CATEGORIES OF RELATIONS WHICH COMPOSE INDEPENDENTLY

— CT 2025 —

Paolo Perrone
Univ. of Oxford

Joint work with
M. Di Meglio C. Heunen
JS Lemay D. Stein

1. MOTIVATION
2. $\perp = ⫫$
3. THE EQUIVALENCE

1. MOTIVATION

M. Di Meglio

Hilbert spaces

(& other things)

C. Heunen

Hilbert spaces

(& other things)

JS Lemay

Restriction cat.s

(& other things)

P. Perrone

Probability

(& other things)

D. Stein

Probability

(& other things)

1. MOTIVATION · Markov Chains

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1. MOTIVATION · Markov Chains

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Ordinary relations:

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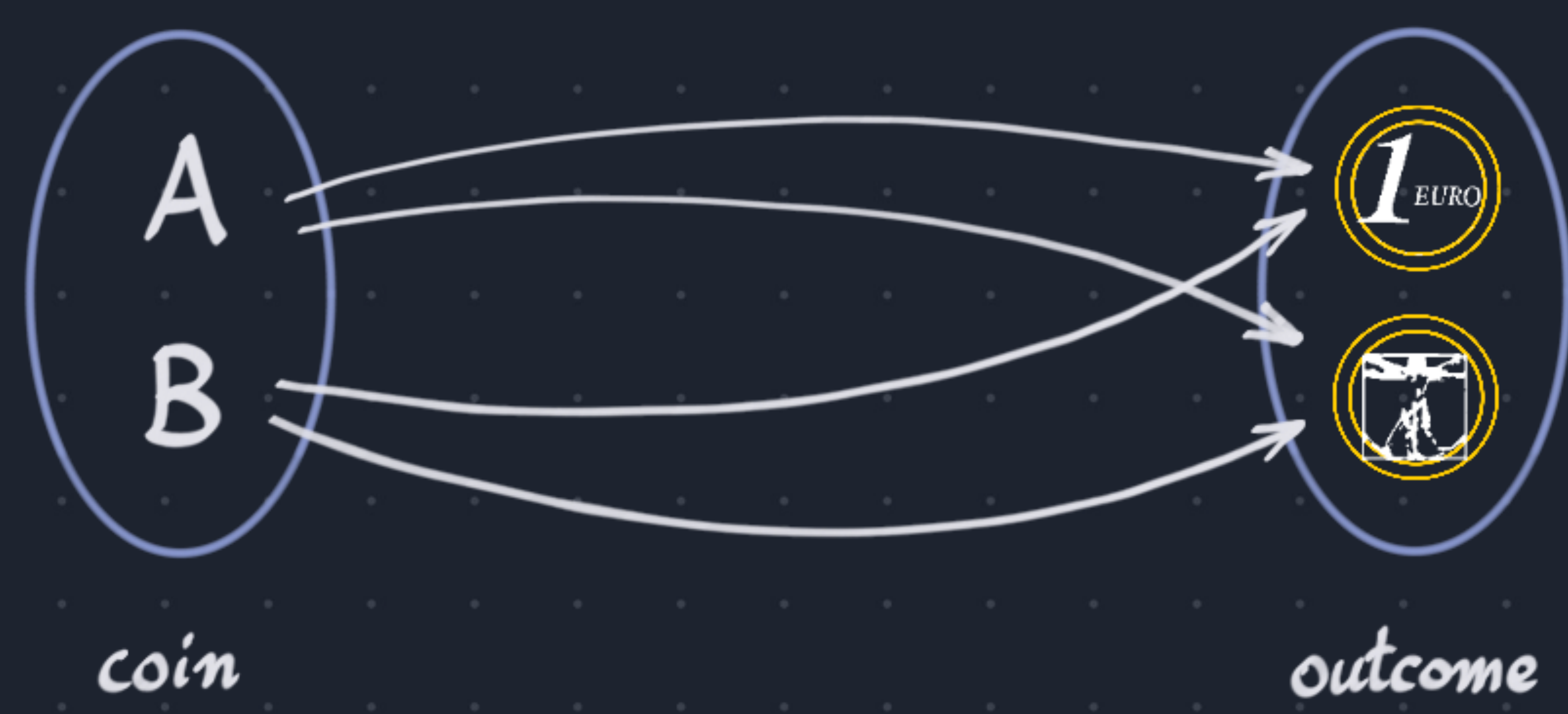

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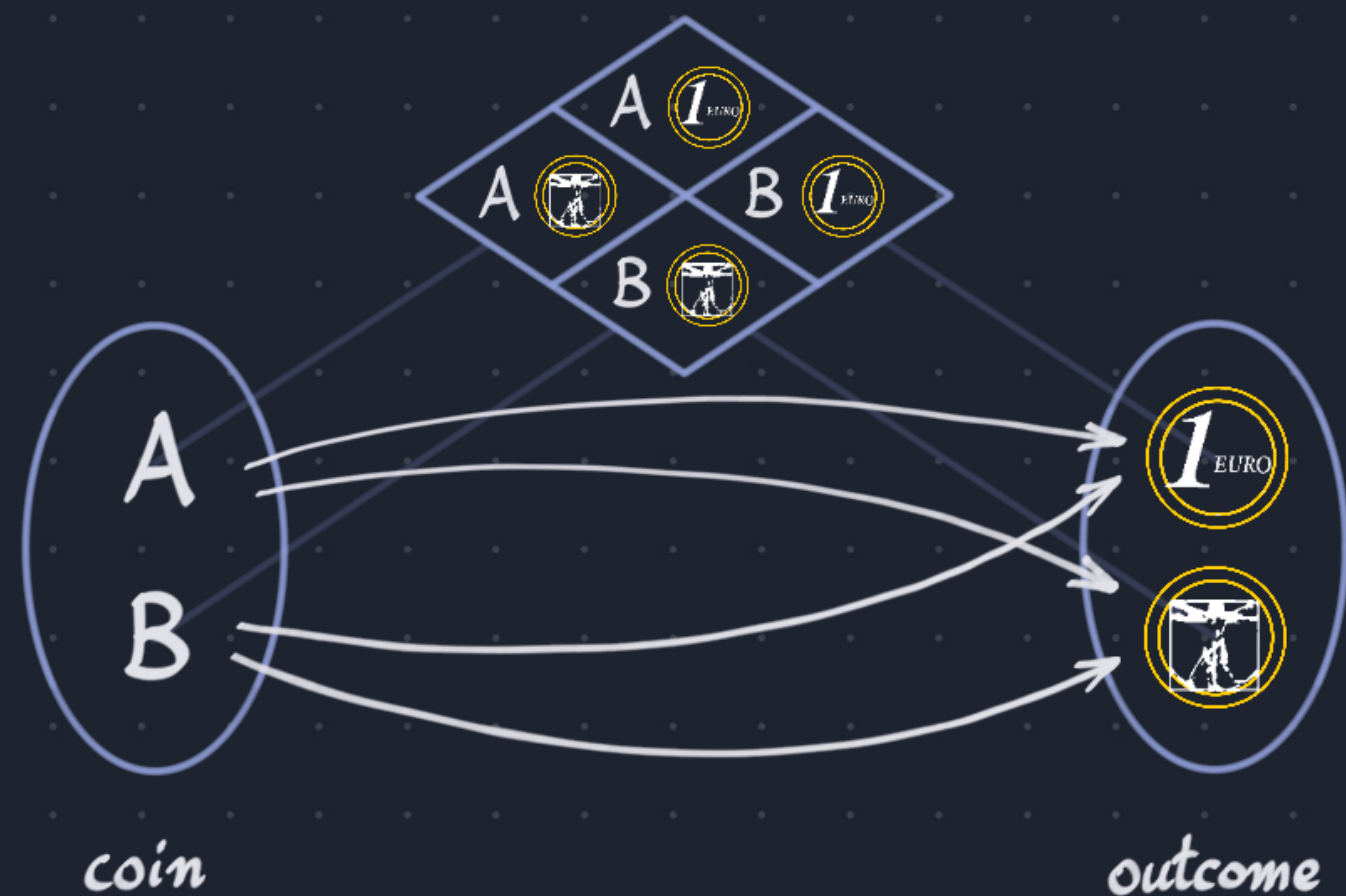

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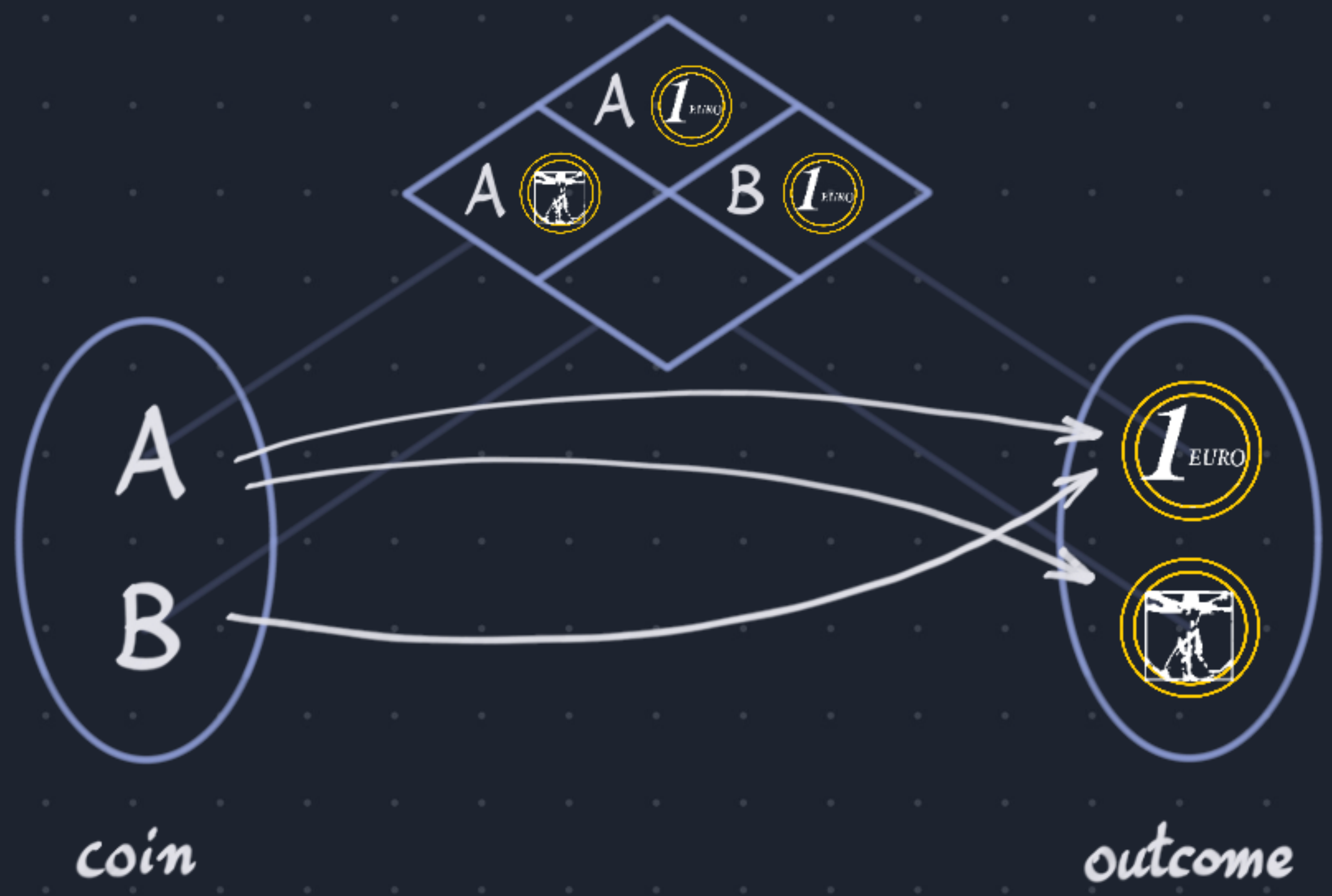

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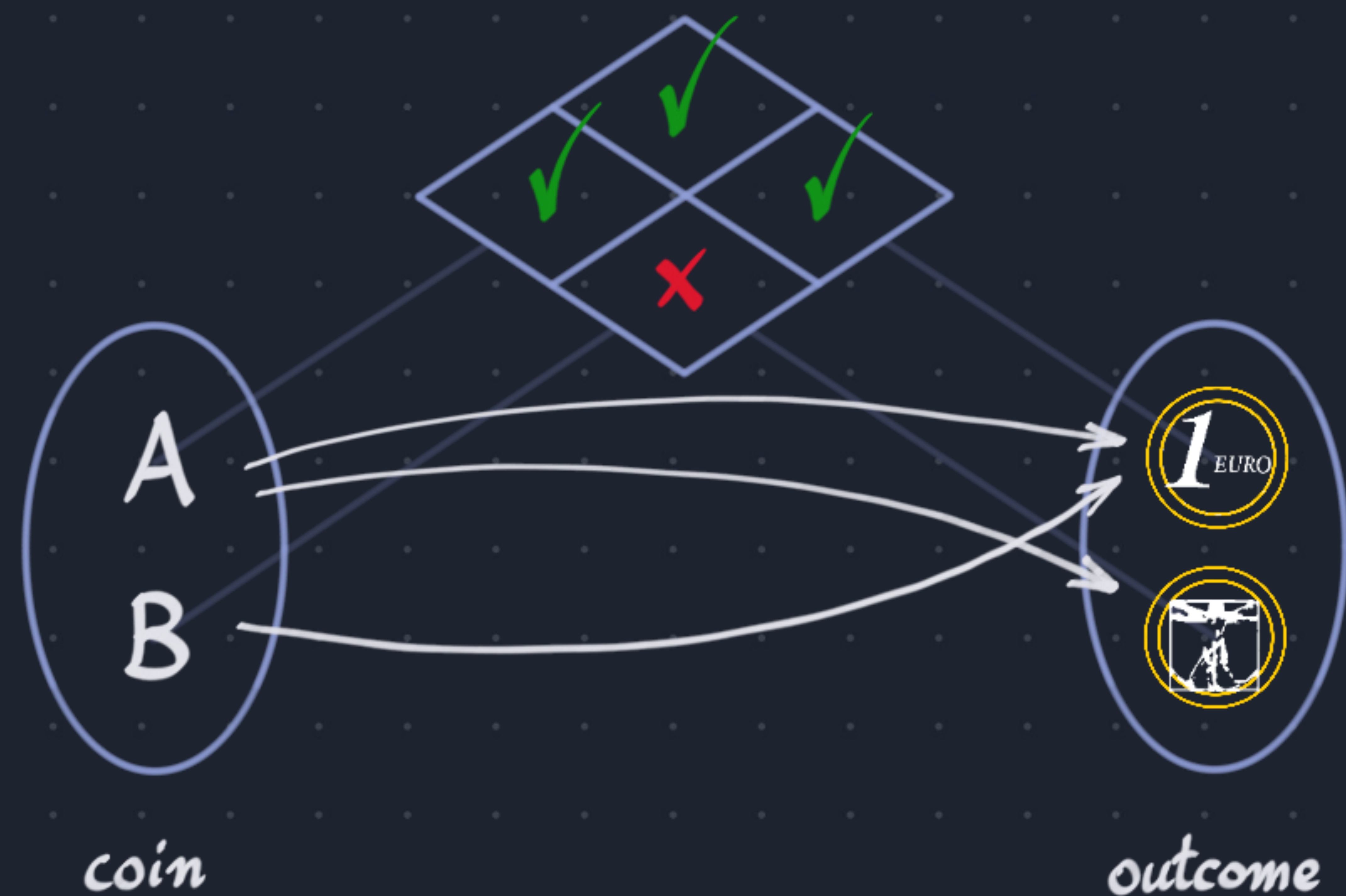

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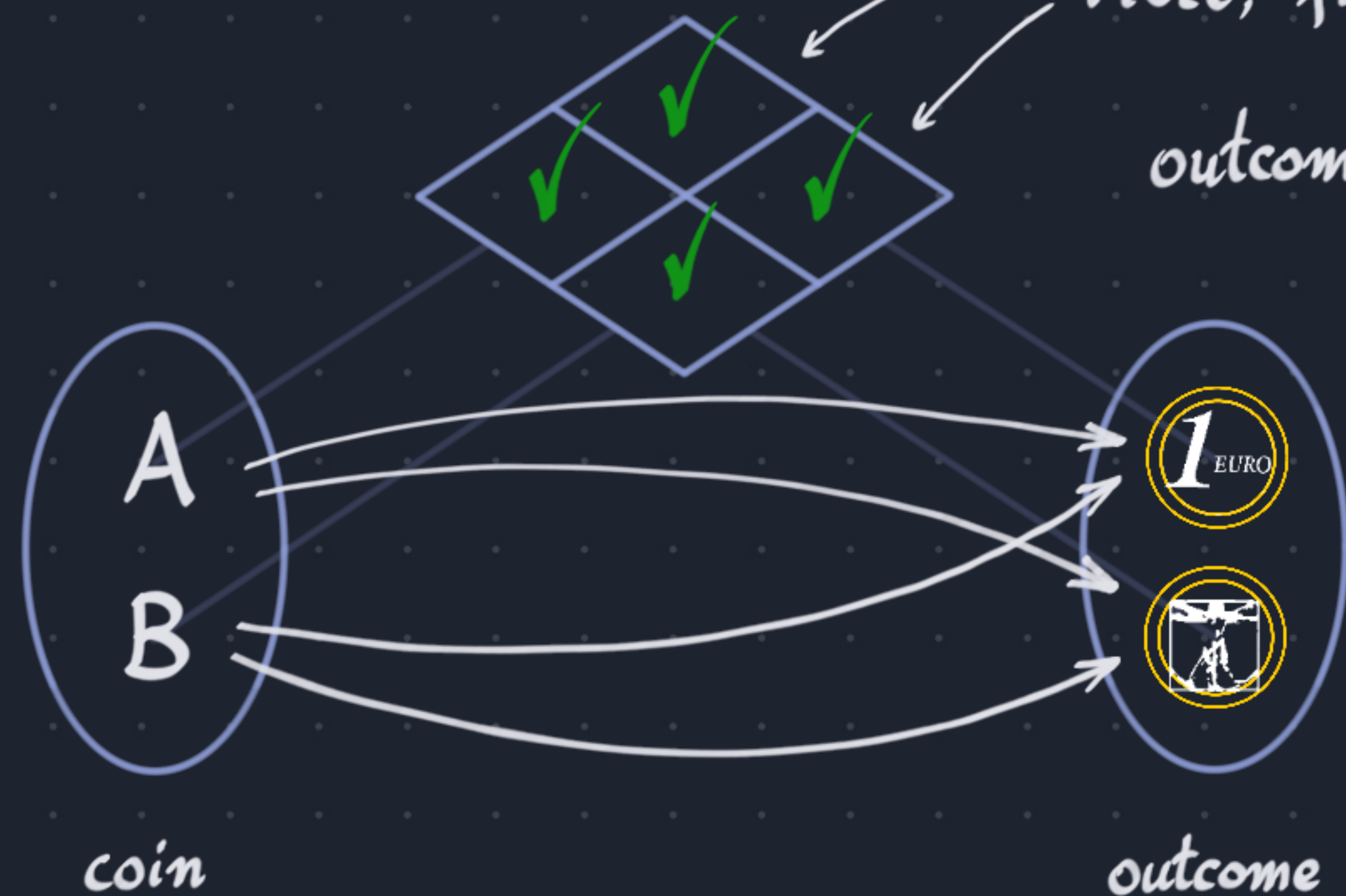

Here, fibers of A, B isomorphic (product!)

outcome does not depend on coin.

