2-dimensional commutativity and Fox's theorem: sketchy approach

(work in progress)

Tomáš Perutka

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Prelude: commutative 1-theories

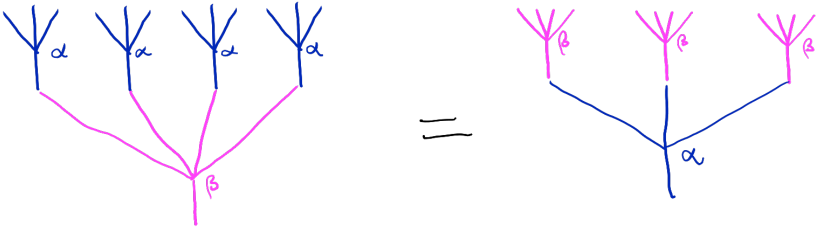
Definition (not the best one)

A Lawvere theory $\mathbb{T}$ is commutative if for each model $X\colon\ \mathbb{T} \to \mathcal{C}$ and for each $\alpha\colon n \to 1$ in $\mathbb{T}$, the n-ary operation $X(\alpha)\colon X(1)^n \to X(1)$ is a homomorphism of models.

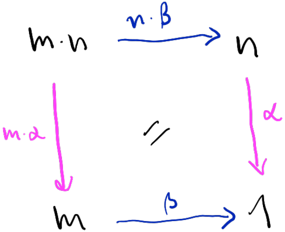
Example (justifying the name): if G is a group, then the multiplication $m\colon G \times G \to G$ is a group homomorphism if and only if G is commutative.

$\Rightarrow$ theory of groups is not commutative, theory of commutative groups is commutative.

Better (syntactic) definition [Linton '66]



$\forall \alpha, \beta$ in Π



Why go 2-dimensional? Fox's Theorem

Special case: *Tensor product of commutative rings is their coproduct.*

Thm. (Fox '76)

1. For any two commutative monoids M, N in a symmetric monoidal category $(\mathcal{V}, \otimes, I)$, $M \otimes N$ is also a commutative monoid.
2. The 2-functor $\mathsf{CMon}(-)\colon \mathbf{SMCat} \to \mathbf{SMCat}$ is a comonad.
3. This comonad is colax idempotent, and the 2-category of coalgebras identified with $\mathbf{Cat}^{\sqcup}$.

Rethinking Fox's Theorem

Thm. (Fox)

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- Let $\mathbb{T}$ be a Lawvere 2-theory for pseudocommutative pseudomonoids.
- Models in **Cat** = symmetric monoidal categories.
- (Same phenomenon as before: $\mathcal{V} \times \mathcal{V} \xrightarrow{\otimes} \mathcal{V}$ is a homomorphism of models only for symmetric monoidal categories.)
- Pseudo / lax / colax homomorphisms of models = strong / lax / colax symmetric monoidal functors.
- $\mathsf{CMon}(\mathcal{V}) = \mathsf{Lax}(1, \mathcal{V})$.

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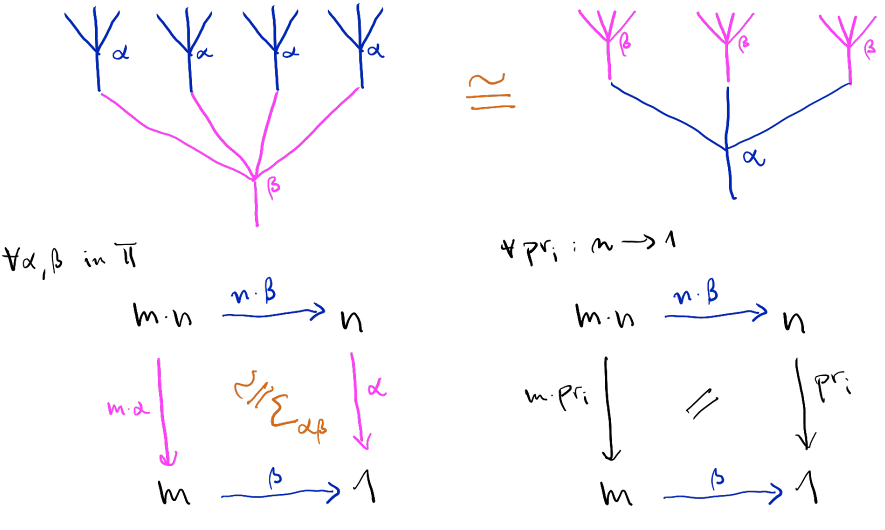
Thm. (P.)

1. For any pseudocommutative Lawvere 2-theory $\mathbb{T}$, there is a natural closed symmetric 2-multicategory structure on $\mathsf{Mod}(\mathbb{T}, \mathbf{Cat})_{lax}$ with internal hom being $\mathsf{Lax}(X, Y)$.

