morphism* of coalgebras $(C, \gamma) \to (D, \delta)$ is a pair (f, ϕ) with $f : C \to D$ in $\mathcal{K}$ and ϕ a 2-cell as on the left (resp. on the right)

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \gamma}\downarrow & \overset{\phi}{\Longleftarrow} & \downarrow{\scriptstyle \delta} \\ TC & \xrightarrow[Tf]{} & TD \end{array} \qquad \begin{array}{ccc} C & \xrightarrow{f} & D \\ {\scriptstyle \gamma}\downarrow & \overset{\phi}{\Longrightarrow} & \downarrow{\scriptstyle \delta} \\ TC & \xrightarrow[Tf]{} & TD \end{array}$$

whose pasting along the naturality square of the pointer is an identity 2-cell.

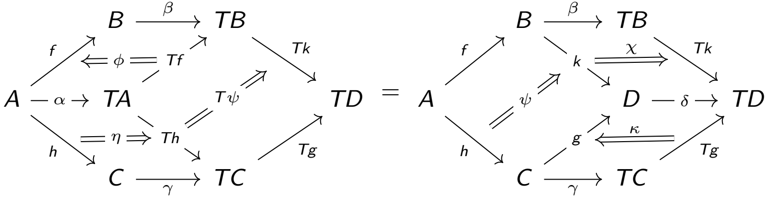
Double-category of coalgebras, lax and colax morphisms

Proposition (After Paré-Grandis [3])

*For any copointed endo-2-functor T, one can form a double-category T-**coAlg** of strict coalgebras, lax morphisms as horizontal cells, colax morphisms as vertical cells, and as 2-cell, the lax squares of the form*

$$\begin{array}{ccc} (A,\alpha) & \xrightarrow{(f,\phi)} & (B,\beta) \\ {\scriptstyle (h,\eta)}\downarrow & \overset{\psi}{\Longrightarrow} & \downarrow{\scriptstyle (k,\chi)} \\ (C,\gamma) & \xrightarrow[(g,\kappa)]{} & (D,\delta) \end{array}$$

The double-cells of this double-category consist hence in 2-cells $\psi : gh \Rightarrow kf$ intertwinning the lax and colax morphism structures in the following coherence



The category $\mathbb{S}\mathcal{C}$

Definition

Define for each category $\mathcal{C}$ the category $\mathbb{S}\mathcal{C}$ as having:

- as objects pairs (c, F) with c an object of $\mathcal{C}$ and F a filter of $\mathrm{Sub}_{\widehat{\mathcal{C}}}\,よ_c$
- as morphisms $(c, F) \to (c', F')$ morphisms $u : c \to c'$ such that

$$F' \leq (u^*)^{-1}(F)$$

This category is fibered over $\mathcal{C}$ with posetal fibers $\mathbb{F}_{\mathcal{C}}(c)$ at each c

$$\mathbb{S}\mathcal{C} \xrightarrow{\pi_{\mathcal{C}}} \mathcal{C}$$

For a functor $f : \mathcal{C} \to \mathcal{D}$, one can define at each filter F of $\mathrm{Sub}_{\widehat{\mathcal{C}}}\,よ_c$ the filter $f[F]$ generated from the set of sieves of the form $f[S]$ for S in F.

Defines the functor $\mathbb{S}f : \mathbb{S}\mathcal{C} \to \mathbb{S}\mathcal{D}$ sending (c, F) to the pair $(f(c), f[F])$.

The copointed endo-2-functor $\mathbb{S}$

Hence this construction is functorial on **Cat**: we have an endo-2-functor

$$\mathbf{Cat} \xrightarrow{\mathbb{S}} \mathbf{Cat}$$

Moreover this endofunctor is copointed through the projections $\pi_{\mathcal{C}} : \mathbb{S}\mathcal{C} \to \mathcal{C}$, whose naturality produce morphisms of fibrations

$$\begin{array}{ccc} \mathbb{S}\mathcal{C} & \xrightarrow{\mathbb{S}f} & \mathbb{S}\mathcal{D} \\ {\scriptstyle \pi_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{D}}} \\ \mathcal{C} & \xrightarrow[f]{} & \mathcal{D} \end{array}$$

Coverages as coalgebra structure

Proposition

A coalgebra structure on $\mathcal{C}$ for the copointed endofunctor $(\mathbb{S}, \pi)$ is a coverage on $\mathcal{C}$.

Indeed a section $J : \mathcal{C} \to \mathbb{S}\mathcal{C}$ of $\pi_{\mathcal{C}}$ picks for each c a filter of sieves on c.

Moreover functoriality says that for any $u : c \to c'$, one has a restriction

$$\begin{array}{ccc} J(c) & \hookrightarrow & \mathrm{Sub}_{\widehat{\mathcal{C}}}\,\text{よ}_c \\ \uparrow & & \uparrow{\scriptstyle u^*} \\ J(c') & \hookrightarrow & \mathrm{Sub}_{\widehat{\mathcal{C}}}\,\text{よ}_{c'} \end{array}$$

expressing that for any R in $J(c')$ the pullback sieve u^*R is in $J(c)$.

This is exactly what a coverage is!

(Co)morphisms of sites as (co)lax morphisms of coalgebras

Proposition

A functor is a lax morphism of coalgebras iff it is cover-preserving.

Having a 2-cell as below

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ {\scriptstyle J}\downarrow & \overset{\phi}{\Longleftarrow} & \downarrow{\scriptstyle K} \\ \mathbb{S}\mathcal{C} & \xrightarrow[\mathbb{S}f]{} & \mathbb{S}\mathcal{D} \end{array}$$

amounts to an inequality at each c

$$f[J(c)] \leq K(f(c))$$

which means that for any S in $J(c)$, $f[S]$ is in $K(f(c))$.

Proposition

A functor is a colax morphism of coalgebras iff it is cover-lifting.

Having a 2-cell as below

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ {\scriptstyle J}\downarrow & \overset{\phi}{\Longrightarrow} & \downarrow{\scriptstyle K} \\ \mathbb{S}\mathcal{C} & \xrightarrow[\mathbb{S}f]{} & \mathbb{S}\mathcal{D} \end{array}$$

amounts to an inequality at each c

$$K(f(c)) \leq f[J(c)]$$

which means that any R in $K(f(c))$ contains some $f[S]$ for a S in $J(c)$.

Further aspects and future directions

There are still several questions regarding this double-categorical approach:

- actually $\mathbb{S}$ bears a structure of 2-comonad, but sites are only *normal lax* coalgebras; it is still unclear what strict coalgebras for the full comonad are.
- this can be fixed by an indexed form of this comonad, where sites correctly corresponds to coalgebras;
- combine this with flatness s morphisms of sites are lax morphisms of coalgebras ?
- in [4] **Top** was shown to be a bilocalization of $\mathbf{Sit}^\flat$ at *dense morphisms of sites*. Is similarly $\mathbf{Top}^\square$ a double-localization of $\mathbf{Sit}^\natural$?
- in a upcomming work we will also show how morphisms and comorphisms are subsumed by *continuous distributors* between sites;
- is $\mathbf{Sit}^\natural$ a good framework to do some formal category theory sites mixing the Yoneda structure of **Cat** with coverages ?

Thank you for your attention !

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