A higher categorical approach to the André-Quillen cohomology of $(\infty, 1)$-categories

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Simplicial categories

- Simplicial categories are categories enriched in the category sSet of simplicial sets. That is, for each pair of objects X, Y, there is a simplicial set of maps $Map(X, Y)$ and a composition map

$$Map(X, Y) \times Map(Y, Z) \to Map(X, Z)$$

  which is associative and unital.

- Simplicial categories are a model of $(\infty, 1)$-categories. They have a model structure whose weak equivalences are called Dwyer-Kan equivalences.

- Notation: $(\mathcal{S},\mathcal{O})$-Cat for simplicial categories with object set $\mathcal{O}$.

Postnikov system for simplicial sets

- For each simplicial set Y there is a tower of fibrations

$$Y \to \cdots \to P^n Y \to P^{n-1} Y \to \cdots \to P^1 Y \to P^0 Y$$

  where $P^n Y$ is an n-type, that is $\pi_i P^n Y = \pi_i Y$ if $i \leq n$ and 0 otherwise.

- $P^n Y$ is the homotopy fiber of the $(n-1)^{st}$ k-invariant map

$$k_{n-1} : P^{n-1} Y \to E_{\hat{\pi}_1 Y}(\pi_n Y, n+1)$$

  into the twisted Eilenberg-MacLane simplicial set, representing the cohomology group $H^{n+1}(P^{n-1} Y, \pi_n Y)$ with local coefficients.

- The homotopy type of Y is determined by the Postnikov sections $\{P^n Y\}$ together with the k-invariants of Y.

Postnikov system for simplicial categories

- Dwyer and Kan showed that this also holds for any simplicial category $\mathcal{X}$.
- $P^n\mathcal{X}$ is a category enriched in simplicial n-types.
- $P^n\mathcal{X}$ is the homotopy fiber of the k-invariant map of simplicial categories

$$\mathcal{K}_{n-1} : P^{n-1}\mathcal{X} \to E_{\hat{\pi}_1\mathcal{X}}(\pi_n\mathcal{X}, n+1)$$

  landing in the Eilenberg-MacLane simplicial category representing the André-Quillen cohomology of $\mathcal{X}$.
- The homotopy information about $\mathcal{X}$ is encoded in the Postnikov truncations and their k-invariants. The latter can be used to extract various higher homotopy invariants of $\mathcal{X}$.

Postnikov truncation and André-Quillen cohomology

- There is an isomorphism

$$H^n_{\mathrm{AQ}}(\mathcal{X};\mathcal{D}) \;:=\; [\mathcal{X},\; E_{\hat{\pi}_1\mathcal{X}}(\mathcal{D},n)] \;\cong\; [P^n\mathcal{X},\; E_{\hat{\pi}_1\mathcal{X}}(\mathcal{D},n)]\;.$$

- So to study the n^{th} André-Quillen cohomology group of a simplicial category it suffices to look at simplicial categories enriched in simplicial n-types.

- We can use the algebraic models of n-types from higher category theory to produce an algebraic replacement for $P^n\mathcal{X}$.

Main goal and approach

- We seek an algebraic description of the André-Quillen cohomology of a simplicial category.
- Our main tools are the model of weak n-categories called weakly globular n-fold categories [P.] and its subcategory of weakly globular n-fold groupoids [Blanc and P.].
- The n-fold nature of this model allows to give an explicit cofibrant replacement for computing the André-Quillen cohomology, coming from a comonad resolution.

n-Fold categories and *n*-fold groupoids

- *n*-Fold groupoids and *n*-fold categories are defined by

$$\mathrm{Gpd}^1 = \mathrm{Gpd} \qquad\qquad \mathrm{Cat}^1 = \mathrm{Cat}$$
$$\mathrm{Gpd}^n = \mathrm{Gpd}(\mathrm{Gpd}^{n-1}) \qquad\qquad \mathrm{Cat}^n = \mathrm{Cat}\,(\mathrm{Cat}^{n-1})$$