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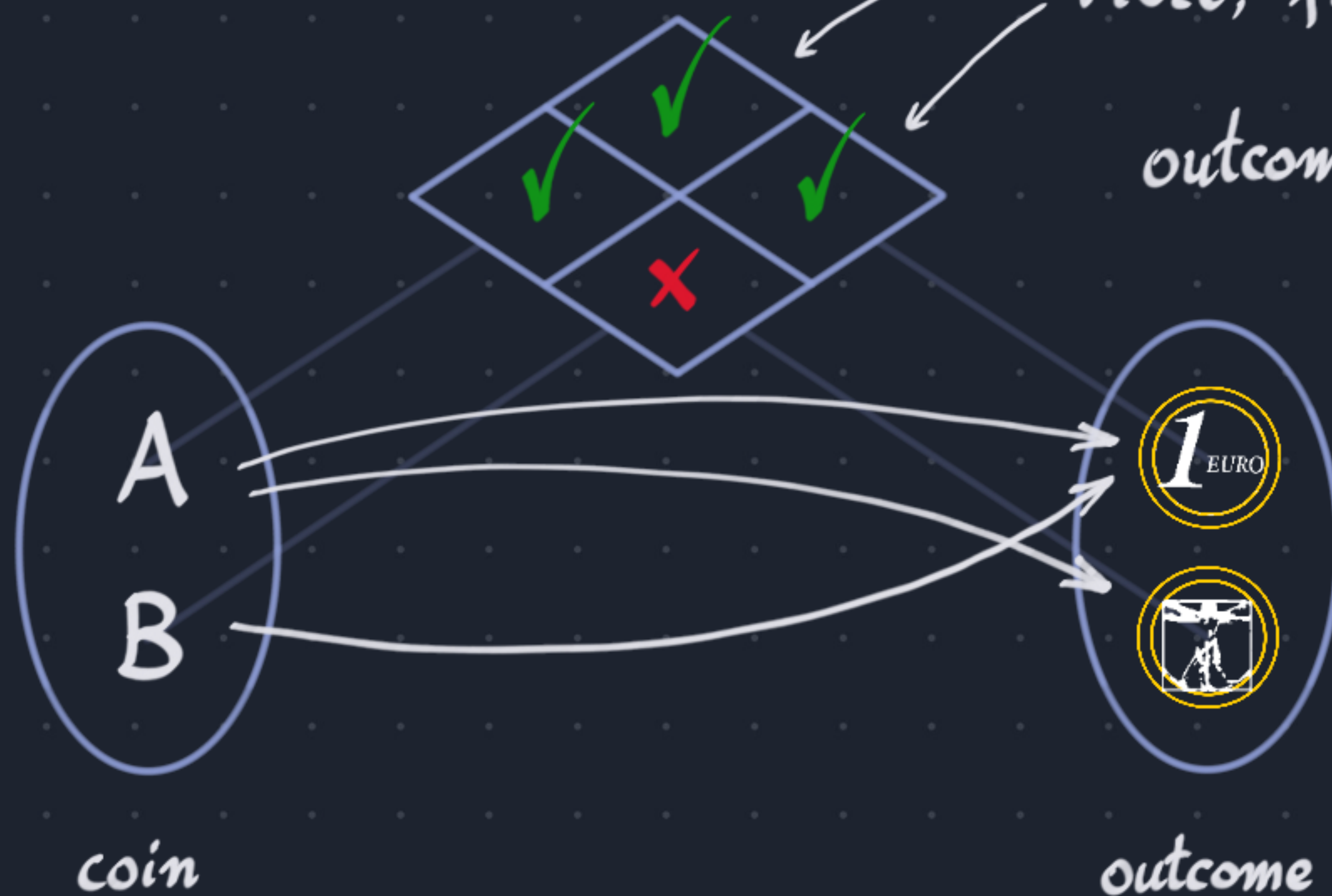

Here, fibers of A, B **not** isomorphic

outcome depends on coin (general case)

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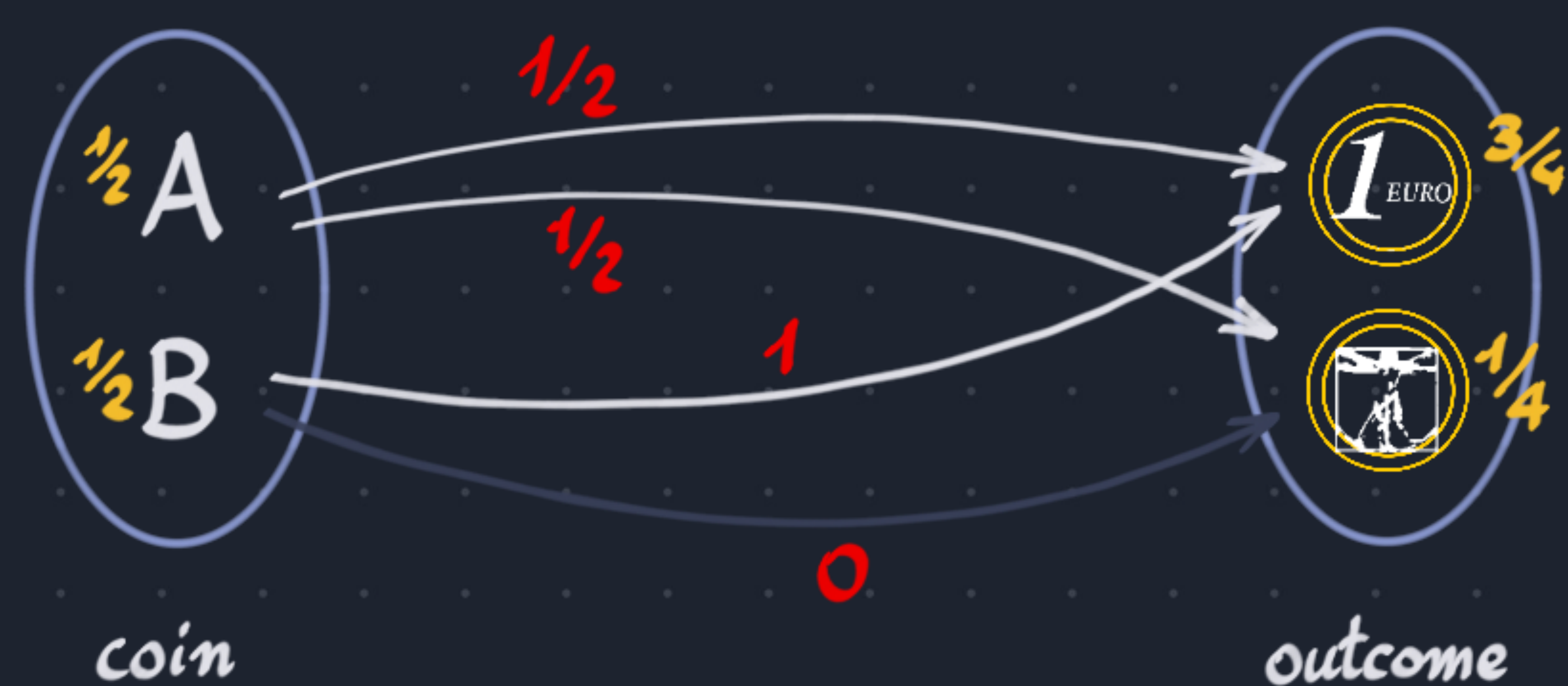
Probability: same but with numbers!



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$$P(A, 1_{\text{EURO}}) = P(A) \cdot P(1_{\text{EURO}} \mid A)$$

$$= \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$$

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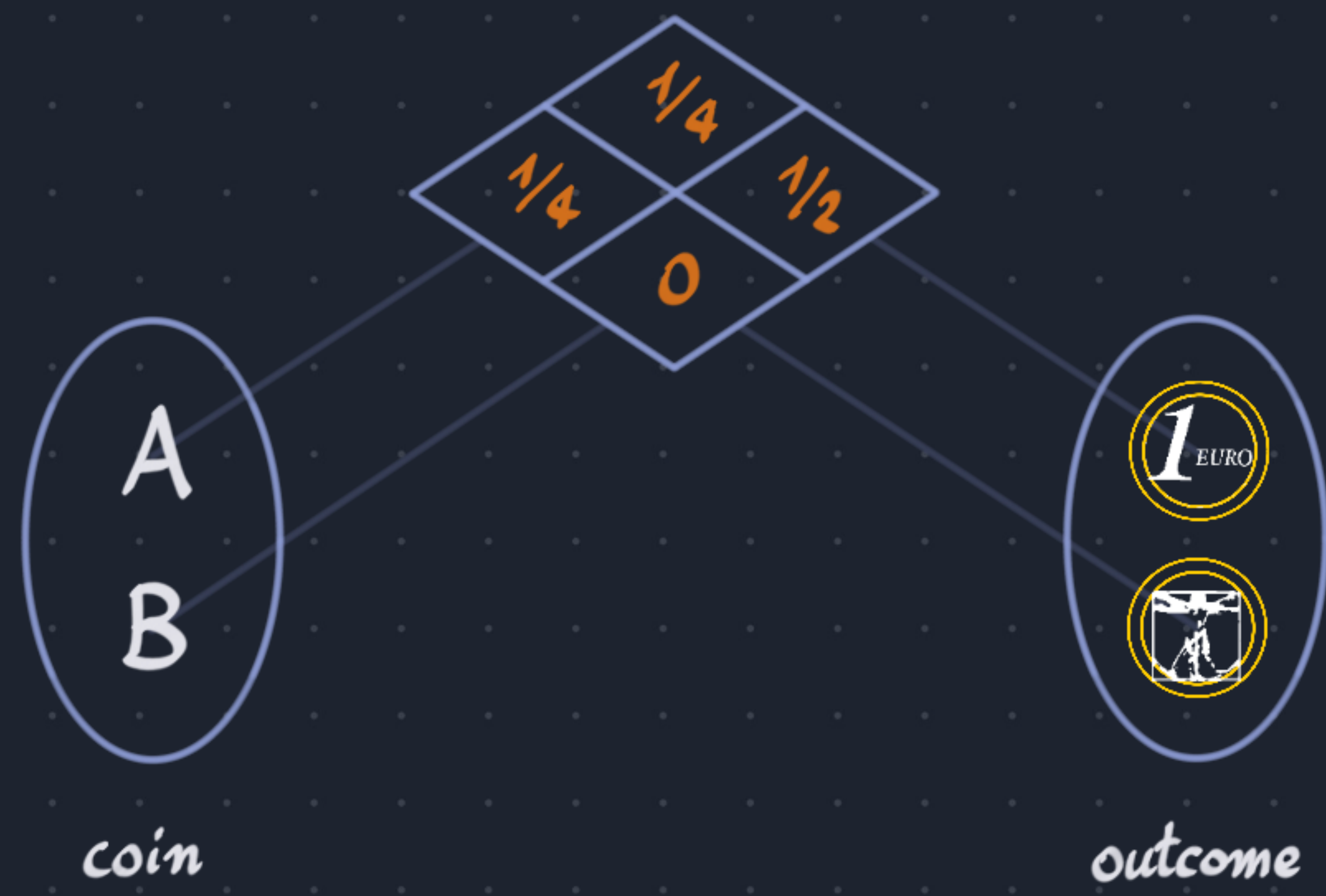

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Again, outcome depends on coin.

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Let's now compose two of them.

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These two transitions are independent

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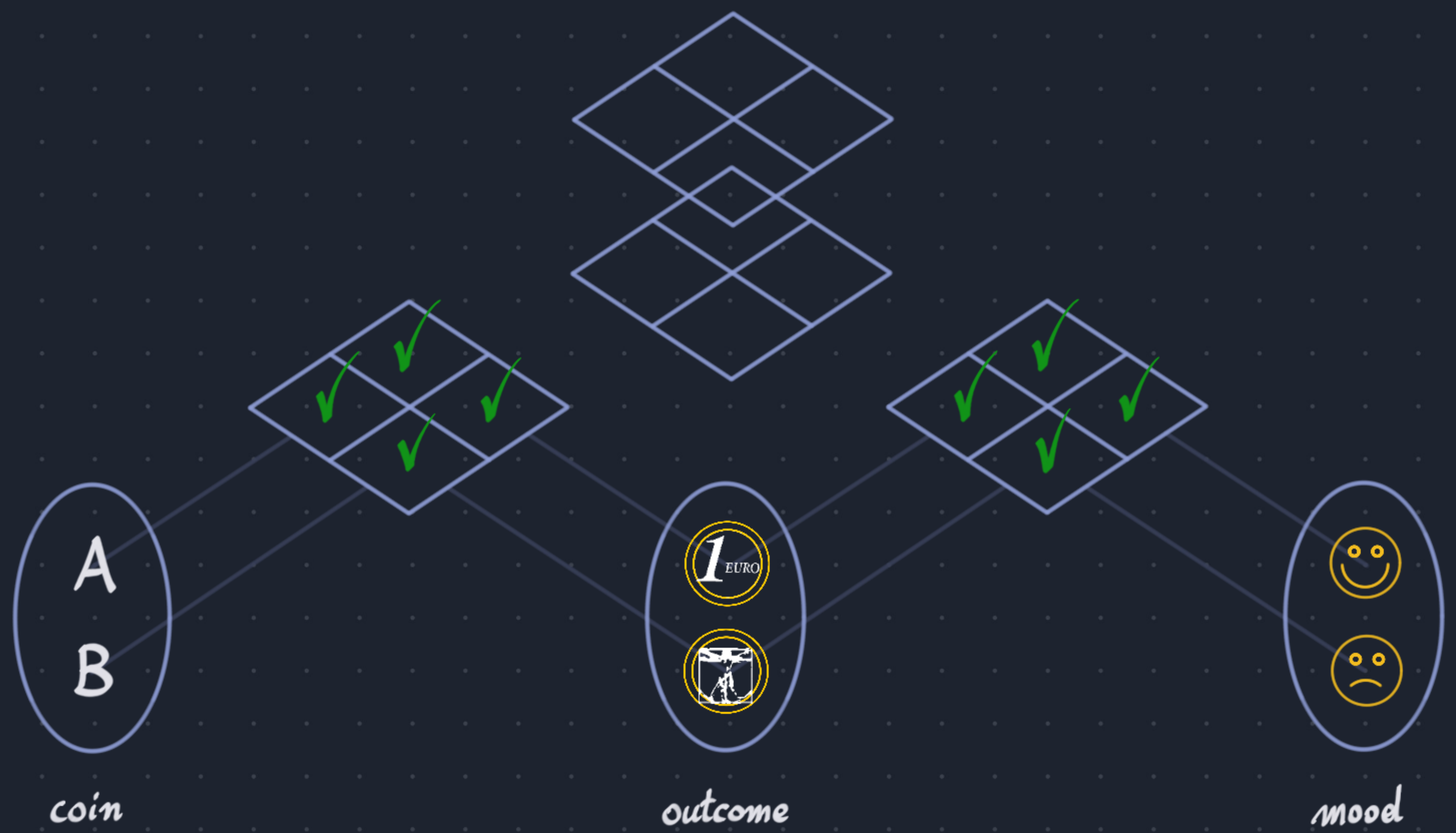

