Pos-pretoposes and compact ordered spaces

Luca Reggio

joint work with

Jérémie Marquès

Dipartimento di Matematica “Federigo Enriques”, Università di Milano, Italy

International Category Theory Conference
Brno, 15 July, 2025

Axiomatic characterisations:
from sets to compact Hausdorff spaces

A common approach in mathematics:

- characterise a category of structures (e.g., **Set** or **Ab**) by isolating its key properties;
- relax these properties to include other categories that behave in a similar way.

- **Set** and *(elementary) toposes*: up to equivalence, **Set** is the unique complete well-pointed topos with a natural number object satisfying AC (Lawvere's ETCS, 1964).
- **Ab** and *abelian categories* (Buchsbaum, Grothendieck 1955–1957).

A common approach in mathematics:

- characterise a category of structures (e.g., **Set** or **Ab**) by isolating its key properties;
- relax these properties to include other categories that behave in a similar way.

- **Set** and *(elementary) toposes*: up to equivalence, **Set** is the unique complete well-pointed topos with a natural number object satisfying AC (Lawvere's ETCS, 1964).
- **Ab** and *abelian categories* (Buchsbaum, Grothendieck 1955–1957).

In a similar spirit, the category **KH** of compact Hausdorff spaces and continuous maps can be characterised among pretoposes (=extensive and Barr-exact categories).

While **Set** is infinitary extensive, to form infinite coproducts in **KH** we need to *compactify*.

Filtrality

For a bounded distributive lattice L, let $\mathcal{C}(L)$ be the Boolean center of L, and $\mathcal{F}(\mathcal{C}(L))$ the filter completion of $\mathcal{C}(L)$. There is a monotone map

$$\varphi\colon L \to \mathcal{F}(\mathcal{C}(L)), \quad x \mapsto {\uparrow}x \cap \mathcal{C}(L).$$

L is a filtral lattice if φ is an isomorphism.

Filtrality

For a bounded distributive lattice L, let $\mathcal{C}(L)$ be the Boolean center of L, and $\mathcal{F}(\mathcal{C}(L))$ the filter completion of $\mathcal{C}(L)$. There is a monotone map

$$\varphi \colon L \to \mathcal{F}(\mathcal{C}(L)), \quad x \mapsto {\uparrow} x \cap \mathcal{C}(L).$$

L is a filtral lattice if φ is an isomorphism.

An object X of a pretopos is filtral if $\mathrm{Sub}\, X$ is a filtral lattice.

Filtrality

For a bounded distributive lattice L, let $\mathcal{C}(L)$ be the Boolean center of L, and $\mathcal{F}(\mathcal{C}(L))$ the filter completion of $\mathcal{C}(L)$. There is a monotone map

$$\varphi\colon L \to \mathcal{F}(\mathcal{C}(L)), \quad x \mapsto {\uparrow}x \cap \mathcal{C}(L).$$

L is a filtral lattice if φ is an isomorphism.

An object X of a pretopos is filtral if $\mathrm{Sub}\, X$ is a filtral lattice.

- The filtral objects in **Set** are the finite sets.
- The filtral objects in **KH** are the Stone spaces (cf. the characterisation of Stone locales as the ideal completions of Boolean algebras).

Filtrality

For a bounded distributive lattice L, let $\mathcal{C}(L)$ be the Boolean center of L, and $\mathcal{F}(\mathcal{C}(L))$ the filter completion of $\mathcal{C}(L)$. There is a monotone map

$$\varphi\colon L \to \mathcal{F}(\mathcal{C}(L)), \quad x \mapsto {\uparrow}x \cap \mathcal{C}(L).$$

L is a filtral lattice if φ is an isomorphism.

An object X of a pretopos is filtral if $\mathrm{Sub}\, X$ is a filtral lattice.

- The filtral objects in **Set** are the finite sets.
- The filtral objects in **KH** are the Stone spaces (cf. the characterisation of Stone locales as the ideal completions of Boolean algebras).

Each $X \in$ **KH** is covered by a filtral object, i.e. there exist a Stone space Y and a (regular) epi $Y \twoheadrightarrow X$. So, **KH** has enough filtral objects. On the other hand, **Set** does not.

Characterising **KH**

In the same way that **Set** can be characterised among toposes, **KH** can be characterised among pretoposes.

