# Substitution for Substructural Theories

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CT2025, Brno, Czech Republic

# Lawvere Theories as Monoids

## Base Category: $\mathbb{F}$

Free strict cocartesian category on one object

Finite cardinals and functions

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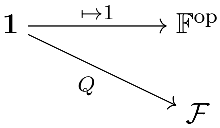

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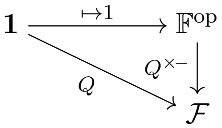

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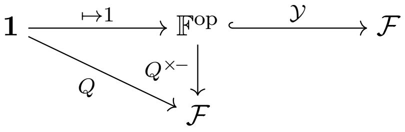

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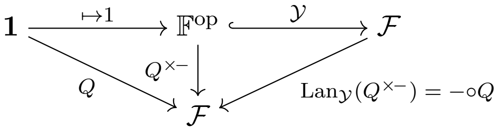

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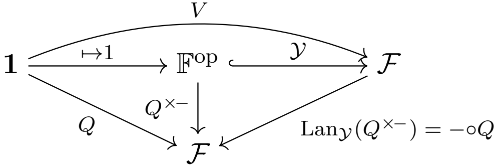


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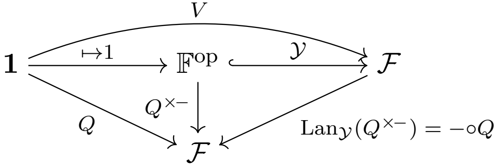


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Fiore-Plotkin-Turi (1999): $\mathrm{Mon}(\mathcal{F}) \cong \mathbf{Law}$

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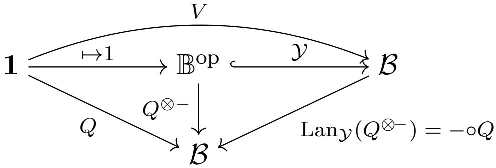

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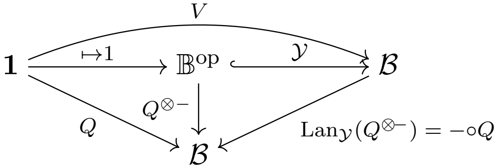


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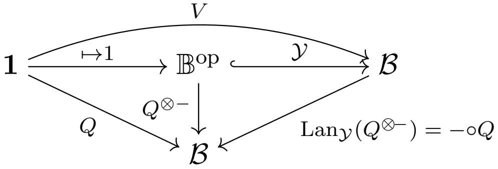


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Kelly (2005): $\mathrm{Mon}(\mathcal{B}) \cong \mathbf{SymOp}$

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Base categories are theories for symmetric monoidal equational presentations

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- Simple cases: Affine and Relevant $\rightsquigarrow$ Day tensor
- Combined cases are more involved

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Linear variables may be coerced into cartesian variables

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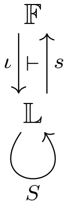

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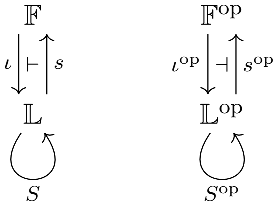

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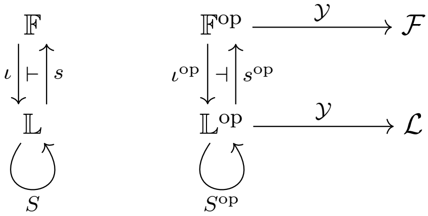

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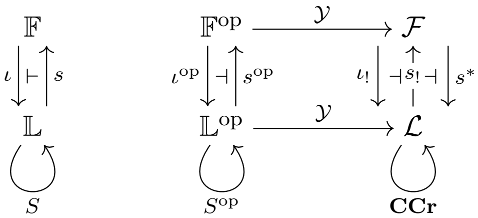

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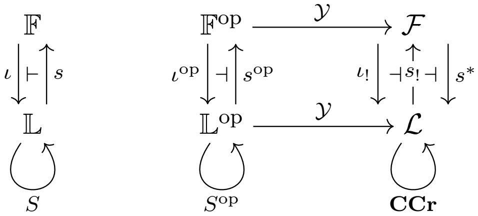


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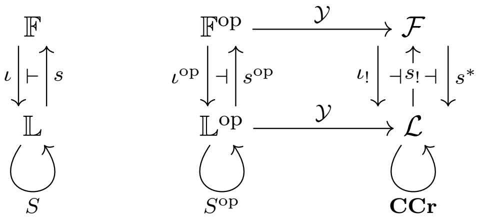