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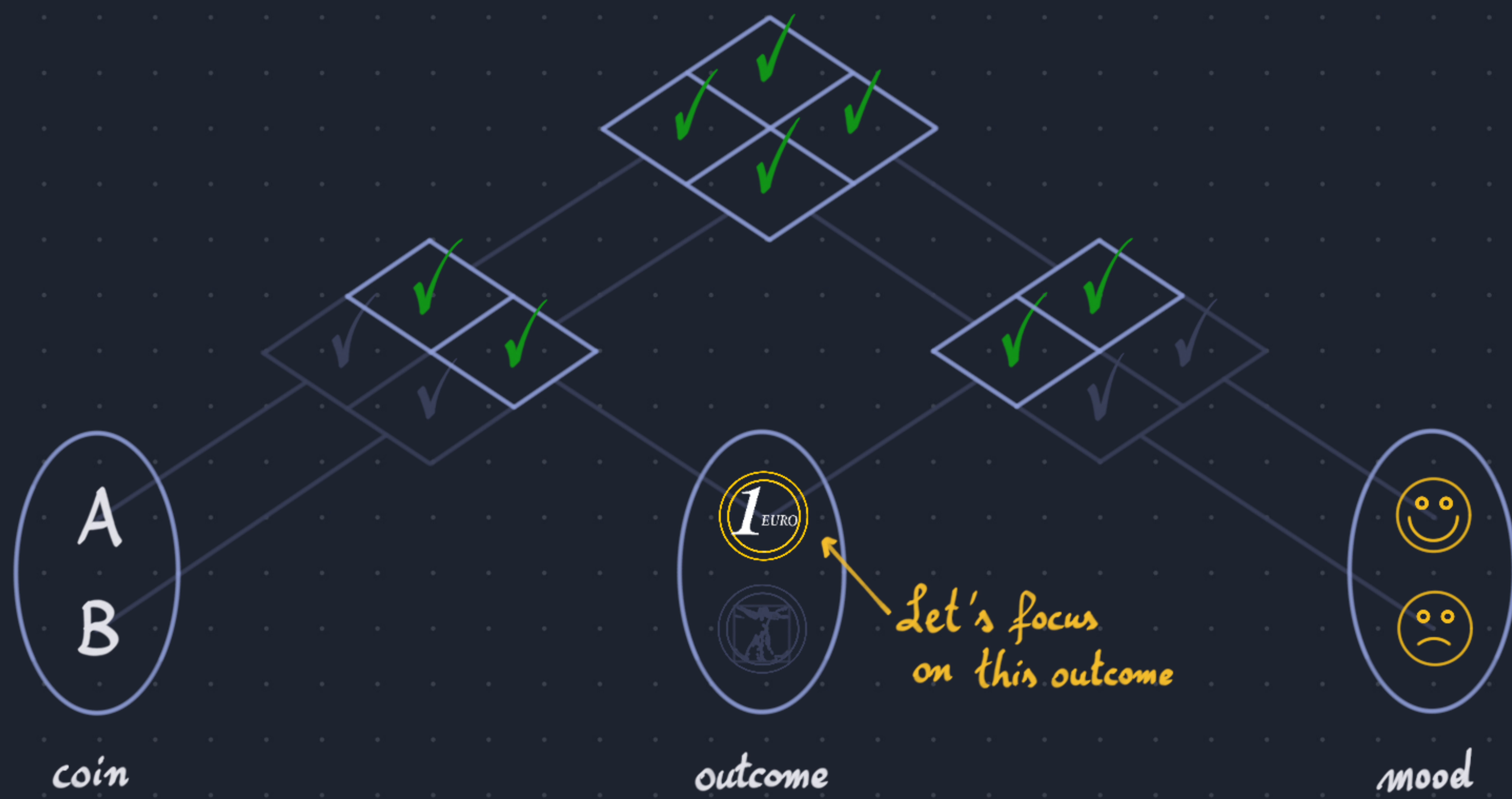

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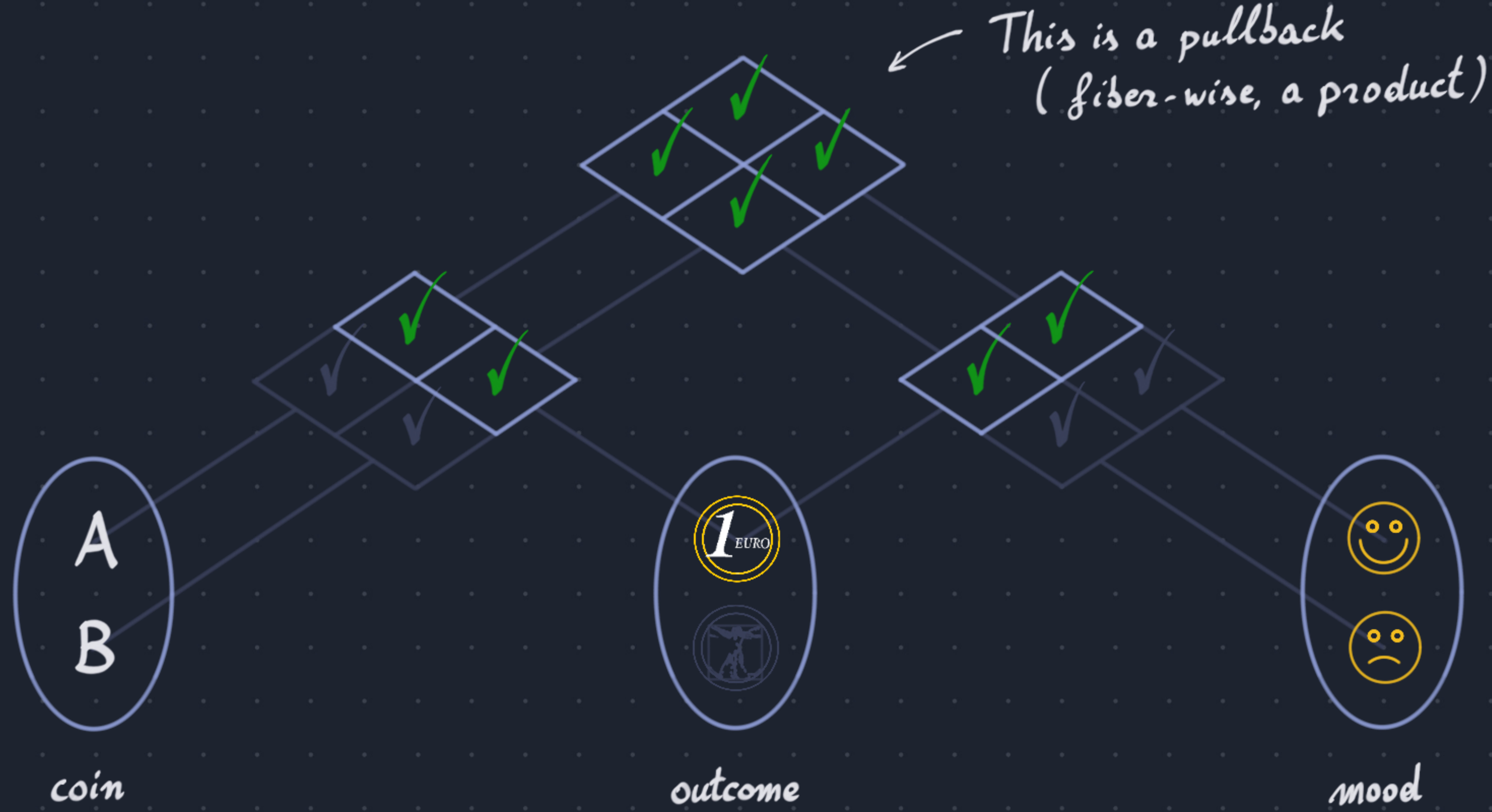

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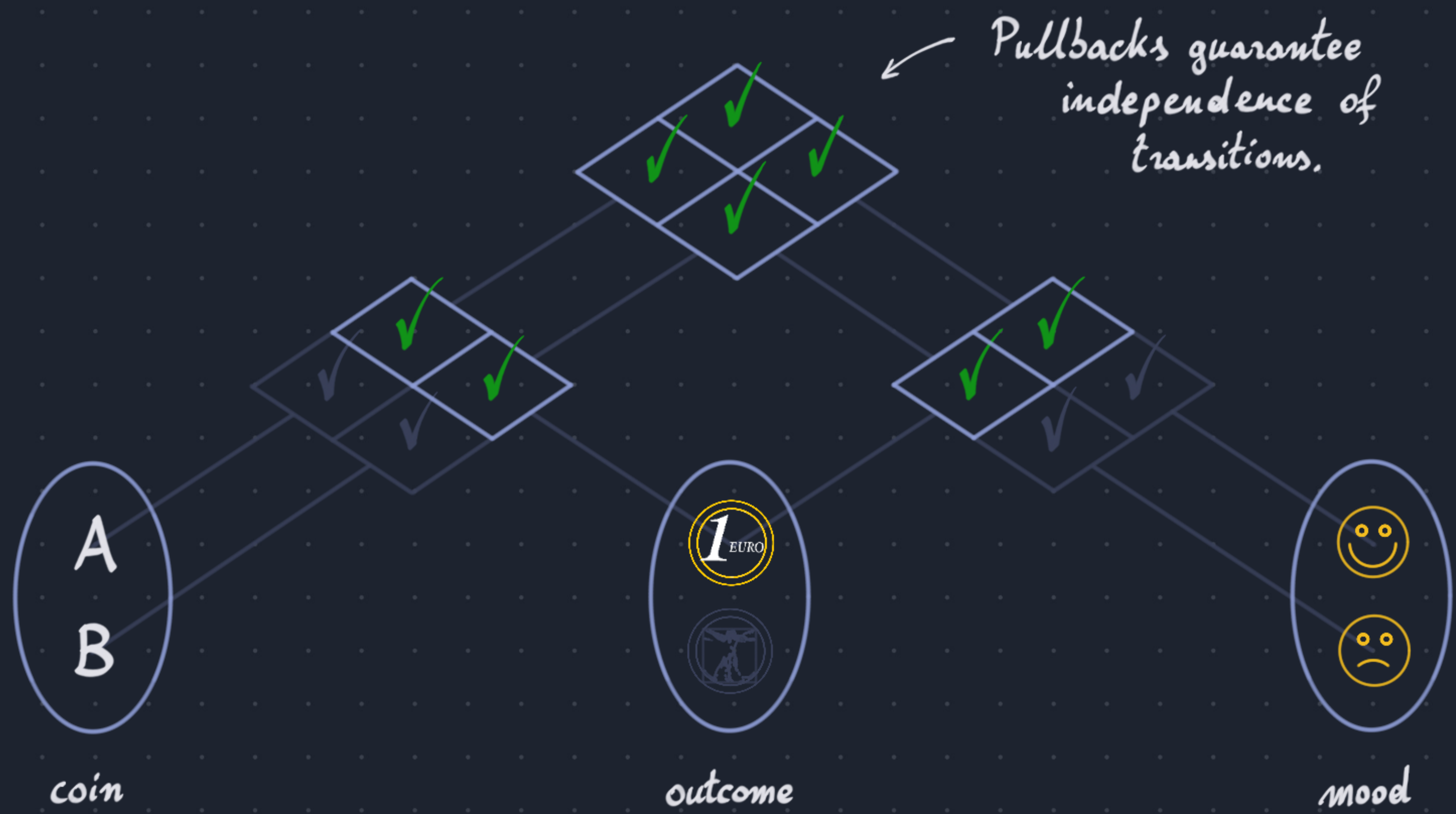

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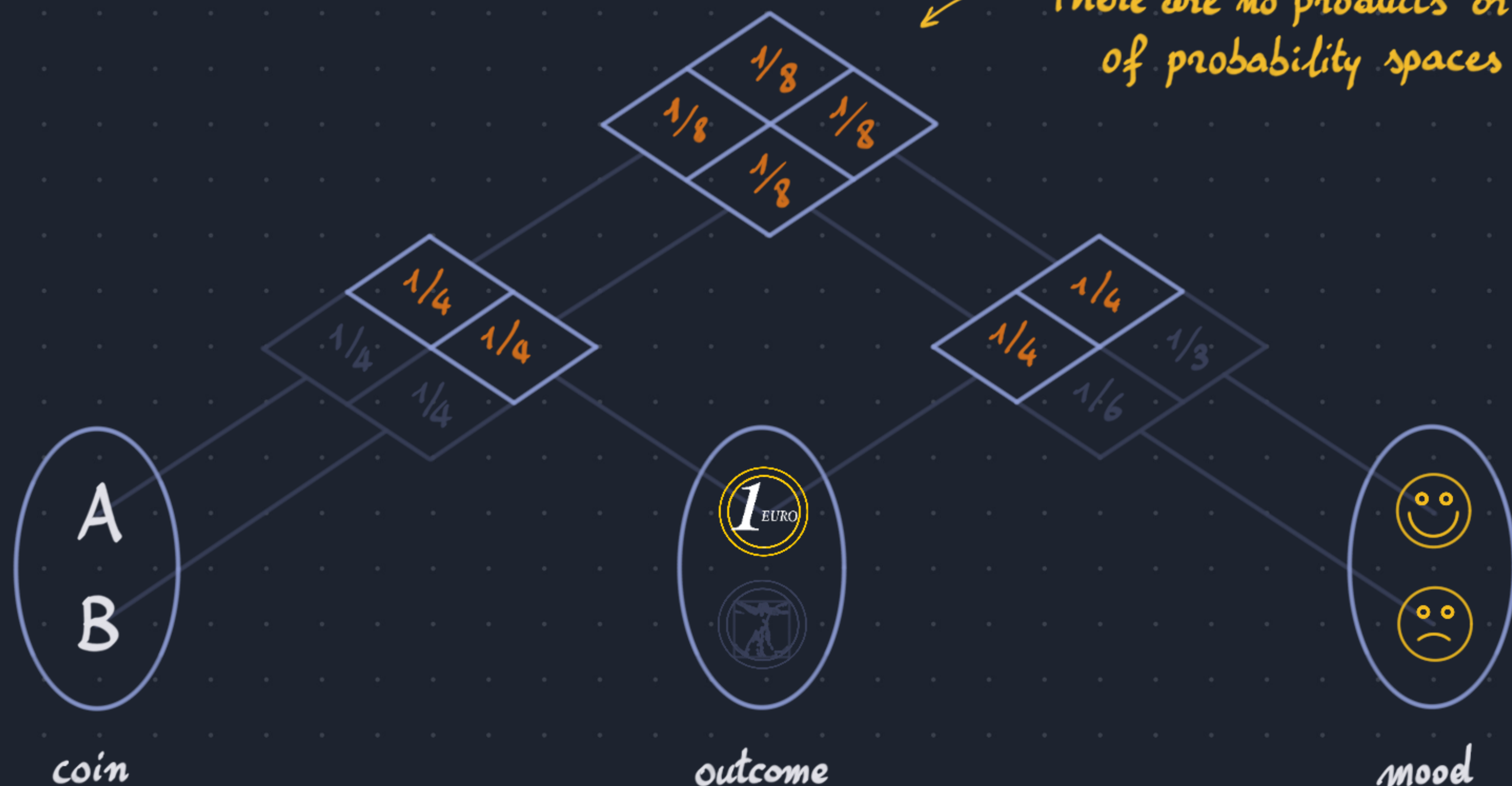

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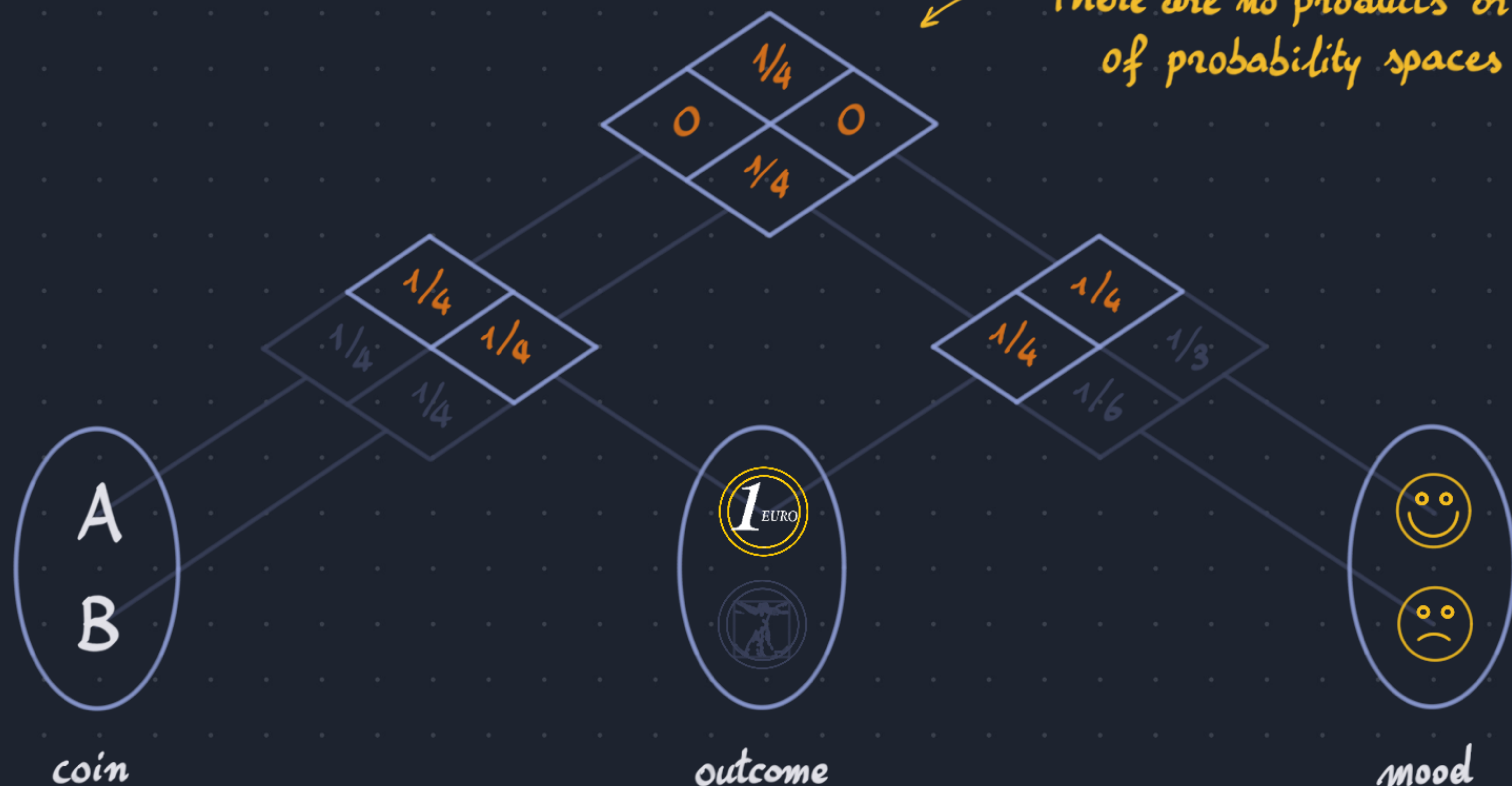

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1. MOTIVATION · Markov Chains

There is a canonical "independent" choice, but it's not a pullback.