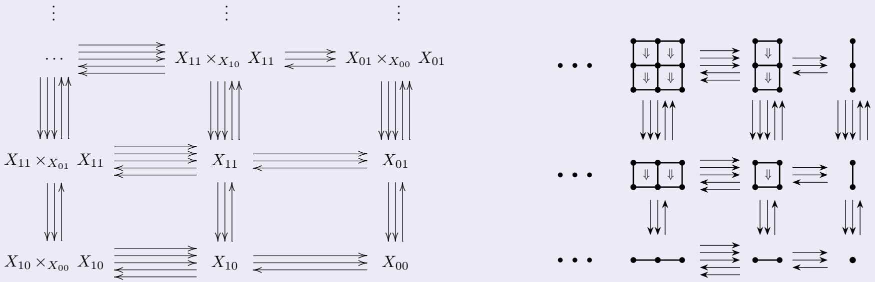
- By iterating the internal nerve construction, we obtain the functor multinerve:

$$\mathrm{Gpd}^n \to [\Delta^{n^{op}}, \mathrm{Set}] \qquad\qquad \mathrm{Cat}^n \to [\Delta^{n^{op}}, \mathrm{Set}]\ .$$

where $\Delta^{n^{op}} = \Delta^{op} \times \cdots^{n} \times \Delta^{op}$

- This affords a more geometric description.

Example: $n = 2$



$\cdots$
$X_{11} \times_{X_{10}} X_{11}$
$X_{01} \times_{X_{00}} X_{01}$
$X_{11} \times_{X_{01}} X_{11}$
X_{11}
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X_{10}
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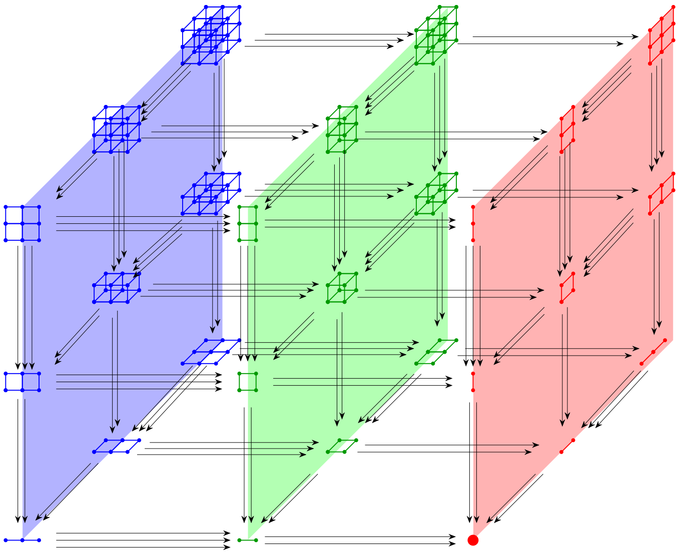
Example: $n = 3$



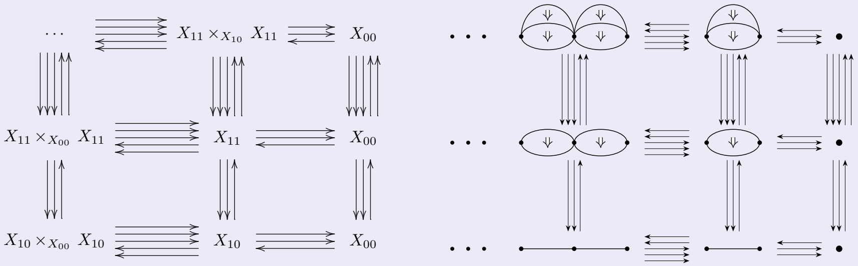
Strict n-categories versus n-fold categories

