CT 2025 - Brno

Relational doctrines, quotient completions and projectives

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Primary doctrines

$$\mathcal{C}^{op} \xrightarrow{P} \mathcal{Pos}$$

$$\underbrace{\times, 1}_{\text{variables}} \qquad \underbrace{\wedge, \top}_{\text{formulas}}$$

Maietti, Rosolini. *Elementary quotient completion.* 2012

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$$d \in P(X \times X) \quad \top \vdash d(x,x) \quad d(x,y) \vdash d(y,x) \quad d(x,y) \wedge d(y,z) \vdash d(x,z)$$

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New objects (X, d) where d is an equivalence relation over X

New arrows $f\colon X \to X'$ in $\mathcal{C}$ s.t. $d(x,y) \vdash d'(f(x), f(y))$

New formulas $\varphi \in P(X)$ such that $\varphi(x) \wedge d(x,y) \vdash \varphi(y)$

Doctrine $\mathcal{Q}_P^{op} \xrightarrow{\widehat{P}} \mathcal{P}os$ <u>d is the equality predicate over (X, d)</u>

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$$d \in P(X \times X) \quad \mathbf{1} \vdash d(x,x) \quad d(x,y) \vdash d(y,x) \quad d(x,y) \otimes d(y,z) \vdash d(x,z)$$

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Proposition (Dozen 96)

$\mathbf{1} \vdash d(x,x)$ and $\varphi(x) \otimes d(x,y) \vdash \varphi(y)$ imply $d(x,y) \dashv\vdash d(x,y) \otimes d(x,y)$

Or also: for f an arrow of $\mathcal{C}$, if $\exists_f \dashv P(f)$, then $\exists_f(\mathbf{1}) = \exists_f(\mathbf{1}) \otimes \exists_f(\mathbf{1})$.

Relational doctrines

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$$\begin{aligned} (\mathcal{C} \times \mathcal{C})^{op} &\xrightarrow{R} \mathcal{P}os \\ (X, Y) &\longmapsto \underbrace{R(X, Y)}_{\text{relations from } X \text{ to } Y} \end{aligned}$$

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$$(X, Y) \longmapsto \underbrace{R(X, Y)}_{\text{relations from } X \text{ to } Y}$$

- with families of monotone functions

$$R(X,Y) \times R(Y,Z) \xrightarrow{-;-} R(X,Z) \qquad 1 \xrightarrow{\mathsf{d}} R(X,X) \qquad R(X,Y) \xrightarrow{(-)^{\perp}} R(Y,X)$$

such that

$$(r;s);t = r;(s;t) \qquad \mathsf{d}_X;r = r = r;\mathsf{d}_Y$$

$$(r;s)^{\perp} = s^{\perp};r^{\perp} \qquad \mathsf{d}_X^{\perp} = \mathsf{d}_X \qquad r^{\perp\perp} = r$$

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$$(r;s)^\perp = s^\perp; r^\perp \qquad \mathsf{d}_X^\perp = \mathsf{d}_X \qquad r^{\perp\perp} = r$$

- reindexing lax preserves operations

$$R_{f,g}(r); R_{g,h}(s) \le R_{f,h}(r;s) \qquad \mathsf{d}_X \le R_{f,f}\mathsf{d}_Y \qquad (R_{f,g}(r))^\perp \le R_{f,g}(r^\perp)$$

Examples

- $[0,\infty]$ is the Lawvere's quantale. $(\mathcal{Set} \times \mathcal{Set})^{op} \xrightarrow{\mathcal{L}_{[0,\infty]}} \mathcal{Pos}$

  $\mathcal{L}_{[0,\infty]}(X,Y) = [0,\infty]^{X\times Y}$ $\qquad$ $\mathcal{L}_{[0,\infty]}(f,g)$ is given by composition

  $$\mathrm{d}_X(x,x') = \begin{cases} 0 & x = x' \\ \infty & x \neq x' \end{cases} \qquad (r;s)(x,z) = \bigwedge_{y\in Y} r(x,y) + s(y,z)$$

- Spans over a category with weak pullbacks
- Jointly monic spans over a (locally) regular category
- **Elementary and existential doctrines (eed)**
- **Ordered categories with involution (oci)**
- ...