What is it ?

1. MOTIVATION

There is a canonical "independent" choice, but it's not a pullback.

What is it ?

In this work, we

- Construct a **categorical theory of independence**
- Use it to define a new **calculus of relations**.

1. MOTIVATION
2. $\perp = \perp\!\!\!\perp$
3. THE EQUIVALENCE

2. $\perp = \perp\!\!\!\perp$ · Towards a theory of independence

Idea. For a good categorical theory, it's ideal to have:

1) Conceptual understanding of the phenomenon

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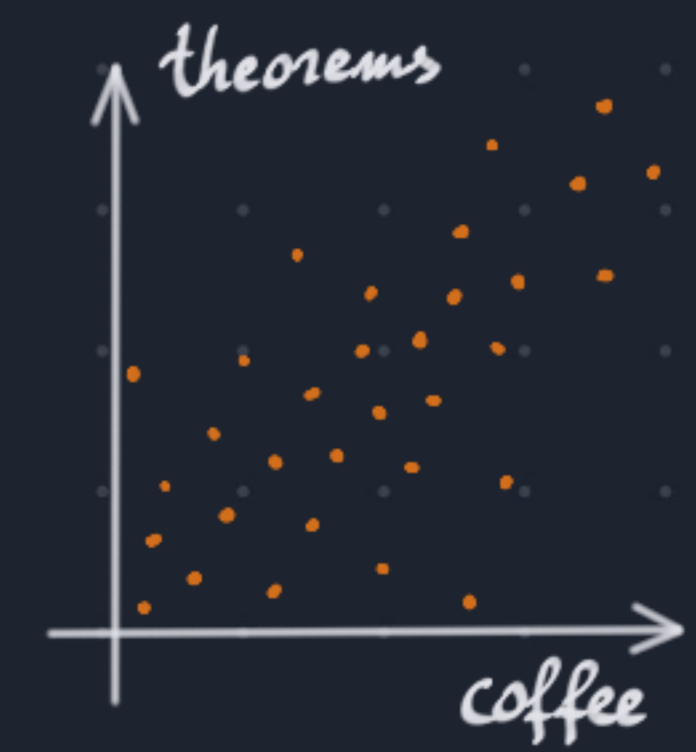

general case

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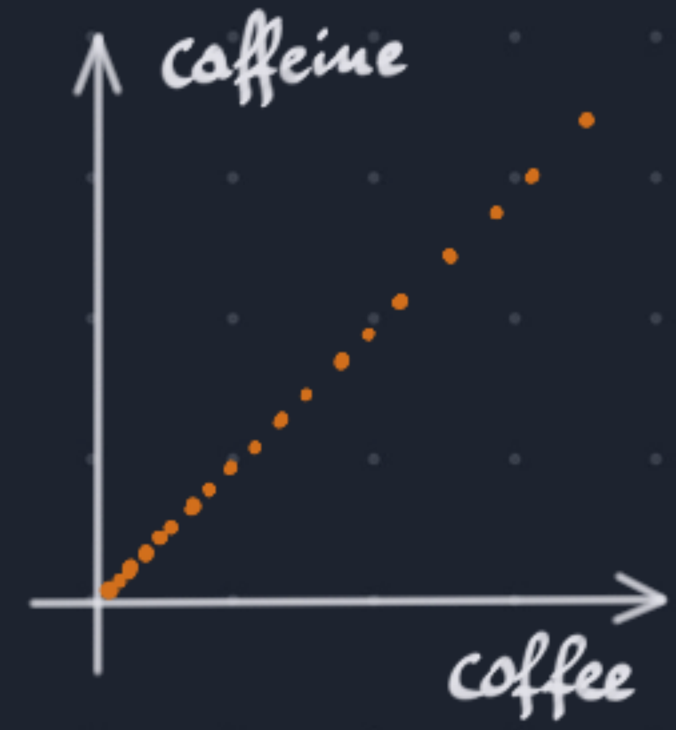
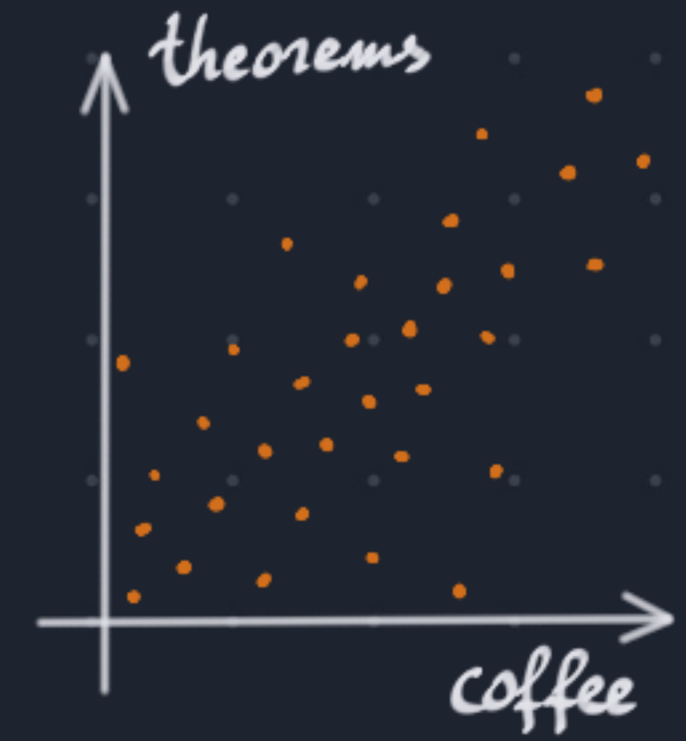


the same



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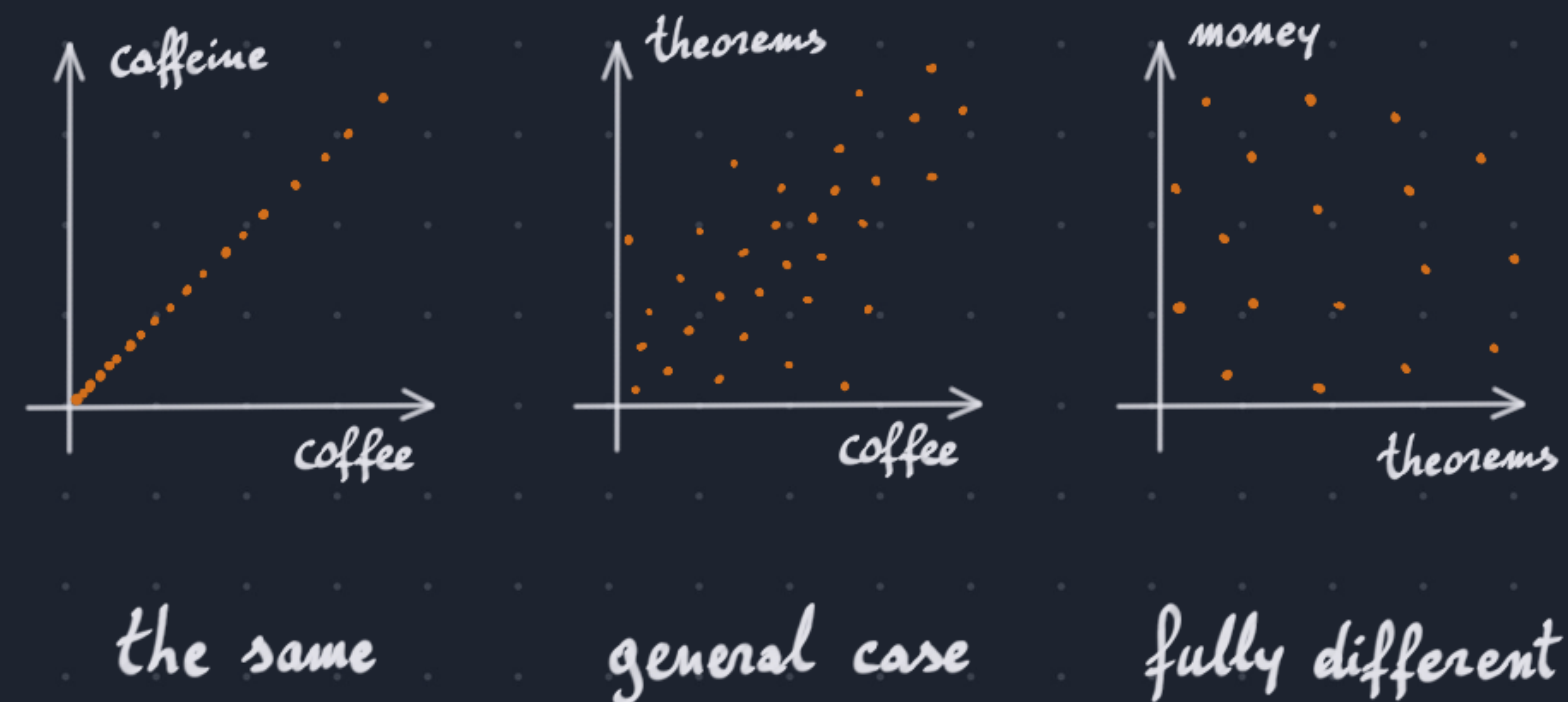

2. $\perp = \amalg$ · Towards a theory of independence

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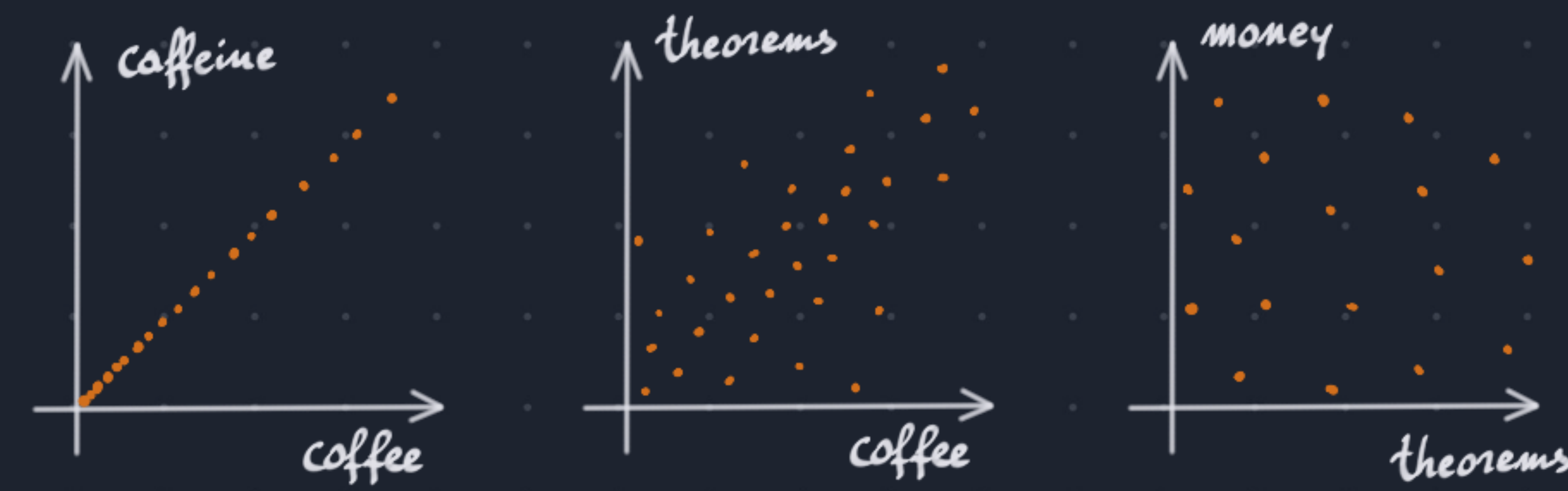
1) Conceptual understanding of the phenomenon

2) An example from a completely different field!



the same — general case — fully different

2. $\perp = \perp\!\!\!\perp$ · Towards a theory of independence

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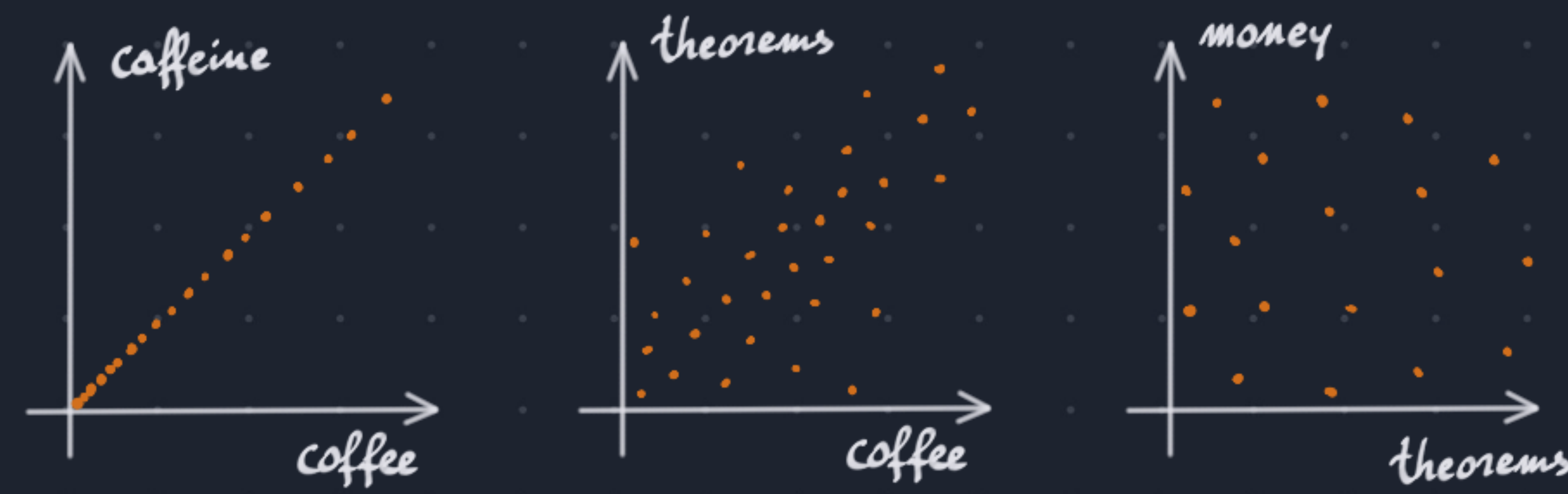


the same · general case · fully different

2) An example from a completely different field!

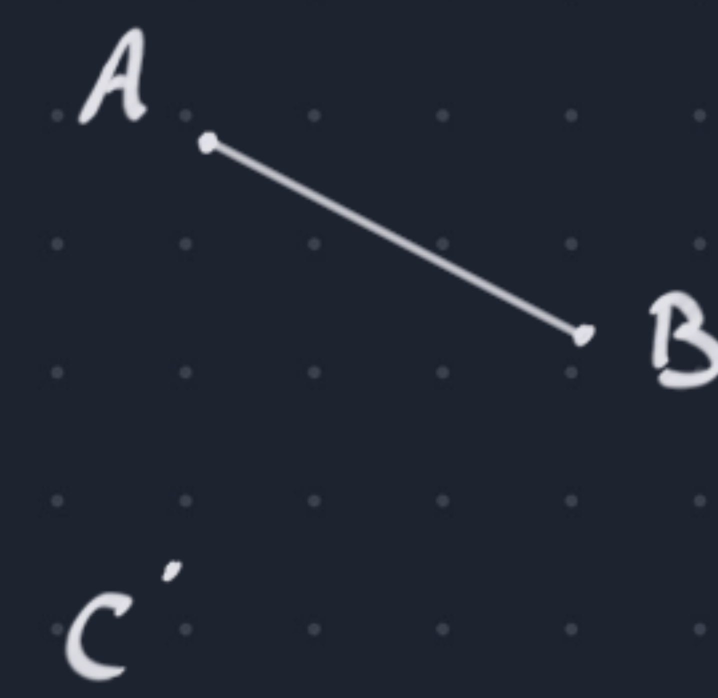
Euclidean Geometry

2. $\perp = \perp\!\!\!\perp$ · Towards a theory of independence

M. Di Meglio	C. Heunen	JS Lemay	P. Perrone	D. Stein
Hilbert spaces	Hilbert spaces	Restriction cat.s	Probability	Probability

†-categories

2. $\perp = \perp\!\!\!\perp$. $\dagger$-categories



Idea.

$$\frac{\text{Dagger categories}}{\text{categories}} = \frac{\text{undirected graphs}}{\text{directed graphs}}$$