- In an n-fold category, the n directions are interchangeable, making it difficult to identify the 'set of k-cells' for $k = 1, \cdots, n$.
- To gain intuition on how remedy this, consider an embedding

$$n\text{-Cat} \hookrightarrow \text{Cat}^n$$

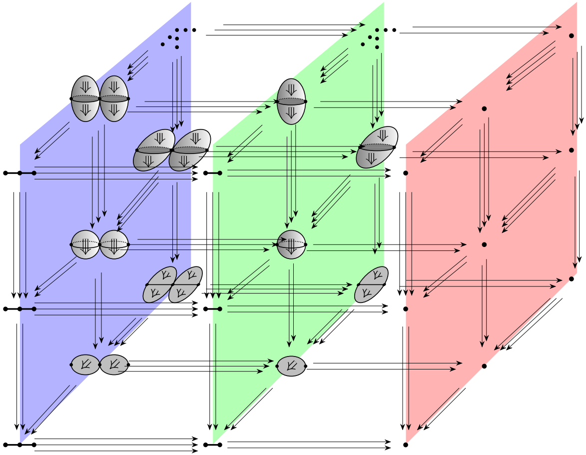
  of strict n-categories into n-fold categories.
- The multinerve of strict n-categories has certain simplicial directions which are constant simplicial sets. This is the globularity condition.

Example: multinerve of strict 2-categories



Geometric picture of the 3-fold nerve of a strict 3-category *X*

$$3\text{-Cat} \xhookrightarrow{N_{(3)}} [\Delta^{3^{op}}, \text{Set}]$$



The idea of weakly globular n-fold categories

- The category $\mathrm{Cat}^n_{\mathrm{wg}}$ of weakly globular n-fold categories is intermediate between strict n-categories and n-fold categories:

$$n\text{-Cat} \hookrightarrow \mathrm{Cat}^n_{\mathrm{wg}} \hookrightarrow \mathrm{Cat}^{\,n}$$

- The simplicial directions which are constant in the multinerve of a strict n-category are only 'homotopically constant' in a weakly globular n-fold category.

- This is the weak globularity condition. The precise formulation uses the notion of homotopically discrete n-fold categories, which are n-dimensional fattening of sets.

A higher categorical model of n-types

Theorem (P. $n \geq 3$, P. and Pronk $n = 2$)

The category Cat^{n}_{wg} *of weakly globular n-fold categories is a model of weak n-categories:*

There is a subcategory $\mathsf{GCat}^{n}_{wg} \subset \mathsf{Cat}^{n}_{wg}$ *of groupoidal weakly globular n-fold categories which is an algebraic model of n-types:*

$$\mathsf{GCat}^{n}_{wg}/{\sim^{n}} \simeq \mathit{Ho}(n\text{-}types)\ .$$

Definition

The category Gpd^{n}_{wg} of weakly globular n-fold groupoids is the full subcategory of GCat^{n}_{wg} whose objects are in Gpd^{n}.

Theorem (Blanc and P.)

There are functors

$$\mathcal{H}_n : n\text{-}types \leftrightarrows \mathsf{Gpd}^{n}_{wg} : B$$

inducing functors

$$\mathcal{H}o(n\text{-}types) \leftrightarrows \mathsf{Gpd}^{n}_{wg}/\sim^{n}$$

with $B\mathcal{H}_n \cong \mathrm{Id}$.

Definition

- An *n-track category* is a category enriched in the category $\mathsf{Gpd}^{\mathsf{n}}_{\mathsf{wg}}$ of weakly globular n-fold groupoids with respect to the cartesian monoidal structure. Notation: $\mathsf{Track}^{\mathsf{n}}_{\mathcal{O}}$.

- We work with $\mathsf{Gpd}^{\mathsf{n}}_{\mathsf{wg}}$ instead of $\mathsf{GCat}^{\mathsf{n}}_{\mathsf{wg}}$ as this allows to more easily build a comonad on n-track categories.

- The functor $D : \mathsf{Track}^{\mathsf{n}}_{\mathcal{O}} \to [\Delta^{op}, \mathsf{Cat}_{\mathcal{O}}] = (\mathcal{S},\mathcal{O})\text{-}\mathsf{Cat}$ collapses all groupoid directions in an n-track catergory into a single simplicial direction.

- Let $\mathsf{Spl}\,\mathcal{C}$ be the category of split epimorphisms in a category $\mathcal{C}$ with finite (co)limits.