Pseudocommutativity

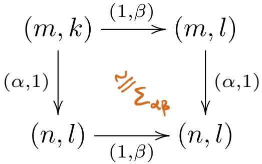


Recall: Gray tensor product

For any $w \in \{\text{strict, pseudo, lax, colax}\}$, let $[\mathcal{D}, \mathcal{E}]_w$ be a category of 2-functors and w-natural transformations.

$$[\mathcal{C} \otimes_{w\text{-Gray}} \mathcal{D}, \mathcal{E}] \cong [\mathcal{C}, [\mathcal{D}, \mathcal{E}]_w]$$

Added 2-cells in $\mathbb{T} \otimes_{ps\text{-Gray}} \mathbb{T}$:



Pseudocommutativity: First try

Let $\theta : \mathbb{F}^{op} \to \mathbb{T}$ be a Lawvere 2-theory. A w-commutativity on $\mathbb{T}$ consists of a stucture $(\mathbb{T}, \mu, u)$ of a monoid in $(\mathbf{Cat}, \otimes_{w\text{-Gray}}, 1)$ on $\mathbb{T}$ such that

1. μ preserves products in each variable,
2. $\ddot{\frown}$ ugly condition,
3. θ is a homomorphism of Gray monoids, i.e. $\mu(m, n) = m \cdot n$ and $u(*) = 1$.

$$\begin{array}{ccc} \mathbb{F}^{op} \otimes \mathbb{F}^{op} & \xrightarrow{\theta \otimes \theta} & \mathbb{T} \otimes \mathbb{T} \\ \Big\downarrow \text{mult.} & & \Big\downarrow \mu \\ \mathbb{F}^{op} & \xrightarrow{\theta} & \mathbb{T} \end{array}$$

Going sketchy

Recall[1]:

- $\mathcal{V}$-sketch is a $\mathcal{V}$-category $\mathcal{S}$ equipped with a set of weighted cones in $\mathcal{S}$.
- $\mathcal{F}$-category is 2-category with tight and loose 1-cells.

$$(\text{Lawvere 2-theories}) \longrightarrow (\mathcal{F}\text{-sketches}) \longleftarrow (\text{2-cats with fin. powers})$$

Here:

- Lawvere theory $\theta\colon \mathbb{F}^{op} \to \mathbb{T}$ is an $\mathcal{F}$-sketch with loose cells those of $\mathbb{T}$, tight cells those in the image of θ, weighted cones those for finite powers.
- 2-category $\mathcal{C}$ with finite powers is an $\mathcal{F}$-sketch with loose cells those of $\mathcal{C}$, tight cells of the form $X^n \to X$, weighted cones those for finite powers.

♡ [1] Nathanael Arkor, John Bourke, Joanna Ko: *Enhanced 2-categorical structures, two-dimensional limit sketches and the symmetry of internalisation*, 2024.

Why sketches?

(Lawvere 2-theories) $\longrightarrow$ ($\mathcal{F}$-sketches) $\longleftarrow$ (2-cats with fin. powers)

Reason 1

$$\mathsf{Mod}_w(\mathbb{T}, \mathcal{C}) \cong \mathcal{F}\text{-}\mathbf{Sk}_{s,w}(\mathbb{T}, \mathcal{C})$$

For any w, we have "enhanced Gray tensor product" $\otimes_{s,w}$ of $\mathcal{F}$-sketches: gives us exactly what we want!

Reason 2: Definition

A w-commutativity on a Lawvere theory $\theta\colon\ \mathbb{F}^{op} \to \mathbb{T}$ is a structure of monoid in $(\mathcal{F}\text{-}\mathbf{Sk}, \otimes_{s,w}, 1)$ on $\mathbb{T}$ such that θ is a homomorphism of monoids.

Short exploration

Definition

A w-commutativity on a Lawvere theory $\theta\colon \mathbb{F}^{op} \to \mathbb{T}$ is a structure of monoid in $(\mathcal{F} - \mathbf{Sk}, \otimes_w, 1)$ on $\mathbb{T}$ such that θ is a homomorphism of monoids.

Examples:

$\mathbb{T}$	commutativity	meaning
pseudomonoids	none	$X \otimes Y \neq Y \otimes X$
braided	lax	$X \otimes Y \to Y \otimes X$
symmetric	pseudo	$X \otimes Y \cong Y \otimes X$

Thm. (P.)

There is a bijection between w-commutativity structures on $\mathbb{T}$ and sections of a forgetful natural transformation $\mathsf{Mod}(\mathbb{T}, \mathsf{Mod}(\mathbb{T}, -)_w)_s \to \mathsf{Mod}(\mathbb{T}, -)_s$.