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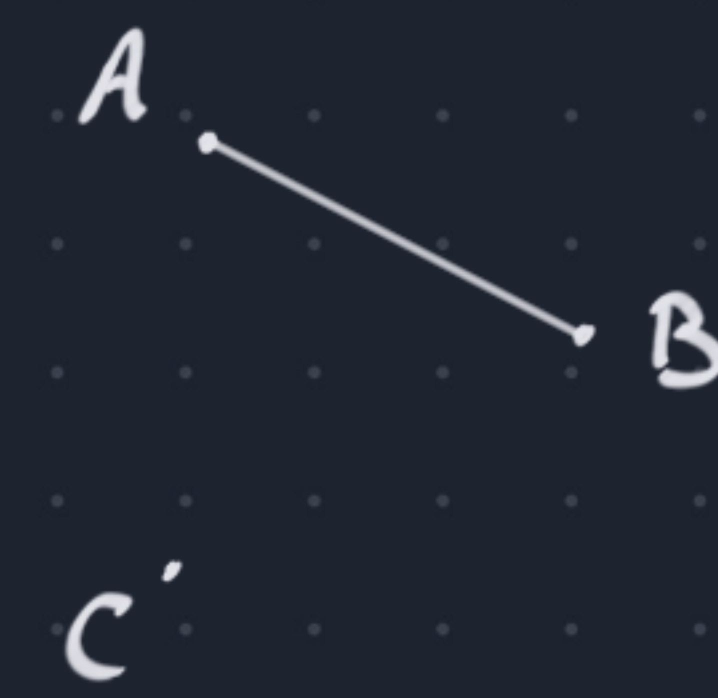
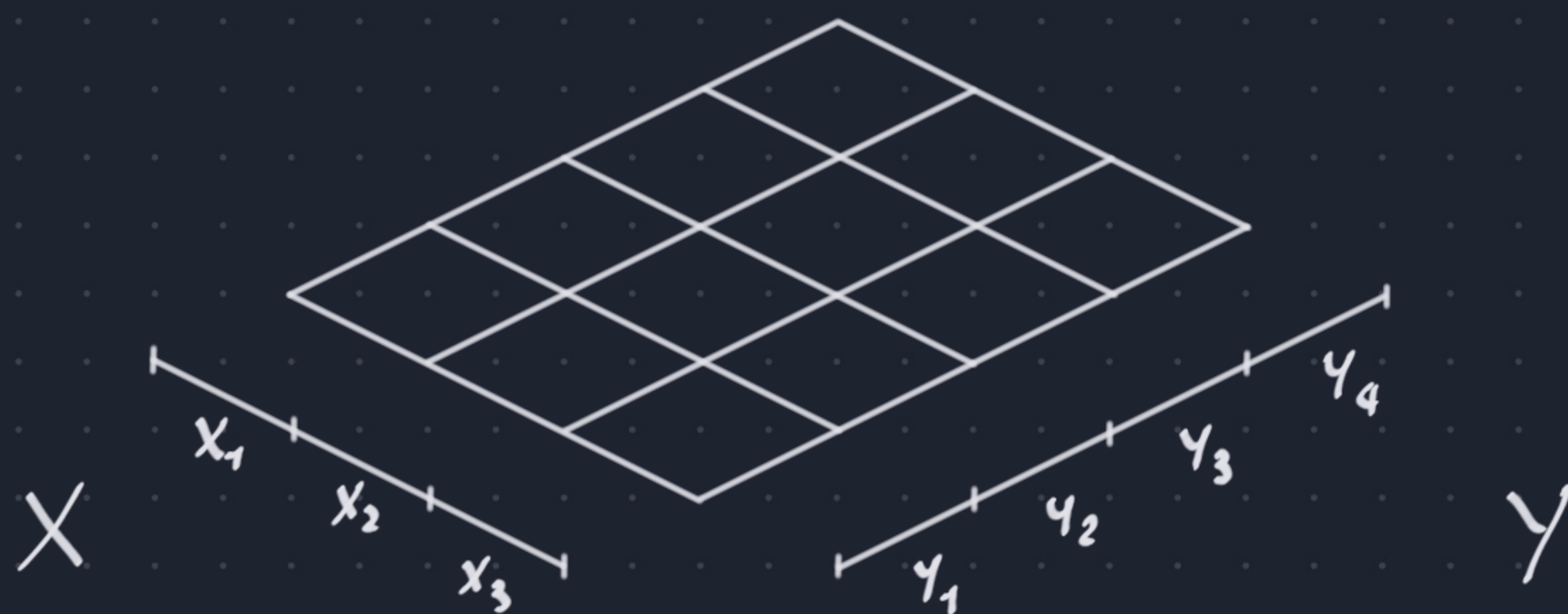
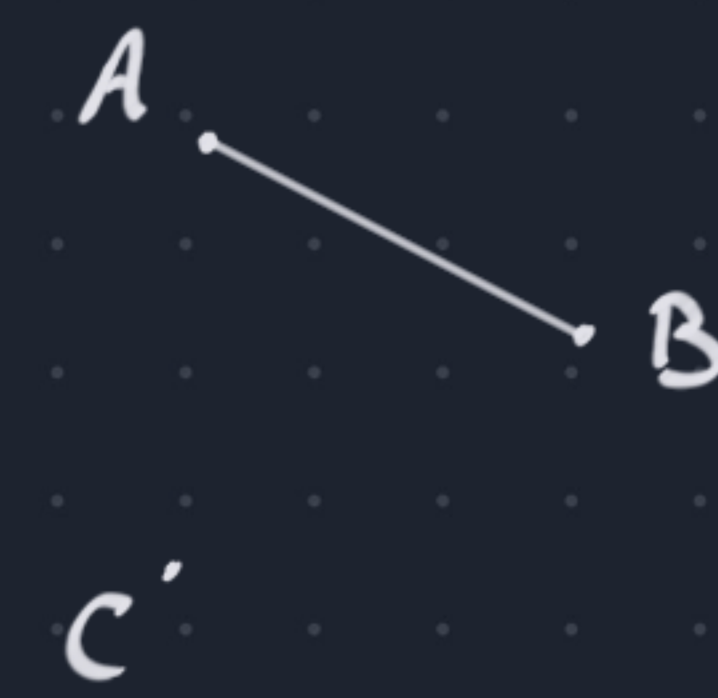


Examples. Rel



2. $\perp = \perp\!\!\!\perp$ · †-categories



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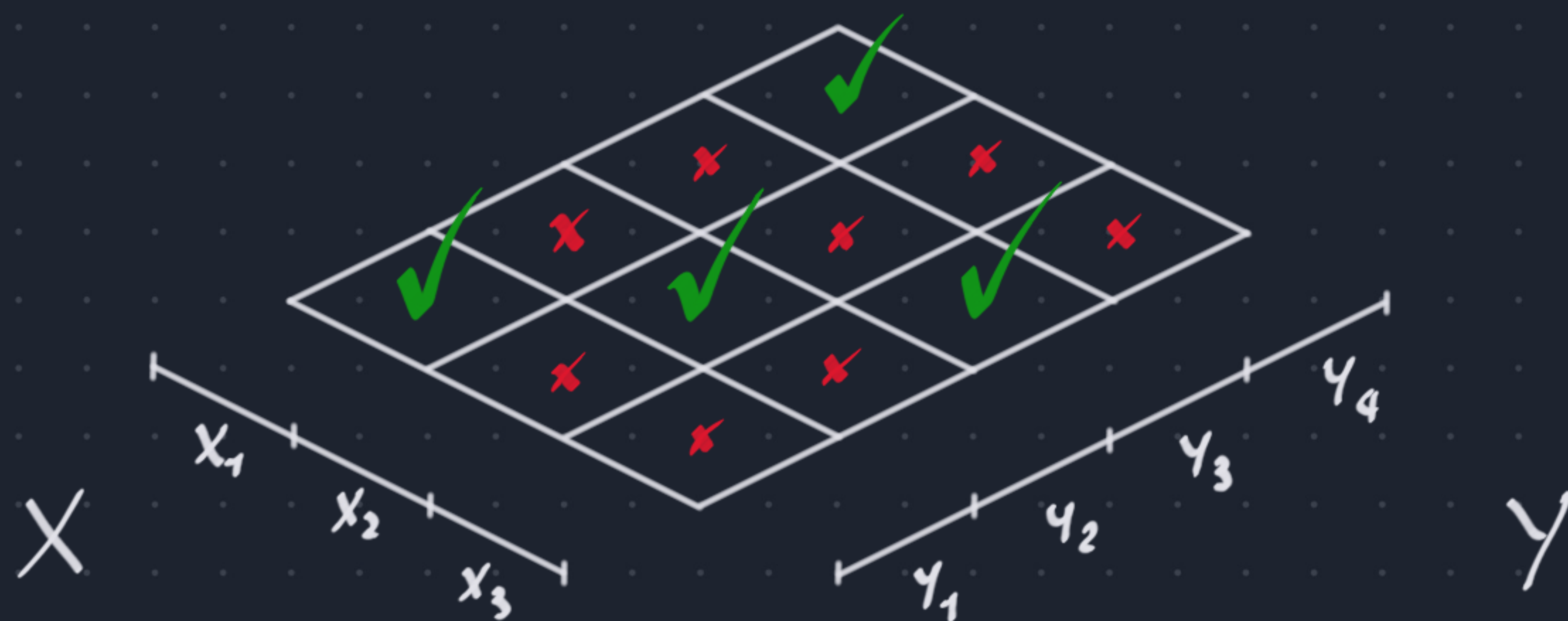

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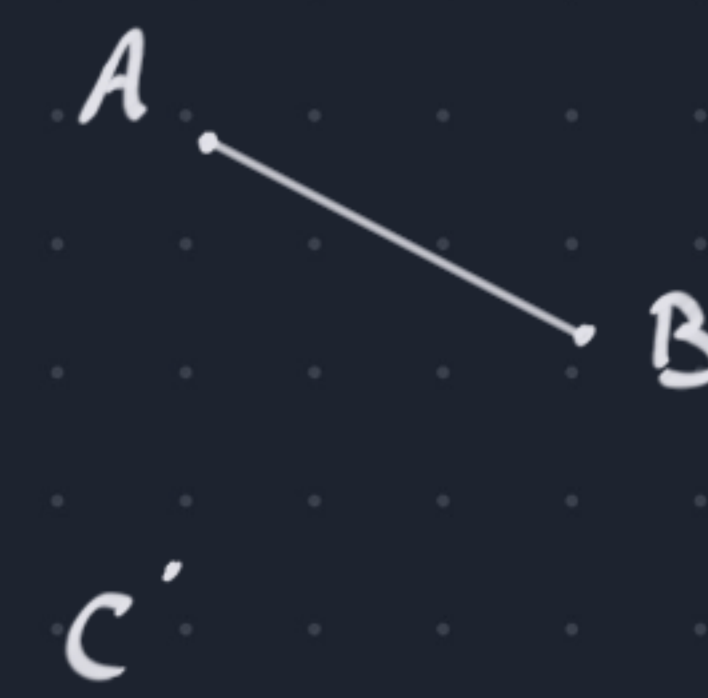

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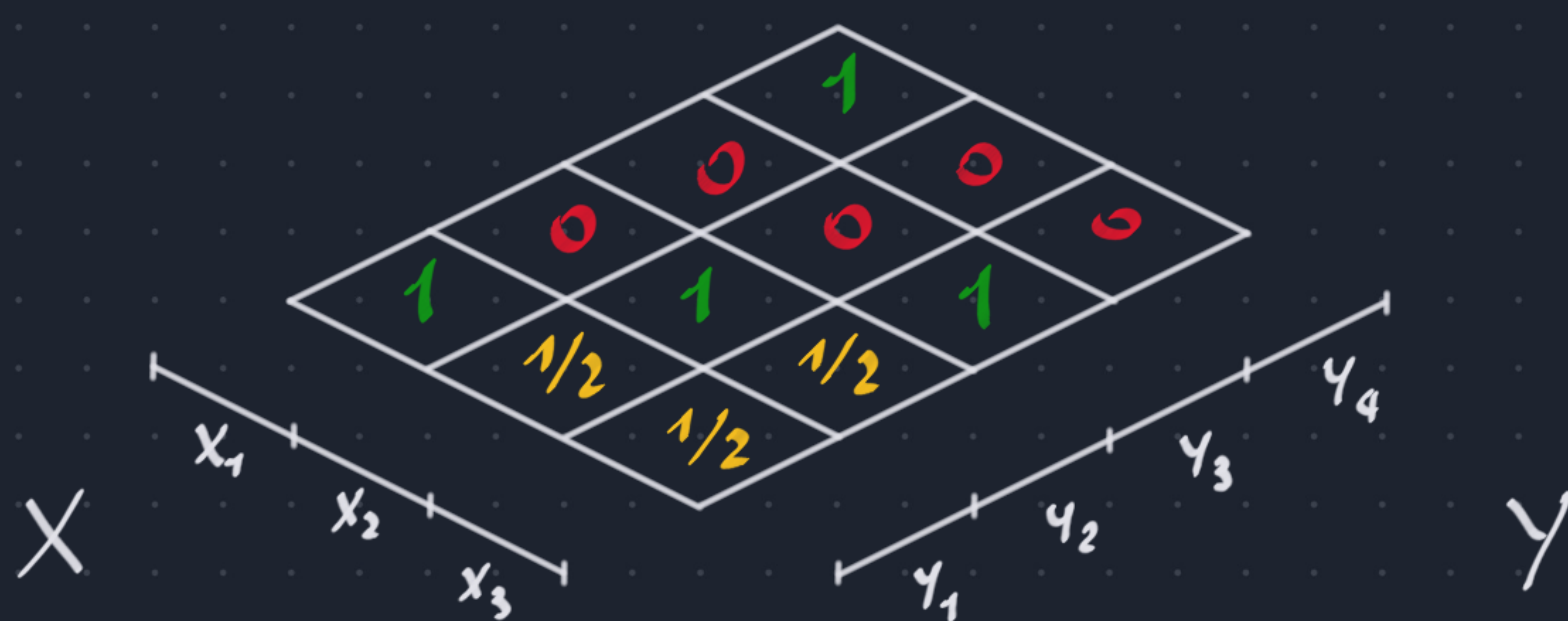
Examples. Rel Mat



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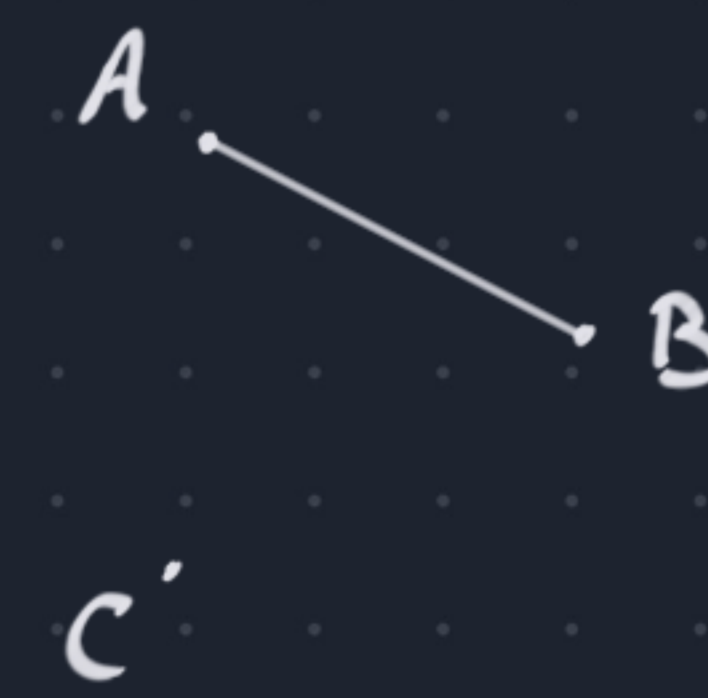

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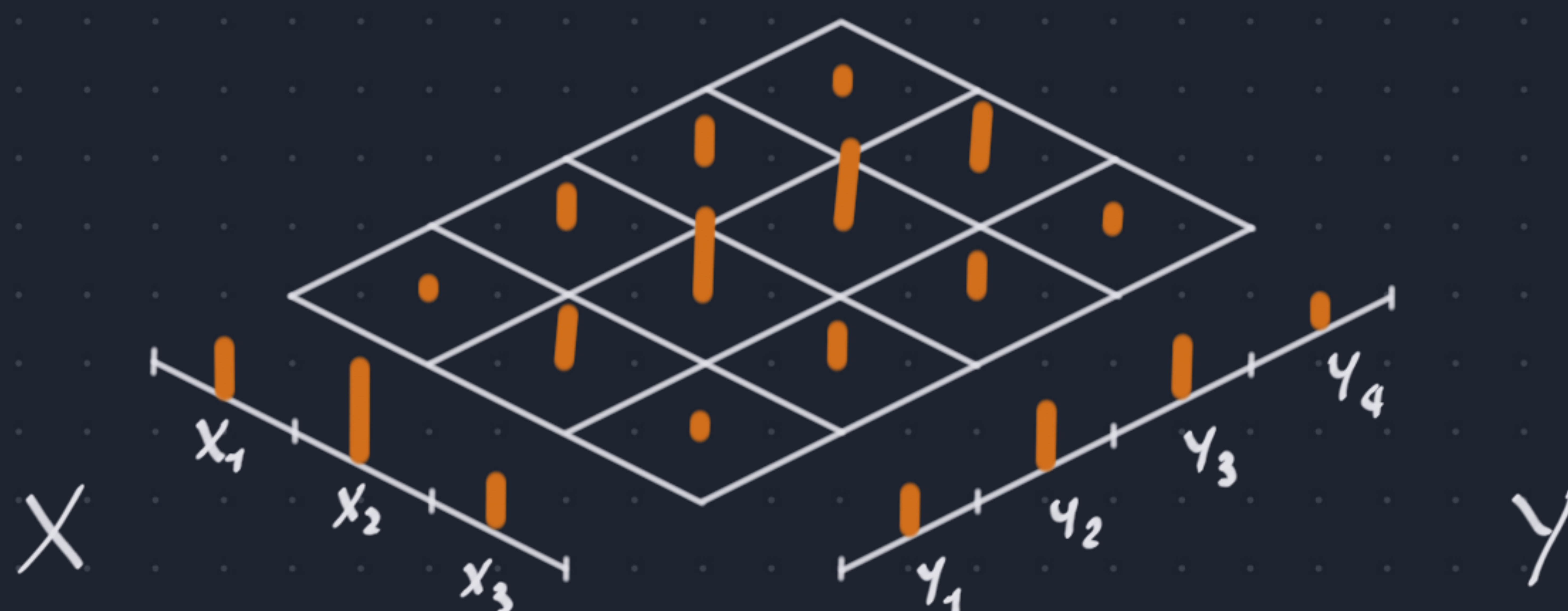
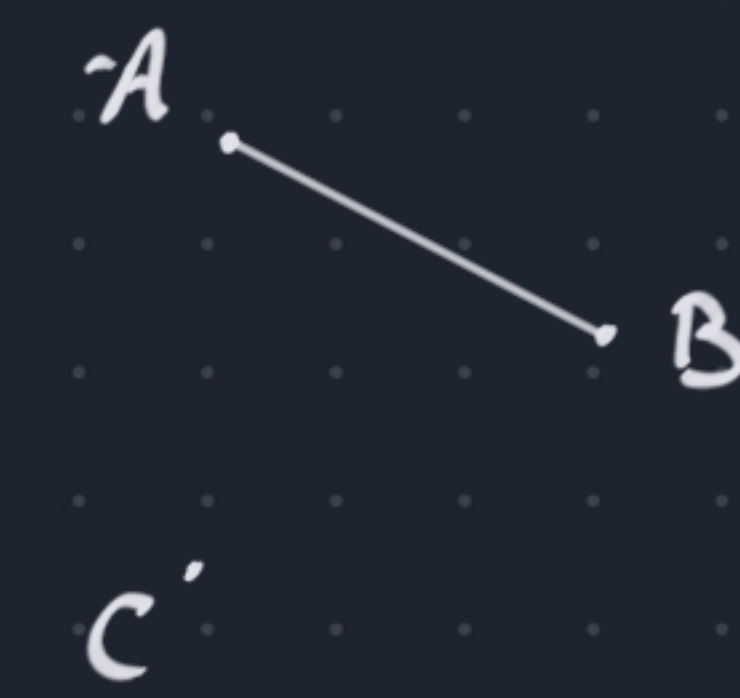
Examples. **Rel** **Mat** **Krn**