- The functor $H : \mathsf{Spl}\,\mathcal{C} \to \mathsf{Gpd}\,\mathcal{C}$ takes $Y : A \underset{t}{\overset{q}{\rightleftarrows}} B$ to the internal groupoid (called internal equivalence relation)

$$A \overset{q}{\times} A \overset{q}{\times} A \xrightarrow{m} A \overset{q}{\times} A \underset{\Delta}{\overset{pr_1,\ pr_2}{\rightrightarrows\!\!\leftarrow}} A$$

where $A \overset{q}{\times} A$ is the kernel pair of q, $\Delta = (\mathrm{Id}_A, \mathrm{Id}_A)$ is the diagonal map and $m = (pr_1, pr_2) : A \overset{q}{\times} A \overset{q}{\times} A \cong (A \overset{q}{\times} A) \times_A (A \overset{q}{\times} A) \to A \overset{q}{\times} A$.

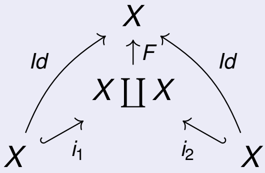
- When $\mathcal{C} = \mathrm{Set}$, HY is a groupoid with no non-trivial loops.

The internal arrow functor

- Let $U : \mathrm{Gpd}\,\mathcal{C} \to \mathcal{C}$ be the internal arrow functor $UY = Y_1$, where Y is

$$Y_1 \times_{Y_0} Y_1 \longrightarrow Y_1 \underset{\longleftarrow}{\overset{\longrightarrow}{\longrightarrow}} Y_0$$

- Let $\mathcal{L}X = H(X \coprod X \underset{i_1}{\overset{F}{\rightleftarrows}} X)$, where F is the map



- Lemma: $\mathcal{L}$ is left adjoint to U.

Some functors on weakly globular n-fold groupoids

- By definition of Gpd^{n}_{wg} there is an inclusion

$$\mathsf{Gpd}^{n}_{wg} \hookrightarrow \mathsf{Gpd}(\mathsf{Gpd}^{n-1}_{wg})$$

- Thus there is the internal arrow functor

$$u_{n-1} : \mathsf{Gpd}^{n}_{wg} \to \mathsf{Gpd}^{n-1}_{wg}$$

with left adjoint

$$\ell_{n-1} : \mathsf{Gpd}^{n-1}_{wg} \to \mathsf{Gpd}(\mathsf{Gpd}^{n-1}_{wg})$$
$$\ell_{n-1} X = H(X \amalg X \rightleftarrows X)$$

- In fact $\ell_{n-1} X \in \mathsf{Gpd}^{n}_{wg}$.

Some functors on n-track categories

Lemma

The functors $\ell_{n-1} : \mathrm{Gpd}^{\mathrm{n-1}}_{\mathrm{wg}} \leftrightarrows \mathrm{Gpd}^{\mathrm{n}}_{\mathrm{wg}} : u_{n-1}$ *induce functors*

$$L_{[n-1]} : \mathrm{Track}^{\mathrm{n-1}}_{\mathcal{O}} \leftrightarrows \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}} : U_{[n-1]}$$

with $L_{[n-1]}$ *left adjoint to* $U_{[n-1]}$*.*

A comonad on n-track categories

- Applying repeatedly the above lemma we obtain an adjunction

$$L_n : \mathrm{Cat}_{\mathcal{O}} \leftrightarrows \mathrm{Track}^n_{\mathcal{O}} : U_n$$

$$U_n : U_{[n-1]} \cdots U_{[1]} U_{[0]}, \qquad L_n = L_{[0]} L_{[1]} \cdots L_{[n-1]}$$

- There is a free-forgetful functors adjunction, where $\mathrm{Graph}_{\mathcal{O}}$ is the category of graphs

$$F : \mathrm{Graph}_{\mathcal{O}} \leftrightarrows \mathrm{Cat}_{\mathcal{O}} : V$$

- By composition, we obtain the adjunction

$$L_n F : \mathrm{Graph}_{\mathcal{O}} \leftrightarrows \mathrm{Track}^n_{\mathcal{O}} : V U_n$$

giving rise to a comonad on $\mathrm{Track}^n_{\mathcal{O}}$

$$\mathcal{K} = L_n F V U_n : \mathrm{Track}^n_{\mathcal{O}} \to \mathrm{Track}^n_{\mathcal{O}} \, .$$

Comonad resolution

- Given $X \in \mathrm{Track}^n_{\mathcal{O}}$, the simplicial object $\mathcal{K}_\bullet X \in [\Delta^{op}, \mathrm{Track}^n_{\mathcal{O}}]$ is augmented over X via $\varepsilon : \mathcal{K}_\bullet X \to X$.

