A study of Kock's fat Delta

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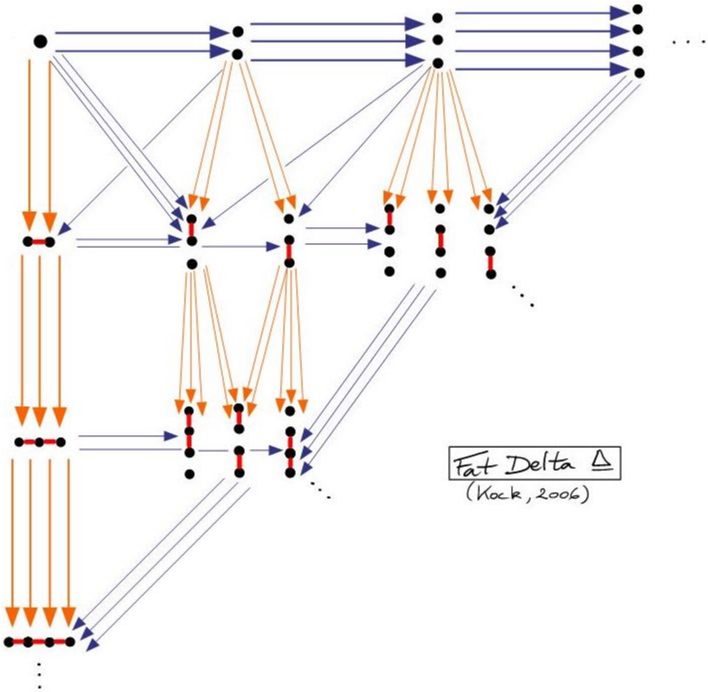
Fat Delta Δ
(Kock, 2006)

Introduction of $\underline{\Delta}$

Motivation: Simpson's conjecture

- Motivation for Kock (2006) to introduce $\underline{\Delta}$:
  - The simplex category Δ with **degeneracies up to homotopy**
  - The identity coherence structure is part of the data as objects
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Motivation: higher categories in homotopical type theories

- Construct Reedy fibrant diagrams over direct categories
- Use simplicial methods

Obstacle: Δ is not a direct category $\rightsquigarrow$ Need a direct replacement

- Introduction of a variation of $\underline{\Delta}$ by Kraus and Sattler (2017)

Study via monads with arities (arXiv:2503.10963)

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Proposition (de Jong, Kraus, Paoli, and P. 2025)

The **free relative non-unital category** *monad on relative graphs* $\mathfrak{f}^+ : \mathrm{RelGraph} \to \mathrm{RelGraph}$ *is strongly cartesian (cartesian + local right adjoint), and hence a* **monad with arities***.*

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Theorem (de Jong, Kraus, Paoli, and P. 2025)

- *The nerve functor* $\underline{\mathcal{N}} : \mathsf{RelSemiCat} \to \widehat{\underline{\Delta}}$ *is* **fully faithful**, *and the essential image is spanned by the presheaves satisfying (generalised) Segal conditions.*
- *The category* $\underline{\Delta}$ *has an* **active-inert** *factorisation system* $(\underline{\Delta}_a, \underline{\Delta}_0)$ *consisting of distance-preserving and endpoint-preserving morphisms.*
- *The category* $\underline{\Delta}$ *is an extentional and unital hypermoment category in the sense of Berger 2022.*

Presentation by generators and relations (arXiv:2506.01717)

The category $\underline{\Delta}$ can be described in terms of Δ as follows.

Definition (Kock 2006)

The objects are epimorphisms $\boldsymbol{\eta} : [n] \twoheadrightarrow [m]$ in Δ and maps $f : \boldsymbol{\kappa} \to \boldsymbol{\eta}$ are commutative squares of the form:

$$\begin{array}{ccc}
[k] & \hookrightarrow & [n] \\
{\scriptstyle \boldsymbol{\kappa}} \big\downarrow & & \big\downarrow {\scriptstyle \boldsymbol{\eta}} \\
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Using the simplicial cofaces and codegeneracies, we can identify 3 classes of elementary morphisms in $\underline{\Delta}$:

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$$\begin{array}{ccc} [m] & = & [m] \\ {\scriptstyle \boldsymbol{\eta}}\downarrow\!\!\downarrow & s_i & \downarrow\!\!\downarrow{\scriptstyle \partial_i \boldsymbol{\eta}} \\ [n] & \underset{\sigma_i}{\twoheadrightarrow} & [n-1] \end{array}$$		

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Degenerated cofaces	Vertical cofaces	Standard cofaces
$\begin{array}{ccc} [m] & = & [m] \\ {\scriptstyle \boldsymbol{\eta}}\downarrow\!\!\downarrow & \mathsf{s}_i & \downarrow\!\!\downarrow{\scriptstyle \mathfrak{d}_i\boldsymbol{\eta}} \\ [n] & \xrightarrow[\sigma_i]{} & [n-1] \end{array}$	$\begin{array}{ccc} [m-1] & \xrightarrow{\delta_i} & [m] \\ {\scriptstyle \partial^i\boldsymbol{\eta}}\downarrow\!\!\downarrow & \mathsf{v}^i \ \lrcorner & \downarrow\!\!\downarrow{\scriptstyle \boldsymbol{\eta}} \\ [n] & = & [n] \end{array}$	$\begin{array}{ccc} [m-1] & \xhookrightarrow{\delta_{\varsigma_i}} & [m] \\ {\scriptstyle \partial_i\boldsymbol{\eta}}\downarrow\!\!\downarrow & \mathsf{d}_i & \downarrow\!\!\downarrow{\scriptstyle \boldsymbol{\eta}} \\ [n-1] & \xhookrightarrow[\delta_i]{} & [n] \end{array}$