2. $\perp = \amalg$ · $\dagger$-categories



Idea. $$\frac{\text{Dagger categories}}{\text{categories}} = \frac{\text{undirected graphs}}{\text{directed graphs}}$$

Examples. $\mathbf{Rel}$ $\quad$ $\mathbf{Mat}$ $\quad$ $\mathbf{Krn}$

Definition. A *dagger structure* on $\mathcal{C}$ is a functor $\mathcal{C}^{op} \xrightarrow[\cong]{\dagger} \mathcal{C}$ such that:

- On objects, $X^\dagger = X$;
- On morphisms, $f^{\dagger\dagger} = f$.

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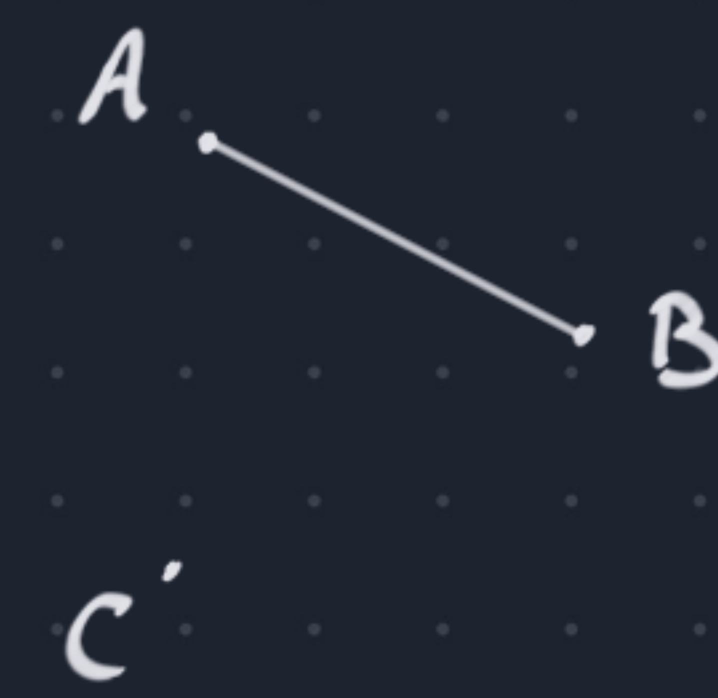

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- On objects, $X^\dagger = X$;
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$$(g \cdot f)^\dagger = f^\dagger \cdot g^\dagger$$

2. $\perp = \amalg$ · "Geometry" of $+$-categories

<u>Idea.</u> $+$-categories encode "geometric" notions. (This is known, but often overlooked.)

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__Examples.__

- The dagger operation itself $\begin{matrix} X \\ \downarrow f \\ Y \end{matrix} \leftrightarrow \begin{matrix} X \\ \uparrow f^\dagger \\ Y \end{matrix}$

  can encode a metric (e.g. in Physics)

Mat	Vect_{BF}	Hilb	Krn
$M \leftrightarrow M^T$		$\lvert v\rangle \leftrightarrow \langle v\rvert$	
	$v_i \leftrightarrow v^i$		$p(y\mid x) \leftrightarrow p(x\mid y)$

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- An iso $f : X \xrightarrow{\cong} Y$ is called

  a $\dagger$-iso or unitary if $f^{-1} = f^\dagger$.

Mat	Vect_{BF}	Hilb	Krn
$M \leftrightarrow M^T$	$v_i \leftrightarrow v^i$	$\lvert v\rangle \leftrightarrow \langle v \rvert$	$p(y\mid x) \leftrightarrow p(x\mid y)$
Orthogonal matrix	(Hyp.) rotation, symplectic map...	Unitary operator	(Deterministic) a.s. isomorphism

2. $\perp = \amalg$ · "Geometry" of $\dagger$-categories

Idea. $\dagger$-categories encode "geometric" notions. (This is known, but often overlooked.)

Examples.

- A mono $f : X \hookrightarrow Y$ is called a $\dagger$-mono or isometry if $f^\dagger f = \mathrm{id}$.

$\underbrace{\quad \mathbf{Mat} \qquad \mathbf{Vect}_{BF} \qquad \mathbf{Hilb} \quad}$	$\mathbf{Krn}$
Embedding / isometry	Proper kernel

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Examples.

- A mono $f : X \hookrightarrow Y$ is called a †-mono or isometry if $f^\dagger f = id$.

  $f^\dagger$ is called a †-epi or coisometry.

$\underbrace{\textbf{Mat} \quad \textbf{Vect}_{BF} \quad \textbf{Hilb}}$	**Krn**
Embedding / isometry	Proper kernel
	Det. map (coarse-graining)

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  $f^\dagger$ is called a $\dagger$-epi or coisometry.

- An idempotent $X \xrightarrow{e} X$ is called a $\dagger$-idempotent or projector if $e^\dagger = e$.

$\underbrace{\textbf{Mat} \quad \textbf{Vect}_{BF} \quad \textbf{Hilb}}$	**Krn**
Embedding / isometry	Proper kernel
	Det. map (coarse-graining)
Orthogonal projection	Conditional expectation

2. $\perp = \amalg$ · Independent squares

Definition (new). A commutative square in a $+$-category

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] & D \end{tikzcd}$$

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is called *orthogonal* $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$

or *independent* $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp\!\!\!\perp"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$

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if $gf^\dagger = v^\dagger u$.

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or independent $\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp\!\!\!\perp"] & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd}$

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if $gf^\dagger = v^\dagger u$.

- We are mostly interested in squares of "maps" (e.g. $\dagger$-epis).

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Examples. · In **Rel**, with functions:

$$
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "v"] \\
B \arrow[r, "u"'] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] & D
\end{tikzcd}
$$

$\forall\, b \in B,\, c \in C:$

$\exists\, a \in A : \ f(a) = b,\ \ g(a) = c$

$\Updownarrow$

$\exists\, d \in D : \ u(b) = d = v(c)$

2. $\perp = \perp\!\!\!\perp$ · Independent squares

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$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] & D \end{tikzcd}$$

$\forall\, b \in B,\ c \in C:\ u(b) = v(c),$

$\exists\ a \in A:\ f(a) = b,\ \ g(a) = c$

i.e. weak pullback (BC)

2. $\perp = \perp\!\!\!\perp$ · Independent squares

Examples. · In **Rel**, with functions:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] & D \end{tikzcd}$$

$\forall\, b \in B,\ c \in C :\ u(b) = v(c),$

$\exists\, a \in A :\ f(a) = b,\ \ g(a) = c$

i.e. **weak pullback (BC)**

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i.e. weak pullback (BC)

2. $\perp = \perp\!\!\!\perp$ · Independent squares

Examples. · In **Rel**, with functions: weak pullback

· In **Mat**, for vectors, $D = 0$:

$$
\begin{tikzcd}
A \arrow[r, "w"] \arrow[d, "v"'] & 1 \arrow[d, "!"] \\
1 \arrow[r, "!"'] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] & 0
\end{tikzcd}
$$

2. $\perp = \perp\!\!\!\perp$ · Independent squares

__Examples.__

- In **Rel**, with functions: weak pullback

- In **Mat**, for vectors, $D = 0$: usual orthogonality

$$\begin{tikzcd}
A \arrow[r, "w"] \arrow[d, "v"'] & 1 \arrow[d, "!"] \\
1 \arrow[r, "!"'] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] & 0
\end{tikzcd}
\qquad\qquad v^{\dagger} w = 0$$

2. $\perp = \amalg$ · Independent squares

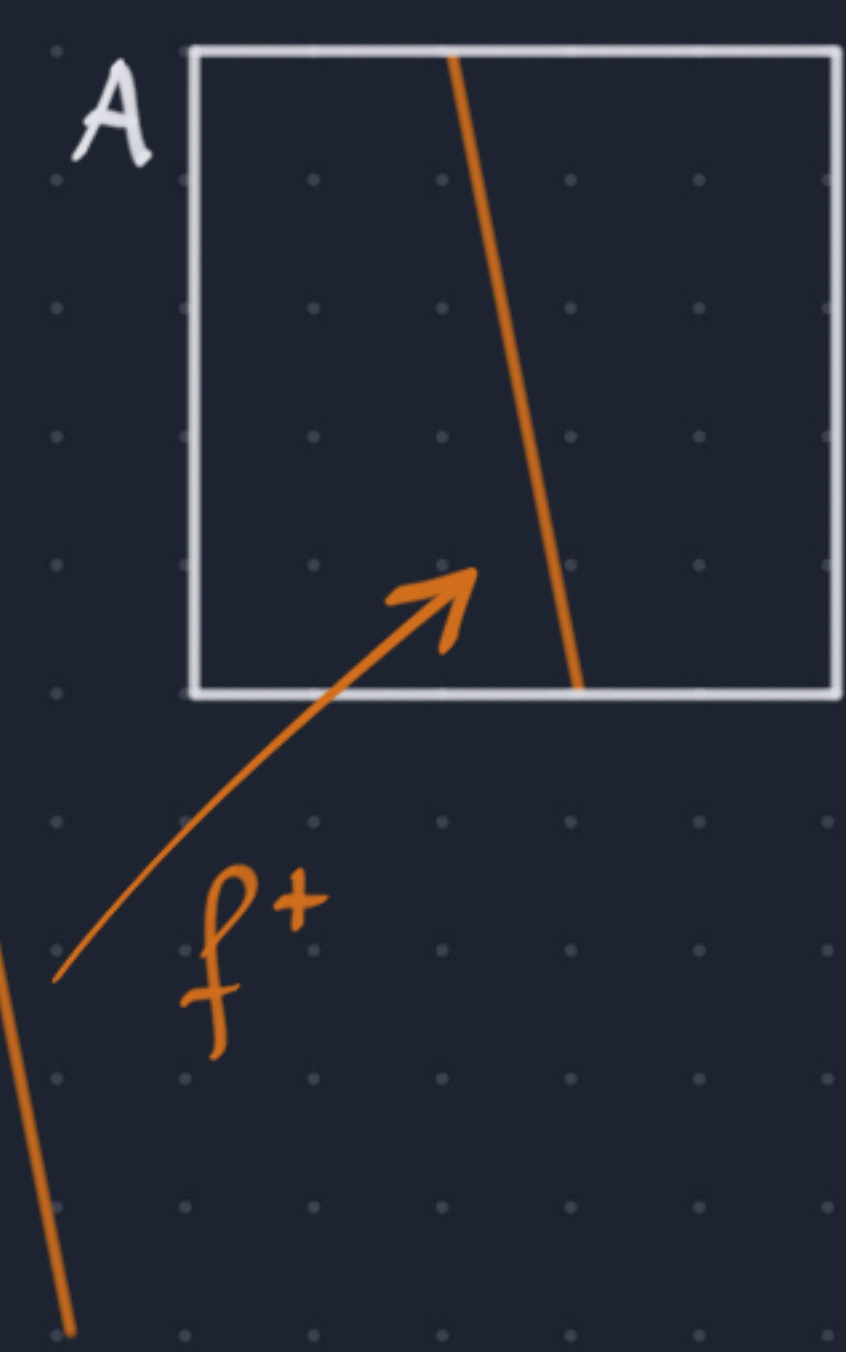
Examples.

- In **Rel**, with functions: weak pullback

- In **Mat**, for vectors, $D = 0$: usual orthogonality

  for $\dagger$-epis, $D = 0$: orthogonal subspaces

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "!"] \\
B \arrow[r, "!"'] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] & 0
\end{tikzcd}$$



2. $\perp = \perp\!\!\!\perp$ · Independent squares

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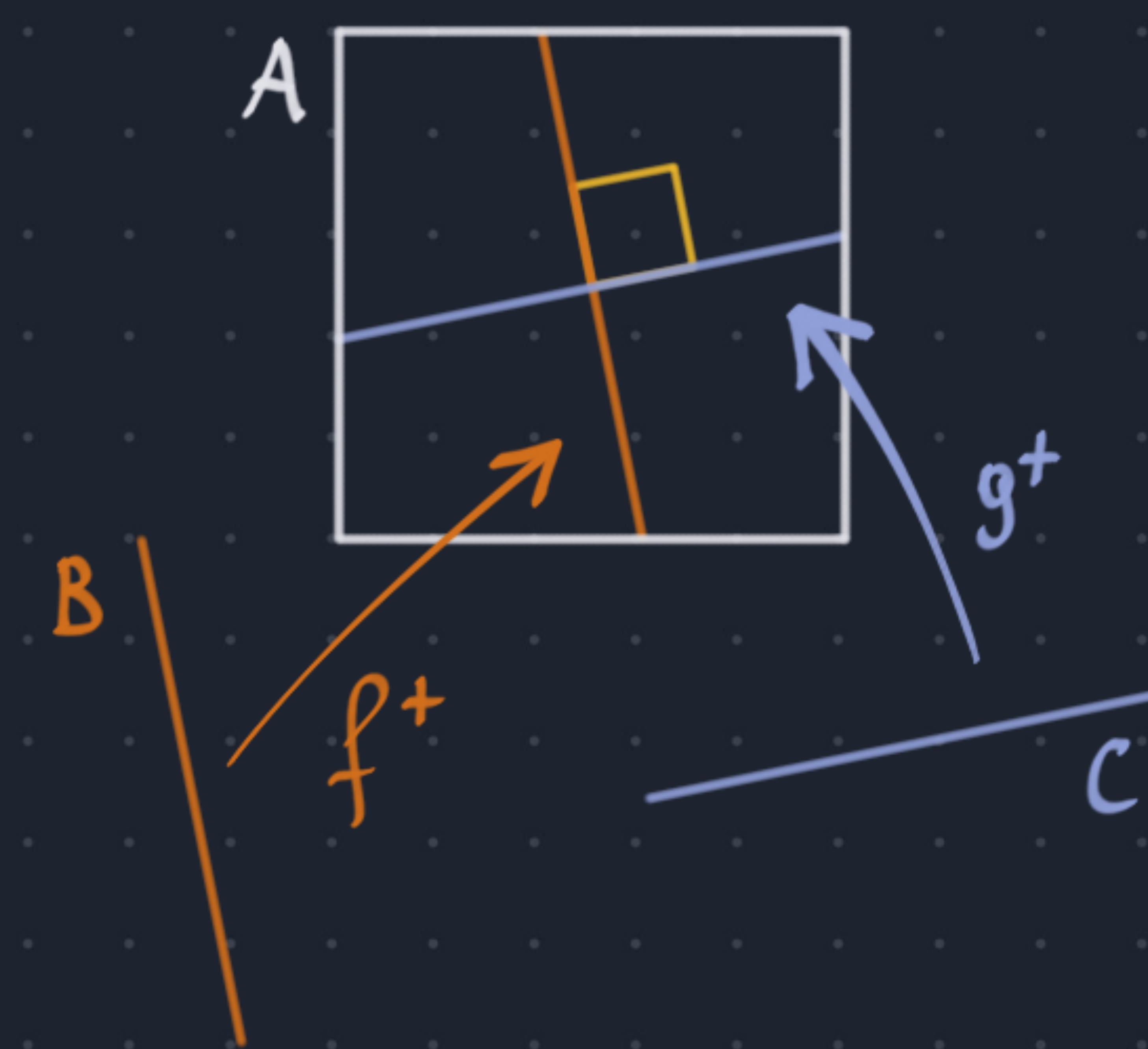