In other words: *operations are w-homomorphisms in a coherent way.*

Lax multimorphisms

Let $\mathbb{T}$ be a pseudocommutative Lawvere 2-theory, $X_1, \ldots, X_u, Y$ are $\mathbb{T}$-models in $\mathbf{Cat}$. Then a lax $\mathbb{T}$-multimap $X_1, \ldots, X_u \to Y$ is a lax natural transformation f

$$
\begin{array}{ccc}
\mathbb{T} \otimes \cdots \otimes \mathbb{T} & \xrightarrow{(X_1, \ldots, X_u)} & \mathbf{Cat} \otimes \cdots \otimes \mathbf{Cat} \\
 & & \downarrow \\
\mu^u \downarrow & \Downarrow f & \mathbf{Cat} \times \cdots \times \mathbf{Cat} \\
 & & \downarrow \prod_{i=1}^{u} \\
\mathbb{T} & \xrightarrow[Y]{} & \mathbf{Cat}
\end{array}
$$

Closed multicategory structure

Let $\mathbb{T}$ be a pseudocommutative Lawvere 2-theory, $X_1, \ldots, X_u, Y$ are $\mathbb{T}$-models in **Cat**. Then a lax $\mathbb{T}$-multimap $X_1, \ldots, X_u \to Y$ is a lax natural transformation f which gives on components

$$f_{n_1,\ldots,n_u} \colon \prod_i X_i(n_i) \to Y(n_1 \cdots n_u).$$

For the internal hom, define $[X, Y]_{lax}(n) := \mathsf{Mod}_{lax}(\mathbb{T})(X, Y^n)$ and use the fact that pseudcommutativity promotes $Y(\alpha)$ to a pseudo-homomorphism of models, so we can put

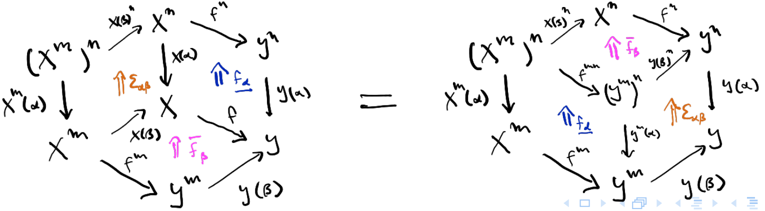
$$[X, Y]_{lax}(\alpha) := \mathsf{Mod}_{lax}(\mathbb{T})(X, Y^n) \xrightarrow{-\circ Y(\alpha)} \mathsf{Mod}_{lax}(\mathbb{T})(X, Y).$$

Interesting phenomenon: bilax maps

Recall: for braided / symmetric monoidal categories, one can study bilax monoidal functors, which is a compatible pair of a lax and colax structure on a functor. We have $\mathsf{Bimon}(\mathcal{V}) \cong \mathsf{Bilax}(1, \mathcal{V})$.

Definition (bilax morphisms of $\mathbb{T}$-models)

Let $\mathbb{T}$ be lax-commutative 2-theory, $X, Y\colon\ \mathbb{T} \to \mathbf{Cat}$ models, $f\colon X(1) \to Y(1)$ a functor. Then a bilax structure on f is a structure of a lax homomorphism $\{\overline{f}_\alpha\}_{\alpha \in mor\,\mathbb{T}}$ and a colax homomorphism $\{\underline{f}_\alpha\}_{\alpha \in mor\,\mathbb{T}}$ such that each $\overline{f}_\alpha$ is a colax natural transformation – or, equivalently, each $\underline{f}_\alpha$ is a lax natural transformation.



Thanks!

Connection to 2-monads

Corresponding notion of a pseudocommutativity for 2-monads[2], involving (co)strengths, 7 axioms, and the following invertible 2-cells:

$$\begin{array}{ccccc}
TA \times TB & \xrightarrow{t^*} & T(A \times TB) & \xrightarrow{Tt} & T^2(A \times B) \\
{\scriptstyle t}\downarrow & & \Downarrow \gamma_{AB} & & \downarrow{\scriptstyle \mu} \\
T(TA \times B) & \xrightarrow[Tt^*]{} & T^2(A \times B) & \xrightarrow[\mu]{} & T(A \times B)
\end{array}$$

If $TX = \int^n X^m \, \mathbb{T}\, m$ corresponds to a Lawvere theory $\mathbb{T}$, we have an endofunctor $SX = \int^{m,n} X^{mn} \, \mathbb{T}\, m \times \mathbb{T}\, n$, γ_{XY} can be rewritten as

$$TX \times TY \xrightarrow{d_{XY}} S(X \times Y) \underset{\otimes_2}{\overset{\otimes_1}{\rightrightarrows}} T(X \times Y) \quad (\Downarrow)$$

[2] M. Hyland, J. Power: *Pseudo-commutative monads and pseudo-closed 2-categories*, JPAA 175, p. 141-185, 2002.