Theorem (Marra & LR, 2020)

Up to equivalence, **KH** *is the unique non-trivial pretopos such that:*

1. *the terminal object* $\mathbf{1}$ *is a generator (and set-indexed copowers of* $\mathbf{1}$ *exist);*
2. *every object is covered by a filtral object.*

Nachbin's compact ordered spaces

Compact ordered spaces

A compact ordered space is a pair $(X, \leq)$ where X is compact and $\leq \subseteq X \times X$ is a partial order that is closed in the product topology (Nachbin, 1965).

Note: $\leq \cap \geq = \Delta_X$ is closed, hence X is Hausdorff.

Compact ordered spaces

A compact ordered space is a pair $(X, \leq)$ where X is compact and $\leq \subseteq X \times X$ is a partial order that is closed in the product topology (Nachbin, 1965).

Note: $\leq \cap \geq = \Delta_X$ is closed, hence X is Hausdorff.

Let **KOrd** be the category of compact ordered spaces and continuous *monotone* maps.

KOrd is *not* a pretopos: it is extensive but not exact.

In a pretopos, (f epi & f mono $\Rightarrow$ f iso). This fails in **KOrd**, cf. $\mathrm{id}\colon (X, =) \to (X, \leq)$.

KOrd as a **Pos**-enriched category

Hom-sets in **KOrd** are naturally ordered: for all $f, g \colon (X, \leq_X) \rightrightarrows (Y, \leq_Y)$,

$$f \leq g \iff \forall x \in X \ \ f(x) \leq_Y g(x).$$

KOrd is enriched in the category **Pos** of posets and monotone maps.

KOrd as a **Pos**-enriched category

Hom-sets in **KOrd** are naturally ordered: for all $f, g \colon (X, \leq_X) \rightrightarrows (Y, \leq_Y)$,

$$f \leq g \iff \forall x \in X \;\; f(x) \leq_Y g(x).$$

KOrd is enriched in the category **Pos** of posets and monotone maps.

Most good properties of **KH** extend to **KOrd** if we are mindful of the order-enrichment.

Theorem (Aravantinos-Sotiropoulos, 2022)

The category **KOrd** *is* **Pos***-exact.*

This suggests attempting to characterise **KOrd** among **Pos**-pretoposes (=(1, 2)-pretoposes).

Pos-pretoposes

Epi-diagonals

Let **C** be an *ordinary* category with finite limits, and let $X \in \mathbf{C}$.

- $\Delta_X \colon X \hookrightarrow X \times X$ represents the sub-presheaf of those pairs (f, g) such that $f = g$.

Epi-diagonals

Let **C** be an *ordinary* category with finite limits, and let $X \in \mathbf{C}$.

- $\Delta_X \colon X \hookrightarrow X \times X$ represents the sub-presheaf of those pairs (f, g) such that $f = g$.

Suppose now that **C** is a *poset-enriched* category with finite limits.

C has epi-diagonals if, for every $X \in \mathbf{C}$, the poset-enriched presheaf

$$F_X \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Pos}, \quad Z \mapsto \{f, g \colon Z \rightrightarrows X \mid f \leq g\}$$

is representable. If this is the case, we denote the representing object by $[\leq_X] \hookrightarrow X \times X$.

Epi-diagonals

Let $\mathbf{C}$ be an *ordinary* category with finite limits, and let $X \in \mathbf{C}$.

- $\Delta_X \colon X \hookrightarrow X \times X$ represents the sub-presheaf of those pairs (f, g) such that $f = g$.

Suppose now that $\mathbf{C}$ is a *poset-enriched* category with finite limits.

$\mathbf{C}$ has epi-diagonals if, for every $X \in \mathbf{C}$, the poset-enriched presheaf

$$F_X \colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Pos}, \quad Z \mapsto \{f, g \colon Z \rightrightarrows X \mid f \leq g\}$$

is representable. If this is the case, we denote the representing object by $[\leq_X] \hookrightarrow X \times X$.

Observation. The following are equivalent for every poset-enriched category $\mathbf{C}$:

1. $\mathbf{C}$ has finite limits and epi-diagonals.
2. $\mathbf{C}$ has finite weighted limits.

The internal language

Let $\mathbf{C}$ be a poset-enriched category with finite weighted limits. The underlying ordinary category $\mathbf{C}_0$ has finite limits, and its internal language is enriched with an order relation $\leq_X$ for each object X.