2. $\perp$ = $⫫$ · Independent squares

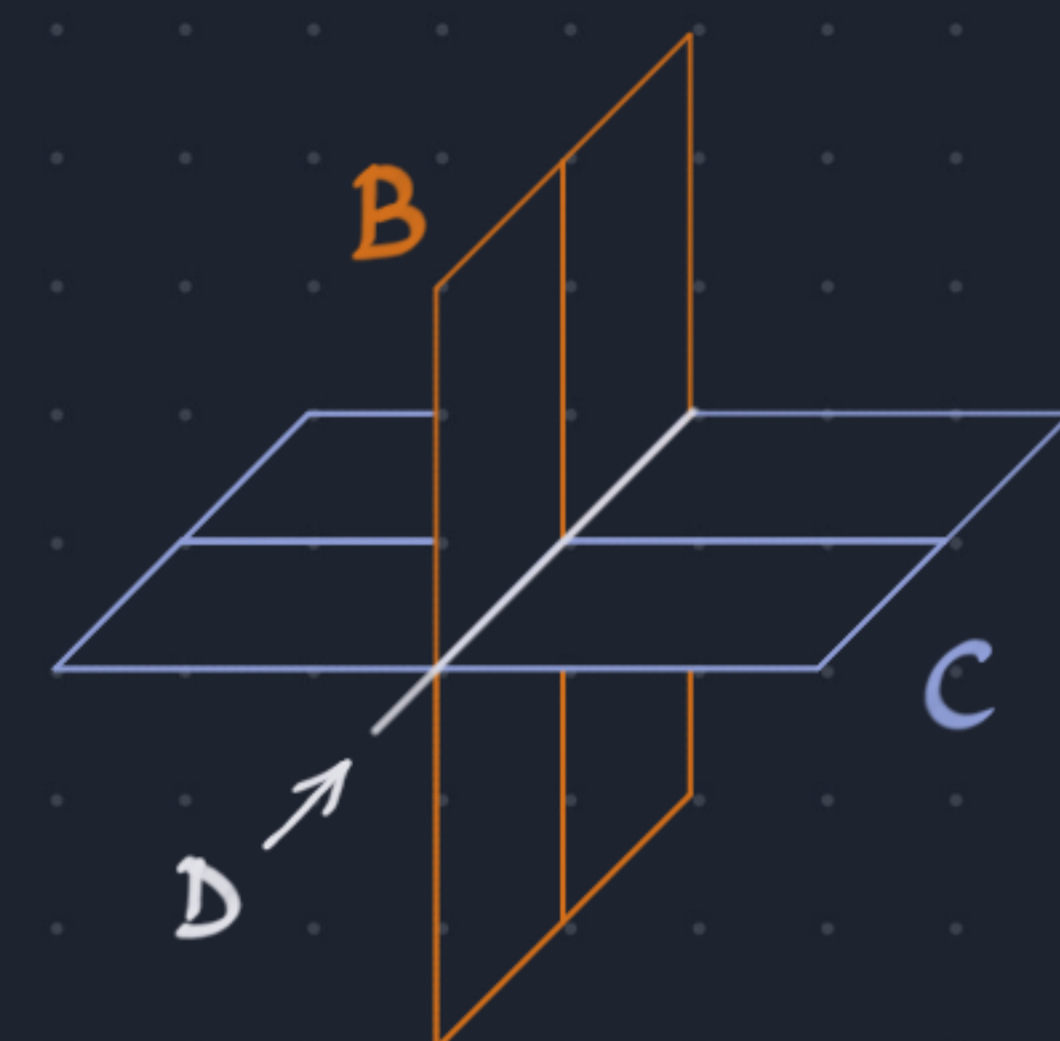
Examples.

- In Rel, with functions: weak pullback
- In Mat, for vectors, $D = 0$: usual orthogonality

  for $\dagger$-epis, $D = 0$: orthogonal subspaces

  for $\dagger$-epis, any D:

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] & D \end{tikzcd}$$



2. $\perp = \perp\!\!\!\perp$ · Independent squares

Examples. · In **Rel**, with functions: weak pullback

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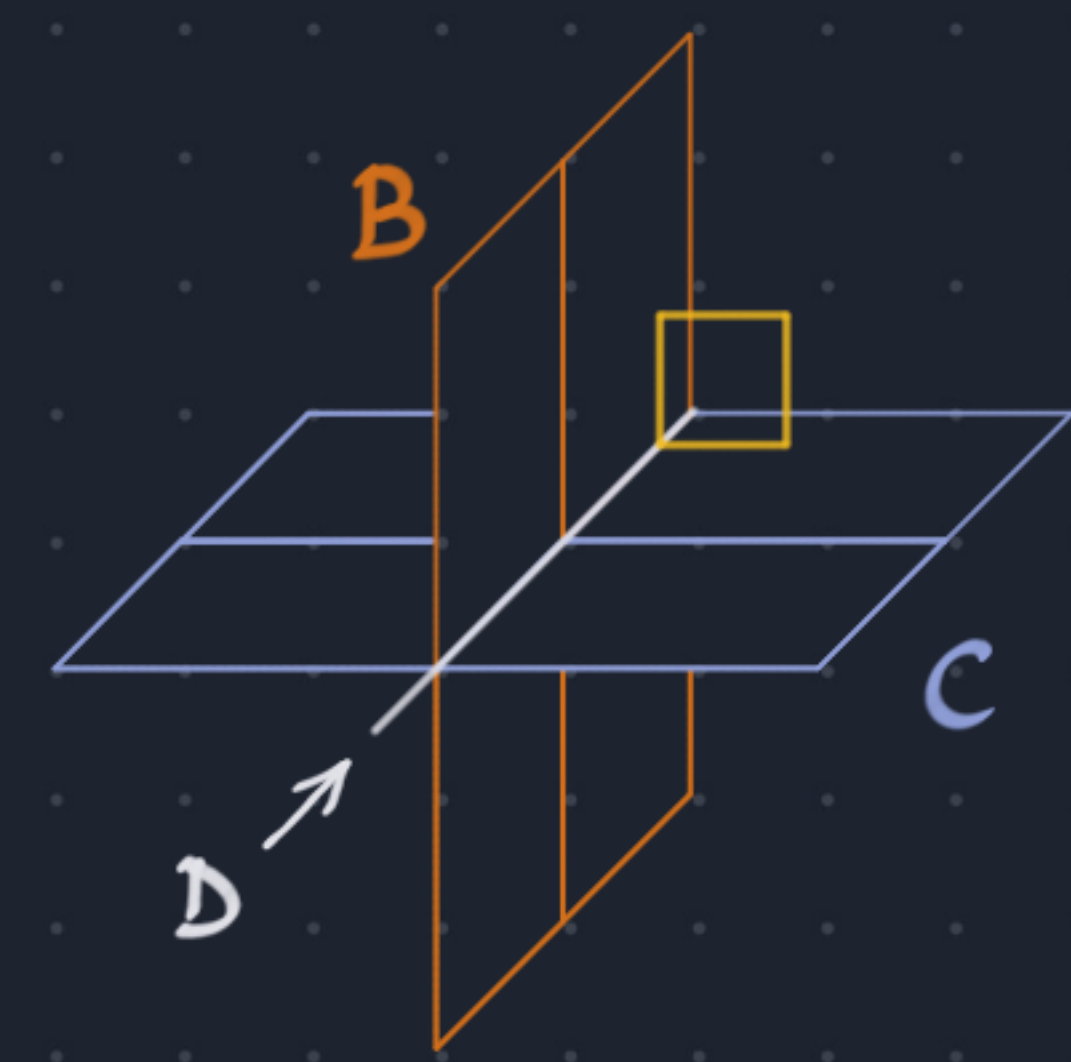

2. $\perp = \perp\!\!\!\perp$ · Independent squares

<u>Examples.</u>

- In **Rel**, with functions: weak pullback

- In **Mat**, for vectors, $D = 0$: usual orthogonality

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  for $\dagger$-epis, any D: relatively orthogonal subspaces

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A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "v"] \\
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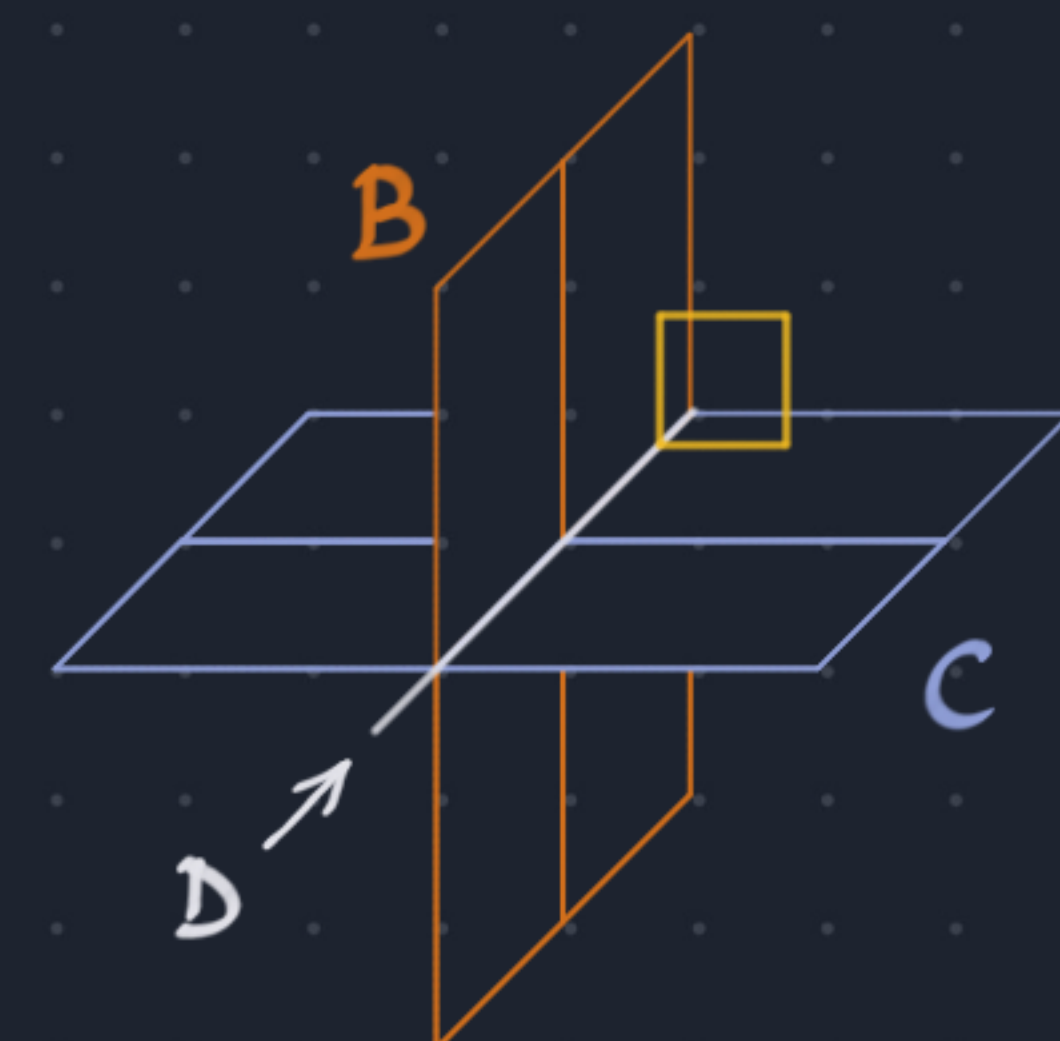
\end{tikzcd}
$$



2. $\perp = \perp\!\!\!\perp$ · Independent squares

Examples.

- In Rel, with functions: weak pullback

- In Mat, for vectors, $D = 0$: usual orthogonality

  for $\dagger$-epis, $D = 0$: orthogonal subspaces

  for $\dagger$-epis, any D: relatively orthogonal subspaces

- In Krn, for $\dagger$-epis (= det. maps / RVs), $D = 0$: independent RVs

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "!"] \\
B \arrow[r, "!"'] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] & O
\end{tikzcd}
\qquad \mathbb{P}[f = b, g = c] = \mathbb{P}[f = b] \cdot \mathbb{P}[g = c]$$

2. $\perp \; = \; \perp\!\!\!\perp$ · Independent squares

Examples.

- In Rel, with functions: weak pullback

- In Mat, for vectors, $D = 0$: usual orthogonality

  for $\dagger$-epis, $D = 0$: orthogonal subspaces

  for $\dagger$-epis, any D: relatively orthogonal subspaces

- In Krn, for $\dagger$-epis (= det. maps / RVs), $D = 0$: independent RVs

  any D: conditionally indep. RVs

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "v"] \\ B \arrow[r, "u"'] \arrow[ur, dashed, bend left] \arrow[ur, dashed, bend right] & D \end{tikzcd}$$

$$\mathbb{P}[f = b, g = c \mid u \cdot f = v \cdot g = d]$$

$$= \mathbb{P}[f = b \mid u \cdot f = d] \cdot \mathbb{P}[g = c \mid v \cdot g = d]$$

2. $\perp = \perp\!\!\!\perp$ · Independent squares

Definition. An *independent pullback** is a universal independent square of $\dagger$-epis.

$$
\begin{tikzcd}
& X \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\perp\!\!\!\perp"] & \\
A \arrow[dr] & & B \arrow[dl] \\
& C &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& X \arrow[d, dashed] \arrow[ddl] \arrow[ddr] & \\
& P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\perp\!\!\!\perp"] & \\
A \arrow[dr] & & B \arrow[dl] \\
& C &
\end{tikzcd}
$$

*not (always) a pullback!

If $C = 0$: *independent product*.

2. $\perp = \perp\!\!\!\perp$ · Independent squares

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$$
\begin{tikzcd}
& X \arrow[ddl] \arrow[ddr] & \\
& & \\
A \arrow[dr] & \perp\!\!\!\perp & B \arrow[dl] \\
& C &
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
& X \arrow[ddl] \arrow[ddr] \arrow[d, dashed] & \\
& P \arrow[dl] \arrow[dr] & \\
A \arrow[dr] & \perp\!\!\!\perp & B \arrow[dl] \\
& C &
\end{tikzcd}
$$

*not (always) a pullback!

If $C = 0$: *independent product*.

Examples.

- In **Mat**, orthogonal (relative) direct sums
- In **Krn**, (conditional) products of probability spaces.

2. $\perp$ = $\perp\!\!\!\perp$ · Independent squares

Remark. There are mathematical structures for "independence":

- Matroids [Whitney '35, Mac Lane '36] $\leftarrow$ combinatorics of linear independence
- Graphoids [Pearl-Paz '85] $\leftarrow$ combinatorics of stochastic independence
- Separoids [Dawid 2001] $\leftarrow$ common generalization

2. $\perp = \perp\!\!\!\perp$ · Independent squares

Remark. There are mathematical structures for "independence":

- Matroids [Whitney '35, Mac Lane '36] ← combinatorics of linear independence
- Graphoids [Pearl-Paz '85] ← combinatorics of stochastic independence
- Separoids [Dawid 2001] ← common generalization

- Independence Stuctures [Simpson 2018] : choice of particular squares

$$\begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, "\mathrm{id}"] & {} \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\perp\!\!\!\perp"] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd} \qquad \begin{tikzcd} {} \arrow[r] \arrow[d] & {} \arrow[d] \\ {} \arrow[r] & {} \end{tikzcd}$$

Question. Do our independent squares form an independence stucture ?

1. MOTIVATION
2. $\perp = \perp\!\!\!\perp$
3. THE EQUIVALENCE

3. THE EQUIVALENCE · Epi-Regular ind. categories

Definition. [Carboni - Walters] A category is regular if :

1) Every cospan has a pullback

$$\begin{tikzcd} & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & \\ A \arrow[dr] & & B \arrow[dl] \\ & C & \end{tikzcd}$$

2) Every span has a

(strong epi - jointly monic) factorization

$$\begin{tikzcd} & S \arrow[dl] \arrow[dr] \arrow[d, "\text{st. epi}"] & \\ A & S' \arrow[l] \arrow[r] & B \end{tikzcd}$$

$$\underbrace{A \leftarrow S' \rightarrow B}_{\text{j. monic}}$$

3) Strong epis are pullback-stable

3. THE EQUIVALENCE · *Epi-Regular ind. categories*

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A & S' \arrow[l] \arrow[r] & B
\end{tikzcd}$$

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3) ~~Strong epis are pullback stable~~

Every morphism is strong epi!

3. THE EQUIVALENCE · Epi-Regular ind. categories

<u>Examples.</u>

- Prob: (std. Borel) prob. spaces, meas.-pres. (det.) maps [Simpson]
- Hilbert spaces, coisometries (same for Mat)

3. THE EQUIVALENCE · Epi-Regular ind. categories

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3. THE EQUIVALENCE · Epi-Regular ind. categories

<u>Examples.</u>

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- $(\text{Sets, injective functions})^{op}$

3. THE EQUIVALENCE · *Epi-Regular ind. categories*

Examples.