Proposition (Blanc and P.)

ε *induces a map* $\alpha : \mathrm{Diag}\, \mathcal{K}_\bullet X \to DX$ *of* $(\mathcal{S},\mathcal{O})$-Cat *which exhibits* $\mathrm{Diag}\, \mathcal{K}_\bullet X$ *as a cofibrant replacement of* DX *in the Dwyer-Kan model category on* $(\mathcal{S},\mathcal{O})$-Cat *.*

Modules

- Let $(\text{Gpd}\,\mathcal{C}, X_0)$ the subcategory of Gpd $\mathcal{C}$ consisting of those Y with $Y_0 = X_0$, and groupoid maps which are the identity on X_0.

Definition

For $X \in (\mathrm{Gpd}\,\mathcal{C}, X_0)$, an *$X$-module* is an abelian group object M in the slice category $(\mathrm{Gpd}\,\mathcal{C}, X_0)\,/X$.

Proposition

There exist objects $E^{(n)}(Q, M) \in \mathrm{Track}^{n}_{\mathcal{O}}$ *such that*

a) $E^{(n)}(Q, M)$ *is an abelian group object in* $(\mathrm{Gpd}^n\text{-}\mathrm{Cat}_{\mathcal{O}}, \mathrm{d}^{(n-1)}\, Q)/\, \mathrm{d}^{(n)}\, Q$.

b) $E^{(n)}(Q, M)$ *is an Eilenberg-Mac Lane object in* $\mathrm{Gpd}^n\text{-}\mathrm{Cat}_{\mathcal{O}}$.

- Recall the functor $D : \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}} \to (\mathcal{S},\mathcal{O})\text{-}\mathrm{Cat}$.

Definition (Dwyer, Kan, Smith)

Let $X \in \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}$, M a module over $p^{(1)}X \in \mathrm{Track}_{\mathcal{O}}$, $E_X(M,n)$ the corresponding twisted Eilenberg-MacLane $(\mathcal{S},\mathcal{O})$-category.
The André-Quillen cohomology of X with coefficients in M is given by

$$H^{n-i}_{\mathrm{AQ}}(DX,M) \;=\; \pi_i \;\mathrm{map}_{(\mathcal{S},\mathcal{O})\text{-}\mathrm{Cat}\;/DX}(\mathrm{Diag}\,\overline{\mathcal{F}}_{\bullet}DX, E_X((,M),n))\;.$$

where $\mathrm{Diag}\,\overline{\mathcal{F}}_{\bullet}DX \to DX$ is the Dwyer-Kan standard free resolution.

- Let $X \in \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}$, M a module over $p^{(1)}X \in \mathrm{Track}_{\mathcal{O}}$. Let $d_I : \mathcal{K}_s X \to X$ be the iterated face map. Let

$$D^s = \mathrm{Hom}_{\mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}/\mathcal{K}_s X}(\mathcal{K}_s X, E^{(n)}(p^{(1)}\mathcal{K}_s X, d_I^* M)) \ .$$

  Then $\{D^s\}_{s \geq 0}$ is a cosimplicial abelian group.

Definition

Let $X \in \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}$, M a $p^{(1)}X$-module. The *n-th algebraic cohomology group* of X with coefficients in M is

$$H^n_{\mathrm{Alg}}(X; M) := \pi^n D^\bullet$$

- Question: How do André-Quillen cohomology and algebraic cohomology of a track category compare?

A long exact sequence for cohomology of track categories

Theorem (Blanc and P.)

For any $X \in \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}$ and module M over $\mathrm{p}^{(1)}X \in \mathrm{Track}_{\mathcal{O}}$, there is a long exact sequence of cohomology groups

$$\to H^s_{\mathrm{AQ}}(DX; M) \to H^s_{\mathrm{AQ}}(DX_0; M) \to H^s_{\mathrm{Alg}}(X; M) \to H^{s-1}_{\mathrm{AQ}}(DX; M) \cdots .$$

An algebraic description of André-Quillen cohomology

Proposition (Blanc and P.)