EEDs and OCIs

- An eed $P\colon \mathcal{C}^{op} \longrightarrow \mathcal{P}os$ models the $(\exists, =, \wedge, \top)$-fragment of FOL.

  $\mathsf{d}(x, x')$ is $x = x'$ $\quad (r;s)(x,z) = \exists_y\,[r(x,y) \wedge s(y,z)] \quad r^{\perp}(y,x) = r(x,y)$

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Proposition. A rel. doc. is an eed if and only if it is cartesian and modular

R is **Cartesian**: $1 \xleftarrow{\;!\;} R \xrightarrow{\;\Delta\;} R \times R$ have a right adjoint in **RD**

R is **Modular**: the (Freyd's) modular laws holds: $\alpha;\beta \wedge \gamma \leq \alpha;(\beta \wedge \alpha^{\perp};\gamma)$

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- An oci is $\mathcal{P}os$-enriched category $\mathcal{C}$ with an involution $\bullet\colon \mathcal{C}^{op} \to \mathcal{C}$

$$\hom_{\mathcal{C}}\colon (\mathcal{C} \times \mathcal{C})^{op} \to \mathcal{P}os$$

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Proposition. A rel. doc. is a oci if and only if it satisfies RUC and Ex.

R satisfies **RUC** if the graph functor $\hom_{\mathcal{C}}(X, Y) \to \mathrm{Map}(X, Y)$ is surjective
R satisfies **Ex** if the graph functor $\hom_{\mathcal{C}}(X, Y) \to \mathrm{Map}(X, Y)$ is injective

Quotients in relational doctrines

$$(\mathcal{C} \times \mathcal{C})^{op} \xrightarrow{R} \mathcal{P}os$$

$r \in R(X, X)$ is an R-equivalence relation over X if r is

reflexive: $\mathsf{d}_X \leq r$ symmetric: $r^{\perp} \leq r$ transitive: $r; r \leq r$

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R has quotients if for every equiv. relation $r \in R(X, X)$ there is

$$X \xrightarrow{q} X/r$$

such that $r = R_{q,q}(\mathrm{d}_{X/r})$ and for every $X \xrightarrow{f} Y$ with $r \leq R_{f,f}(\mathrm{d}_{X/r})$

$$\begin{array}{ccc} X & \xrightarrow{q} & X/r \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle \exists!} \\ & & Y \end{array}$$

Relational quotient completion

$$(\mathcal{C} \times \mathcal{C})^{op} \xrightarrow{R} \mathcal{P}os \quad \longmapsto \quad (Q_R \times Q_R)^{op} \xrightarrow{R_q} \mathcal{P}os$$

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The category $\mathcal{Q}_R$:

Objects: (X, r) — r is an R-equivalence relation on X

Arrows: $(X, r) \xrightarrow{[f]} (Y, s)$ — $r \leq R_{f,f}(s)$

$[f] = [g]$ when $r \leq R_{f,g}(s)$

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The functor R^q:

$$R^q((X, r), (Y, s)) = \{\varphi \in R(X, Y) \mid r; \varphi; s \leq \varphi\} \qquad R^q_{[f],[g]}(\varphi) = R_{f,g}(\varphi)$$

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$$\mathbf{QRD} \underset{\hookrightarrow}{\overset{\leftarrow}{\bot}} \mathbf{RD}$$

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  $(\mathcal{L}_{[0,\infty]})_q$ is $(\mathcal{Met} \times \mathcal{Met})^{op} \xrightarrow{\text{BiMod}} \mathcal{Pos}$

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- $P \colon \mathcal{C}^{op} \to \mathcal{Pos}$ is a eed. A P-equiv. rel. ρ over X is such that

  $$\vdash \rho(x,x) \qquad \rho(x,y) \vdash \rho(y,x) \qquad \rho(x,y) \wedge \rho(y,z) \vdash \rho(x,z)$$

  P_q is Maietti-Rosolini elementary quotient completion.