- Prob: (std. Borel) prob. spaces, meas.-pres. (det.) maps [Simpson]
- Hilbert spaces, coisometries (same for Mat)
- Sets, surjective functions
- $(\text{Sets, injective functions})^{op}$

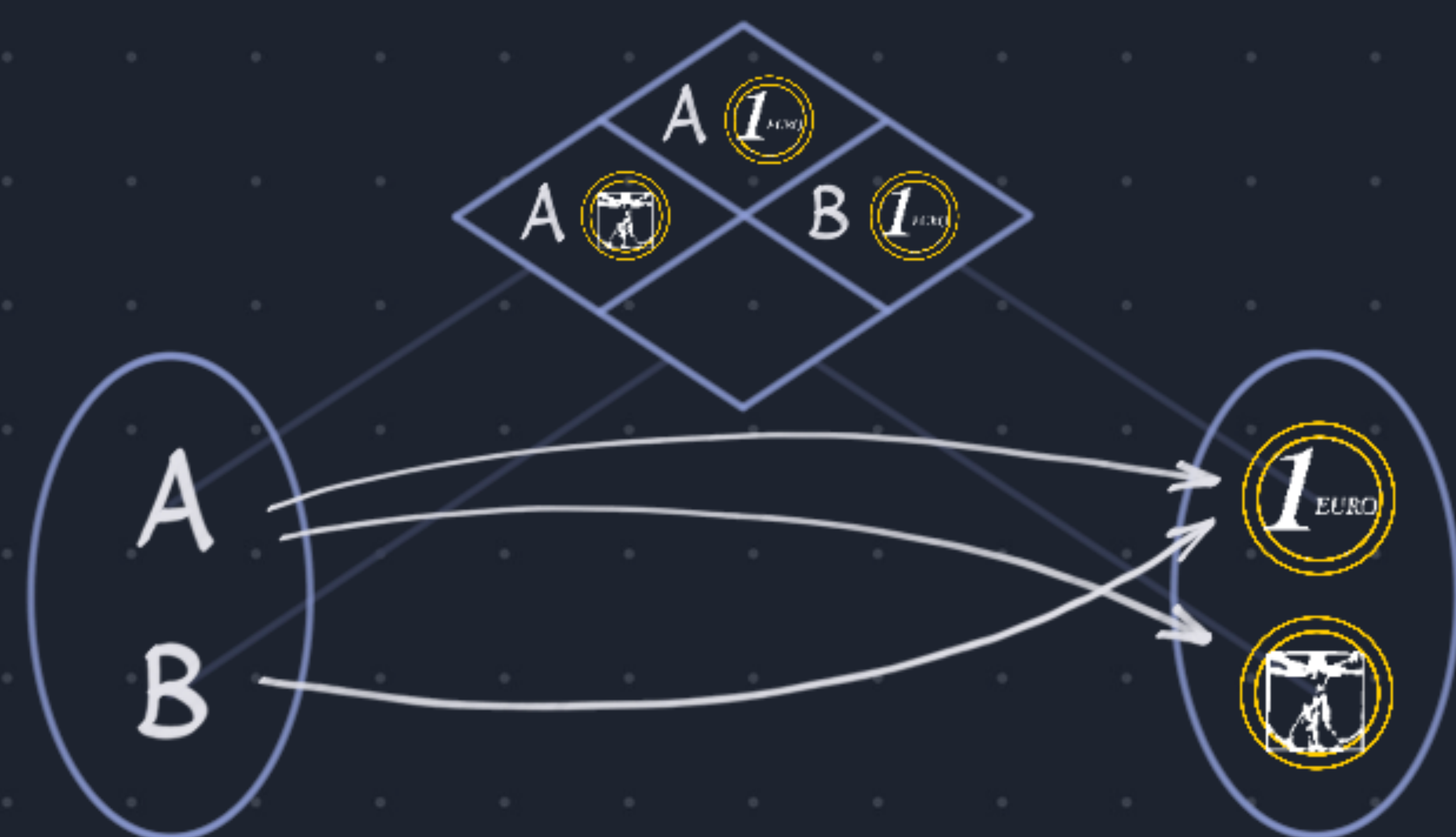
Properties.

- Every morphism is the coequalizer of its independent kernel pair
- f monic $\iff$

$$\begin{tikzcd} & {} \arrow[dl, "id"'] \arrow[dr, "id"] \arrow[dd, phantom, "\perp"] & \\ {} \arrow[dr, "f"'] & & {} \arrow[dl, "f"] \\ & {} & \end{tikzcd}$$

3. THE EQUIVALENCE · Dilations & Dilators

Idea. Expressing a relation as a "table" (= span)



$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ \mathrm{Coin} \arrow[rr, "r"'] & & \mathrm{Outcome} \end{tikzcd}$$

Classically: tabulator (in allegories, double cats...)

3. THE EQUIVALENCE · *Dilations & Dilators*

Definition. Given $A \xrightarrow{r} B$ in a dagger category,

- A *dilation* of r is a span (f, g) of dagger epis s.t. $g f^{\dagger} = r$.

$$
\begin{tikzcd}
 & X \arrow[dl, "f\ (\dagger\text{-epi})"'] \arrow[dr, "g\ (\dagger\text{-epi})"] & \\
A \arrow[rr, "r"'] & & B
\end{tikzcd}
$$

3. THE EQUIVALENCE · Dilations & Dilators

Definition. Given $A \xrightarrow{r} B$ in a dagger category,

- A **dilation** of r is a span (f, g) of dagger epis s.t. $gf^\dagger = r$.

$$
\begin{tikzcd}
& X \arrow[dl, "f\ (\dagger\text{-epi})"'] \arrow[dr, "g\ (\dagger\text{-epi})"] & \\
A \arrow[rr, bend left=60] \arrow[rr, "r"'] \arrow[rr, phantom, bend left=25, "\shortparallel"] & & B
\end{tikzcd}
$$

3. THE EQUIVALENCE · Dilations & Dilators

Definition. Given $A \xrightarrow{r} B$ in a dagger category,

- A **dilation** of r is a span (f, g) of dagger epis s.t. $g f^{\dagger} = r$.

- A **dilator** of r is a universal dilation.

$$\begin{tikzcd}
 & Y \arrow[d, dashed] \arrow[ddl, bend right, "p"'] \arrow[ddr, bend left, "q"] & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A \arrow[rr, "r"'] & & B
\end{tikzcd}$$

3. THE EQUIVALENCE · Dilations & Dilators

Definition. Given $A \xrightarrow{r} B$ in a dagger category,

- A **dilation** of r is a span (f, g) of dagger epis s.t. $g f^{\dagger} = r$.

- A **dilator** of r is a universal dilation.

- A dagger category is **dilatory** if every morphism has a dilator.

In subcat. of $\dagger$-epis, dilator = jointly monic dilation.

$$
\begin{tikzcd}
 & Y \arrow[ddl, bend right, "p"'] \arrow[ddr, bend left, "q"] \arrow[d, dashed] & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A \arrow[rr, "r"'] & & B
\end{tikzcd}
$$

3. THE EQUIVALENCE · Dilations & Dilators

Examples.

- $\mathrm{Mat}_{\leq 1}$ has dilators:

  rel. to Choleski decomposition of matrices

- Krn has dilators:

  bloom-shriek factorization of kernels

$$\begin{tikzcd}
 & Y \arrow[d, dashed] \arrow[ddl, bend right, "p"'] \arrow[ddr, bend left, "q"] & \\
 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
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 & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\
A \arrow[rr, "r"'] & & B
\end{tikzcd}
$$

- The following subcategories of **Rel** have dilators:
  - MSurj - surjective multi-valued functions ($\dagger$-epi: single-valued)
  - PInj - injective partial functions ($\dagger$-epi: co-total)

3. THE EQUIVALENCE · *Main theorems*

Theorem 1. We have an adjoint equivalence of categories

$$\begin{tikzcd} \mathsf{EpiReg} \arrow[rr, bend left, "\substack{\mathsf{Rel} \\ \text{j.m. spans}/\sim}"] \arrow[rr, phantom, "\simeq"] & & \mathsf{Dil} \arrow[ll, bend left, "\substack{\mathsf{Coisom} \\ \text{subcat. of } +\text{-epis}}"] \end{tikzcd}$$

EpiReg

epi-reg. ind. cats

functors preserving
ind. pbs, j.m. spans

Dil

dilatory +-cats

functors preserving
+, dilators

3. THE EQUIVALENCE · *Main theorems*

Theorem 1. We have an adjoint equivalence of categories

$$\mathrm{EpiReg} \xrightarrow[\text{j.m. spans}/\sim]{\mathrm{Rel}} \mathrm{Dil}$$

$$\mathcal{C} \longmapsto \boxed{\mathrm{Rel}(\mathcal{C})}$$

$$\begin{tikzcd}[column sep=small] & R \arrow[dl] \arrow[dr] & & S \arrow[dl] \arrow[dr] & \\ A & & B & & C \end{tikzcd}$$

3. THE EQUIVALENCE · *Main theorems*

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$$\mathrm{EpiReg} \xrightarrow[\text{j.m. spans}/\sim]{\mathrm{Rel}} \mathrm{Dil}$$

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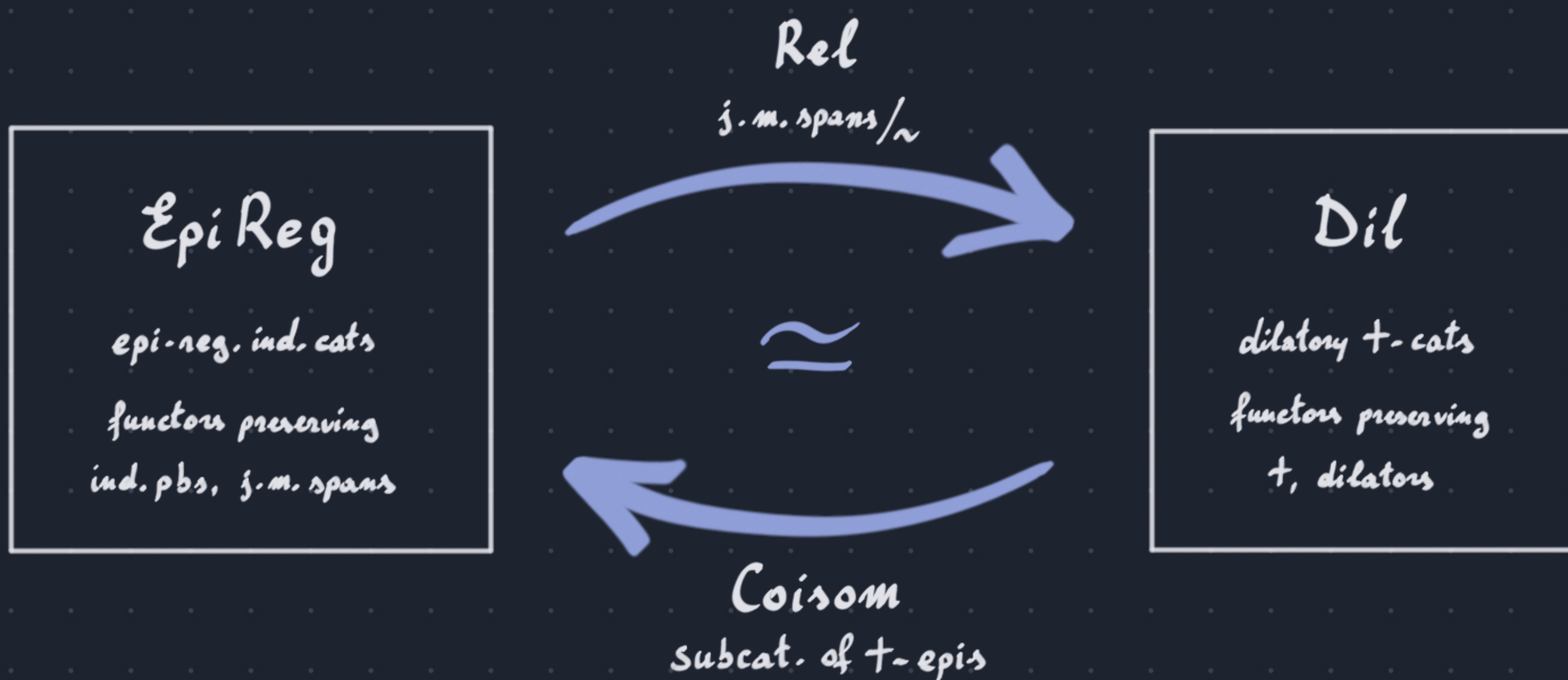
$$\begin{tikzcd}
 & & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" rotate=-45, very near start] & & \\
 & R \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\perp\!\!\!\perp"] & & S \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}$$

3. THE EQUIVALENCE · *Main theorems*

<u>**Theorem 1.**</u> We have an adjoint equivalence of categories



3. THE EQUIVALENCE · *Main theorems*

Theorem 2. We have an adjoint equivalence of 2-categories

$$\mathbf{EpiReg} \underset{\underset{\text{subcat. of } \dagger\text{-epis}}{\mathbf{Coisom}}}{\overset{\overset{\text{j.m. spans}/\sim}{\mathbf{Rel}}}{\rightleftarrows}} \mathbf{Dil} \qquad \simeq$$

EpiReg

epi-reg. ind. cats

functors preserving ind. pbs, j.m. spans

Def. Independent nat. transf.:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd} : \quad \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\perp\!\!\!\perp"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

Rel

j.m. spans/$\sim$

$\simeq$

Coisom

subcat. of $\dagger$-epis

Dil

dilatory $\dagger$-cats

functors preserving $\dagger$, dilators

Def. Natural coisometries:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{D} \arrow[l, bend left, "G"] \end{tikzcd} : \quad \begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] \arrow[dr, phantom, "\dagger\text{-epi}"] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

3. THE EQUIVALENCE · Main theorems

Examples.

- $\mathsf{Krn} \simeq$ "category of couplings"

- Same for $\mathsf{ProbStoch}(\mathcal{C})$, where

  $\mathcal{C}$ = Markov cat. w. conditionals.

- $\mathsf{Hilb}_{\leq 1}$, $\mathsf{Mat}_{\leq 1}$ $\simeq$ cospans of isometries

$\mathsf{Rel}(\mathcal{C})$

$$
\begin{tikzcd}
 & & P \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\perp\!\!\!\perp"] & & \\
 & R \arrow[dl] \arrow[dr] & & S \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}
$$

3. THE EQUIVALENCE · *Main theorems*

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- MSurj $\simeq$ spans of surj. functions $\rightarrow$ True in every (usual) regular cat.

Rel$(\mathcal{C})$

$$
\begin{tikzcd}
 & & P \arrow[dl] \arrow[dr] & & \\
 & R \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\perp\!\!\!\perp"] & & S \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}
$$

3. THE EQUIVALENCE · *Main theorems*

<u>*Examples.*</u>

- $\mathsf{Krn} \simeq$ "category of couplings"
- Same for $\mathsf{ProbStoch}(\mathcal{C})$, where
  $\mathcal{C}$ = Markov cat. w. conditionals.
- $\mathsf{Hilb}_{\leq 1}$, $\mathsf{Mat}_{\leq 1}$ $\simeq$ cospans of isometries
- $\mathsf{MSurj} \simeq$ spans of surj. functions $\rightarrow$ True in every (usual) regular cat.
- $\mathsf{PInj} \simeq$ cospans of inj. functions $\rightarrow$ True in every restriction cat. (Ask JS!)

$\mathrm{Rel}(\mathcal{C})$

$$
\begin{tikzcd}
 & & P \arrow[dl] \arrow[dr] & & \\
 & R \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\perp\!\!\!\perp"] & & S \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}
$$

3. THE EQUIVALENCE · Summary

CONCEPT → SETTING ↓	CATEGORY OF "RELATIONS"	
TRADITIONAL	ALLEGORY	
OURS	+ - CATEGORY	

3. THE EQUIVALENCE · Summary

CONCEPT → SETTING ↓	CATEGORY OF "RELATIONS"	"FUNCTION"	
TRADITIONAL	ALLEGORY	MAP (adjoint) $f \dashv f^+$	
OURS	+ - CATEGORY	+ - EPI (coisometry) $ff^+ = id$	

3. THE EQUIVALENCE · Summary

CONCEPT → / SETTING ↓	CATEGORY OF "RELATIONS"	"FUNCTION"	RELATIONS ARE "TABLES"	
TRADITIONAL	ALLEGORY	MAP (adjoint) $f \dashv f^+$	TABULATION, TABULATOR	
OURS	+ - CATEGORY	+ - EPI (coisometry) $ff^+ = id$	DILATION, DILATOR	

3. THE EQUIVALENCE · Summary

CONCEPT → / SETTING ↓	CATEGORY OF "RELATIONS"	"FUNCTION"	RELATIONS ARE "TABLES"	"GLUING"	
TRADITIONAL	ALLEGORY	MAP (adjoint) $f \dashv f^+$	TABULATION, TABULATOR	PULLBACK	
OURS	$+$-CATEGORY	$+$-EPI (coisometry) $ff^+ = id$	DILATION, DILATOR	INDEPENDENT PULLBACK	

3. THE EQUIVALENCE · *Summary*

CONCEPT → / SETTING ↓	CATEGORY OF "RELATIONS"	"FUNCTION"	RELATIONS ARE "TABLES"	"GLUING"	"EQUIVALENCE" RELATION
TRADITIONAL	ALLEGORY	MAP (adjoint) $f \dashv f^+$	TABULATION, TABULATOR	PULLBACK	CONGRUENCE $id \leq f$, $ff \leq f$, $f = f^+$
OURS	$+$-CATEGORY	$+$-EPI (coisometry) $ff^+ = id$	DILATION, DILATOR	INDEPENDENT PULLBACK	$+$-IDEMPOTENT $ff = f$, $f = f^+$

3. THE EQUIVALENCE · Summary

Open Questions

1) Common generalization?

1a. Are there ~~epi~~-regular independence categories?

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2) [Recall: †-epi ⇔ coequalizer of independent kernel pair]

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3. THE EQUIVALENCE · Summary

Open Questions

1) Common generalization?

1a. Are there ~~epi~~-regular independence categories?

2) [Recall: $\dagger$-epi $\Leftrightarrow$ coequalizer of independent kernel pair]

"Independent descent theory"? (Simpson & Stein have studied the probability case.)

3) "Independent regular logic"? $\Rightarrow$ Would apply to probability!

References

Main results:

- M. Di Meglio, C. Heunen, JS Lemay, P. Perrone, D. Stein, Dagger Categories of Relations: The equivalence of dilatory dagger categories and epi-regular independence categories. COMING SOON.

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- A. Simpson, Category-Theoretic Structure for Independence & Conditional Independence. ENTCS 2018.