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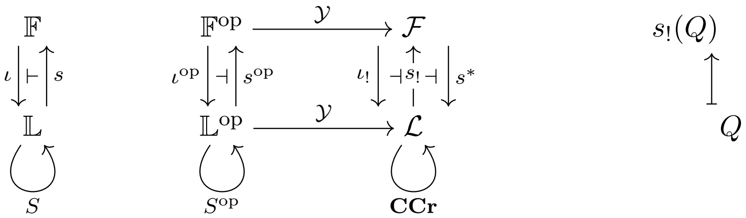


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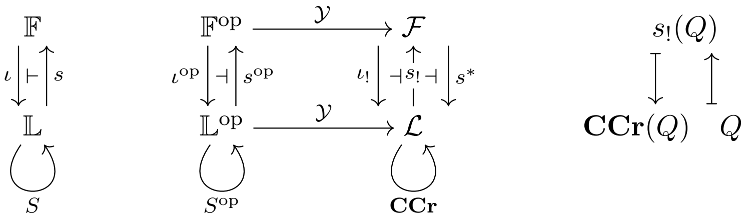


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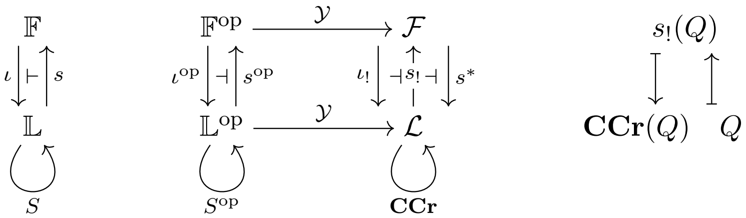


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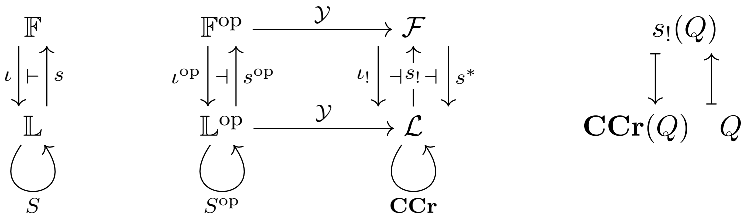


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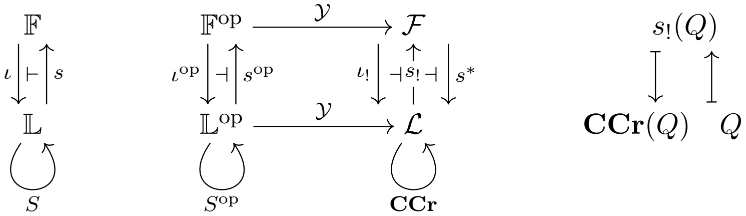


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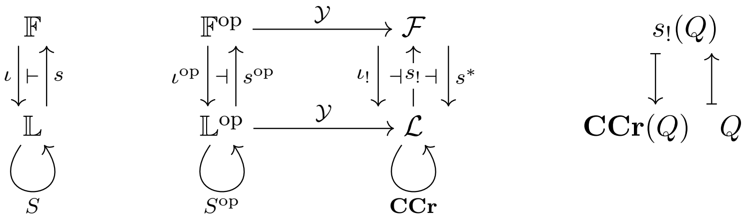


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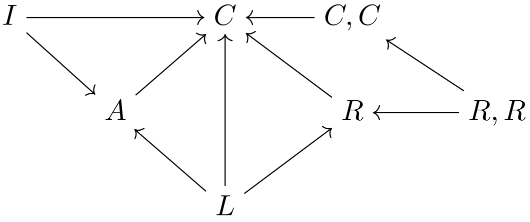
## Equational Presentation: $\mathfrak{M}$



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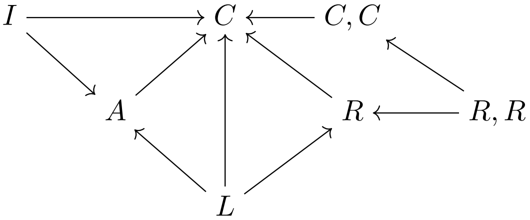


$C$ commutative monoid
$A$ pointed object
$R$ commutative semigroup
Coercions commute and respect operations

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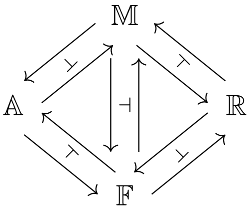

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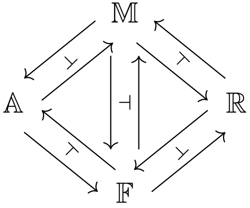
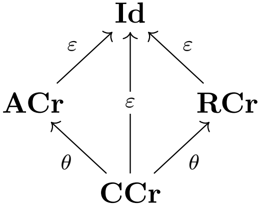



coMonads on Presheaves $\mathcal{M}$:



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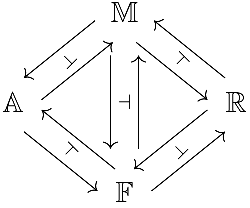


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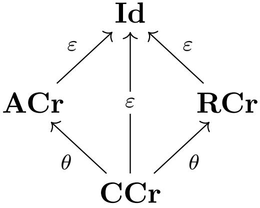


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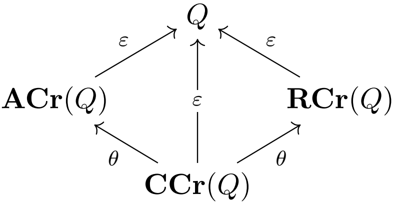

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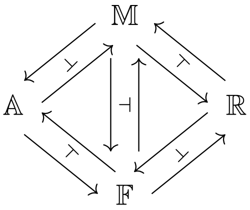


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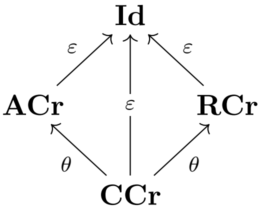


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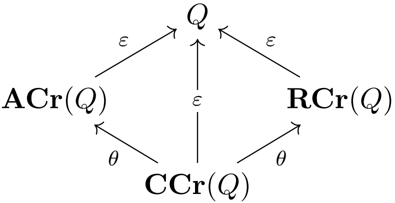


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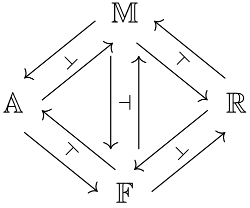


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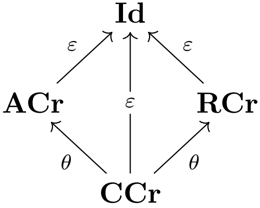


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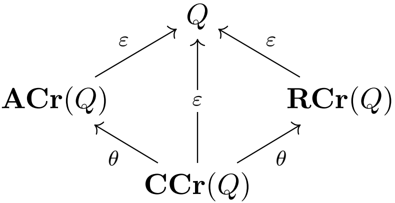


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eg. $\mathfrak{N}$ = total order on $\mathbb{N}$, no operations

Graded Operad: $\mathfrak{N}$-$\mathbf{Op} = \mathrm{Mon}(\mathcal{N})$

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2-monad on $\mathbf{Cat} \rightsquigarrow$ Pseudomonad on $\mathbf{Prof} \rightsquigarrow$ Bicategory

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Solution: Directly construct bicategory

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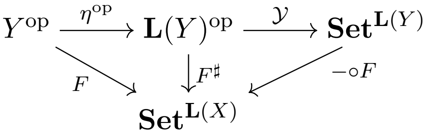


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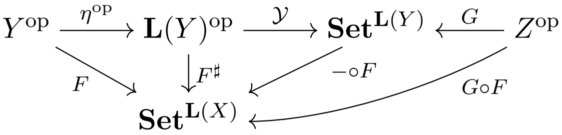


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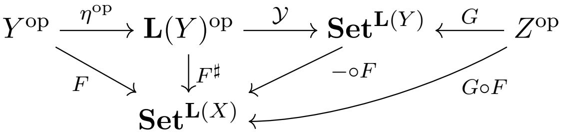


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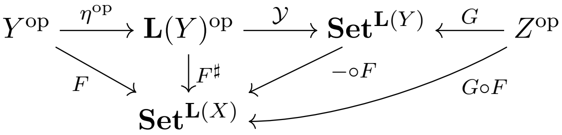


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This construction works for all superspecies

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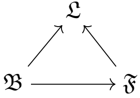

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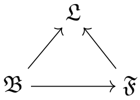


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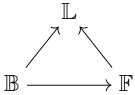


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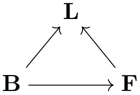


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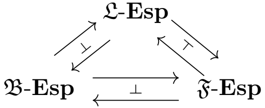

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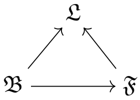


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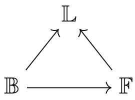


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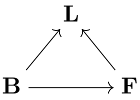


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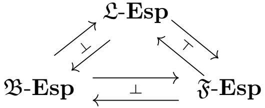


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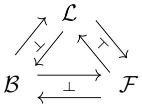

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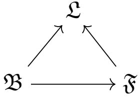


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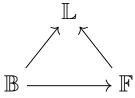


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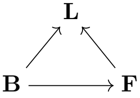


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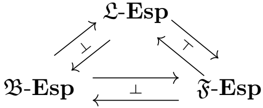


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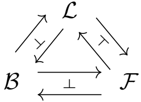
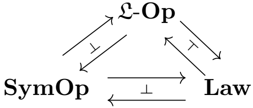



Monoids/Operads:



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