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$$\mathsf{v}^j\mathsf{s}_i = \begin{cases} \mathsf{s}_i\mathsf{s}_{i+1-\epsilon}\mathsf{d}_{i+1} & j = \varsigma_{i+1} \\ \mathsf{s}_i\mathsf{v}^j & j \neq \varsigma_{i+1} \end{cases}$$
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Theorem (P. 2025)

Any map in $\underline{\Delta}$ factors uniquely up to these relations as a composition of degenerated, vertical and standard cofaces.

Complete fat Segal spaces (work in progress)

Let $\mathcal{S}_{\underline{\Delta}} = [\underline{\Delta}^{\mathrm{op}}, \mathrm{sSet}_{QK}]$ be the category of $\underline{\Delta}$-spaces equipped with the Reedy model structure.

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Vertical condition	
Left Bousfield localisation at the vertical morphisms, i.e. morphisms induced by maps $\boldsymbol{\kappa} \to \boldsymbol{\eta}$ in $\underline{\Delta}$ of the form	

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Proposition

*A Reedy fibrant $\underline{\Delta}$-space $X \in \mathcal{S}_{\underline{\Delta}}$ satisfies the vertical condition if and only if it is local w.r.t. v-**horns** $\Lambda^{\boldsymbol{\eta}}_k \hookrightarrow \underline{\Delta}^{\boldsymbol{\eta}}$ (k is a marked vertex of $\boldsymbol{\eta} \in \underline{\Delta}$).*

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Vertical condition	Segal condition
Left Bousfield localisation at the vertical morphisms, i.e. morphisms induced by maps $\boldsymbol{\kappa} \to \boldsymbol{\eta}$ in $\underline{\Delta}$ of the form $\begin{array}{ccc} [k] & \hookrightarrow & [n] \\ {\scriptstyle \boldsymbol{\kappa}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\eta}} \\ [m] & = & [m] \end{array}$	Left Bousfield localisation at the spine inclusions $\mathrm{Sp}^{\boldsymbol{\eta}} \hookrightarrow \underline{\Delta}^{\boldsymbol{\eta}}$, where the spine $\mathrm{Sp}^{\boldsymbol{\eta}}$ consists of simplices $f : \boldsymbol{\kappa} \to \boldsymbol{\eta}$ such that the image of $\overline{f} : [k] \to [n]$ is either of the form $\{j\}$ or $\{j, j+1\}$.

Proposition

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Complete fat Segal spaces (work in progress)

Conjecture

A Reedy fibrant $\underline{\Delta}$-space $X \in \mathcal{S}_{\underline{\Delta}}$ satisfies the vertical and Segal conditions if and only if it is local w.r.t. **fat horns** $\Lambda^{\boldsymbol{\eta}}_k \hookrightarrow \underline{\Delta}^{\boldsymbol{\eta}}$ *(k is a marked vertex of $\boldsymbol{\eta} \in \underline{\Delta}$ or $0 < k <$ length $\boldsymbol{\eta}$). Call such objects* **fat Segal spaces**.

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A fat Segal space X is complete if the map

$$\mathrm{Eqv}_X \to \underline{\mathrm{Map}}(\underline{\Delta}^{\sigma_0^0}, X)$$

is an equivalence of simplicial sets.

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Conjecture

There is a Quillen equivalence between the model category of quasi-unital ∞-categories (Harpaz 2015) and the model category of complete fat Segal spaces.

Further reading:

- de Jong, Tom, Kraus, Nicolai, Paoli, Simona, and P., Stiéphen (2025). *A study of Kock's fat Delta*. arXiv: 2503.10963.
- P., Stiéphen (2025). *On combinatorial aspects of fat Delta*. arXiv: 2506.01717.

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- Berger, Clemens (2022). "Moment Categories and Operads". In: *Theory and Application of Categories* 38.39, pp. 1485–1537.
- Harpaz, Yonatan (2015). "Quasi-unital ∞–categories". In: *Algebraic & Geometric Topology* 15.4, pp. 2303–2381. DOI: 10.2140/agt.2015.15.2303.
- Kock, Joachim (2006). "Weak Identity Arrows in Higher Categories". In: *International Mathematics Research Papers*. DOI: 10.1155/IMRP/2006/69163.
- Kraus, Nicolai and Sattler, Christian (2017). "Space-Valued Diagrams, Type-Theoretically (Extended Abstract)". In: DOI: 10.48550/arXiv.1704.04543.
- Paoli, Simona (2025). "Weakly globular double categories and weak units". In: *Higher Structures* 9 (1), pp. 269–328. DOI: 10.21136/HS.2025.06.

For more see: `https://www.stiephenpradal.com`

Thank you!