Given $X \in \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}$ *for* $n \geq 2$*, there exists* $S(X) \in \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}$ *with* $p^{(0)}(S(X))_0 \in \mathrm{Cat}_{\mathcal{O}}$ *a free category, and an n-equivalence* $v_X : S(X) \to X$*.*

Corollary

Given $X \in \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}$ *a module* M *over* $p^{(1)}X$ *and* $S(X)$ *as in the above Proposition, for each* $s > 1$ *we have an isomorphism*

$$H^{s+1}_{\mathrm{AQ}}(DX; M) \;\cong\; H^{s}_{\mathrm{Alg}}(S(X); M)\ .$$

Proof of corollary

- The n-equivalence $v_X : S(X) \to X$ induces a Dwyer-Kan equivalence $Dv_X : DS(X) \to DX$ and therefore an isomorphism
$$H^s_{\mathrm{AQ}}(DX; M) \cong H^s_{\mathrm{AQ}}(DS(X); M)$$
- Since $S(X)_0$ is homotopically discrete, for each $s > 1$
$$H^s_{\mathrm{AQ}}(DS(X)_0; M) \cong H^s_{\mathrm{AQ}}(p^{(0)}S(X)_0; M) = 0$$
where the last equality holds since $p^{(0)}S(X)_0$ is a free category.
- The long exact sequence of the theorem applied to $S(X)$ yields
$$H^{s+1}_{\mathrm{AQ}}(DS(X); M) \ \cong \ H^s_{\mathrm{Alg}}(S(X); M) \ .$$
- In conclusion $H^{s+1}_{\mathrm{AQ}}(DX; M) \cong H^{s+1}_{\mathrm{AQ}}(DS(X); M) \cong H^s_{\mathrm{Alg}}(S(X); M)$.

Sketch of proof of theorem

- Let $Z = H\,(Z_0 \underset{t}{\overset{q}{\rightleftarrows}} \pi_0)$ be an object of $\mathrm{Spl}\,\mathcal{C}$, M a Z-module. There is a short exact sequence of abelian groups

$$\pi_1\,\mathrm{map}_{\mathrm{Gpd}(\mathcal{C})/Z}(Z, M) \rightarrowtail \mathrm{Hom}_{\mathcal{C}/Z_0}(Z_0, j^* M_1) \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}/Z}(Z, M).$$

- When $Z = \mathcal{K}_\bullet X$, $X \in \mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}$, M a module over $p^{(1)}X \in \mathrm{Track}_{\mathcal{O}}$, we obtain a short exact sequence of cosimplicial abelian groups

$$\begin{aligned} 0 &\to \pi_1\,\mathrm{map}_{\mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}/\mathcal{K}_\bullet X}(\mathcal{K}_\bullet X,\ E^n(p^{(1)}\mathcal{K}_\bullet X, \delta^* M)) \to \\ &\to \pi_1\,\mathrm{map}((\mathcal{K}_\bullet X)_0, E^n(p^{(1)}\mathcal{K}_\bullet X, \delta^* M)) \to \\ &\to \mathrm{Hom}_{\mathrm{Track}^{\mathrm{n}}_{\mathcal{O}}/\mathcal{K}_\bullet X}(\mathcal{K}_\bullet X, E^n(p^{(1)}\mathcal{K}_\bullet X, \delta^* M)) \to 0\,. \end{aligned}$$

- The corresponding long exact cohomotopy sequence yields the result.

Summary

- Simplicial categories have Postnikov systems whose k-invariants are André-Quillen cohomology classes. In fact it is enough to work with categories enriched in simplicial n-types.
- To treat the latter algebraically we need higher category theory. Using the model of weakly globular n-fold categories leads to the notion of n-track category.
- There is a comonad on n-track categories built from techniques of internal category theory.
- The corresponding comonad resolution on n-track categories gives a cofibrant replacement.
- Using the latter leads to a long exact cohomology sequence for the André-Quillen cohomology and then to an algebraic description of the latter.

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