  ($\mathcal{Equ}$, $\mathcal{Asm}$, setoids, pers, ex/wlex completion of cartesian categories)

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  ($\mathcal{Equ}$, $\mathcal{Asm}$, setoids, pers, ex/wlex completion of cartesian categories)

- $\mathcal{C}$ is weakly lex. $(\mathcal{C} \times \mathcal{C})^{op} \xrightarrow{Spn} \mathcal{Pos}$. Spn-equivalence relations are pseudo equivalence relations

  Spn_q is $(\mathcal{C}_{\text{ex/wlex}} \times \mathcal{C}_{\text{ex/wlex}})^{op} \xrightarrow{Spn_{jm}} \mathcal{Pos}$

A (bit off-topic) remark

OCI is **RD** such that $\hom_{\mathcal{C}}(X, Y) \equiv \mathrm{Map}(X, Y)$. **RD** $\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}$ **OCI**

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Bimod. over V-cats $\mapsto$ Bimod. over the Cauchy-complete V-cats

$$(\mathcal{Top} \times \mathcal{Top})^{op} \xrightarrow{cl_\beta} \mathcal{Pos} \quad \mapsto \quad (\mathcal{KH} \times \mathcal{KH})^{op} \xrightarrow{cl} \mathcal{Pos}$$

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(\mathcal{Top} \times \mathcal{Top})^{op} \xrightarrow{cl_\beta} \mathcal{Pos} \quad &\mapsto \quad (\mathcal{KH} \times \mathcal{KH})^{op} \xrightarrow{cl} \mathcal{Pos}
\end{aligned}$$

$$\mathbf{RD} \xrightarrow{\text{quot.}} \mathbf{QRD} \subseteq \mathbf{RD} \xrightarrow{\text{ruc.}} \mathbf{OCI}$$

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Maietti, Rosolini. *Unifying exact completions.* 2013

$$(\mathcal{Set} \times \mathcal{Set})^{op} \xrightarrow{\mathcal{L}_{[0,\infty]}} \mathcal{Pos} \quad \mapsto \quad (\mathcal{CMet} \times \mathcal{CMet})^{op} \xrightarrow{\mathrm{BiMod}} \mathcal{Pos}$$

$$(\mathcal{Vec} \times \mathcal{Vec})^{op} \xrightarrow{\mathrm{SN}} \mathcal{Pos} \quad \mapsto \quad (\mathcal{Ban} \times \mathcal{Ban})^{op} \xrightarrow{\mathrm{SN}_b} \mathcal{Pos}$$

Projectives

The graph of $f\colon X \to Y$ is $\Gamma_f = R_{f,\mathrm{id}_Y}(\mathsf{d}_Y)$. We say that f is surjective if

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Let $T\colon R \to R$ be a monad in **RD**: the rel. doc. $(\mathcal{C}^T \times \mathcal{C}^T)^{op} \xrightarrow{R^T} \mathcal{P}os$

$$(A,\alpha)\,,\,(B,\beta) \mapsto \{r \in R(A,B) \mid \alpha^{\perp};T(r);\beta \leq r\}$$

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Suppose $R\colon (\mathcal{C} \times \mathcal{C})^{op} \to \mathcal{P}os$ has quotients and $T\colon R \to R$ is a monad:

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- $\mathbb{P}$ is a projective cover of $\mathcal{C}$ iff $\mathbb{P}_T$ is a projective cover of $\mathcal{C}^T$
- if surjections in $\mathcal{C}$ split then $\mathcal{C}_T$ is a prjojective cover

Thank you!