Every morphism $f\colon X \to Y$ in $\mathbf{C}$ is monotone: $x \leq x' \vdash f(x) \leq f(x')$.

The internal language

Let $\mathbf{C}$ be a poset-enriched category with finite weighted limits. The underlying ordinary category $\mathbf{C}_0$ has finite limits, and its internal language is enriched with an order relation $\leq_X$ for each object X.

Every morphism $f\colon X \to Y$ in $\mathbf{C}$ is monotone: $x \leq x' \vdash f(x) \leq f(x')$.

- f is an embedding if $f(x) \leq f(x') \vdash x \leq x'$. (ff-morphism)
- f is injective if $f(x) = f(x') \vdash x = x'$. (monomorphism)

The internal language

Let **C** be a poset-enriched category with finite weighted limits. The underlying ordinary category $\mathbf{C}_0$ has finite limits, and its internal language is enriched with an order relation $\leq_X$ for each object X.

Every morphism $f\colon X \to Y$ in **C** is monotone: $x \leq x' \vdash f(x) \leq f(x')$.

- f is an embedding if $f(x) \leq f(x') \vdash x \leq x'$. (ff-morphism)
- f is injective if $f(x) = f(x') \vdash x = x'$. (monomorphism)

We **restrict** the internal language by only allowing predicates represented by *embeddings*.

(Since embeddings are stable under pullbacks, all formulas constructed using $\wedge, =, \top$ are interpreted as embeddings.)

When X is an object of a poset-enriched category, by a subobject of X we mean an embedding $Y \hookrightarrow X$ (modulo isomorphism).

Regular and coherent categories

C a poset-enriched category with all finite weighted limits, $f\colon A \to B$ an arrow in **C**.

- If it exists, the image of f is the largest subobject $\mathrm{im}(f)$ of B through which f factors.
- f is a surjection if $\mathrm{im}(f) = B$. (so-morphism)

Regular and coherent categories

C a poset-enriched category with all finite weighted limits, $f\colon A \to B$ an arrow in **C**.

- If it exists, the image of f is the largest subobject $\mathrm{im}(f)$ of B through which f factors.
- f is a surjection if $\mathrm{im}(f) = B$. (so-morphism)

C is regular if every morphism factors as a surjection followed by an embedding, and surjections are stable under pullbacks.

If **C** is regular, its internal language interprets $\exists$. We get a (non-enriched) Lawvere doctrine

$$\mathrm{Sub}\colon \mathbf{C}^{\mathrm{op}} \to \mathsf{MSLat}.$$

Regular and coherent categories

$\mathbf{C}$ a poset-enriched category with all finite weighted limits, $f\colon A \to B$ an arrow in $\mathbf{C}$.

- If it exists, the image of f is the largest subobject $\mathrm{im}(f)$ of B through which f factors.
- f is a surjection if $\mathrm{im}(f) = B$. (so-morphism)

$\mathbf{C}$ is regular if every morphism factors as a surjection followed by an embedding, and surjections are stable under pullbacks.

If $\mathbf{C}$ is regular, its internal language interprets $\exists$. We get a (non-enriched) Lawvere doctrine

$$\mathrm{Sub}\colon \mathbf{C}^{\mathrm{op}} \to \mathsf{MSLat}.$$

$\mathbf{C}$ is coherent if each $\mathrm{Sub}(X)$ has finite joins, and these are stable under pullbacks. (Equivalently, if the latter functor factors through the inclusion $\mathsf{DL} \hookrightarrow \mathsf{MSLat}$.)

The internal language of a coherent category is enriched further with $\bot$ and $\vee$.

Exact categories and pretoposes

Let **C** be a poset-enriched regular category, and $X \in \mathbf{C}$.

- A congruence on X is a relation $R \subseteq X^2$ that is transitive and satisfies $x \leq y \vdash R(x, y)$.
- A quotient of X by R is a surjection $X \twoheadrightarrow X/R$ whose lax kernel is R.

C is exact if it is regular and every congruence has a quotient.

Exact categories and pretoposes

Let **C** be a poset-enriched regular category, and $X \in \mathbf{C}$.

- A congruence on X is a relation $R \subseteq X^2$ that is transitive and satisfies $x \leq y \vdash R(x, y)$.
- A quotient of X by R is a surjection $X \twoheadrightarrow X/R$ whose lax kernel is R.

C is exact if it is regular and every congruence has a quotient.

To define a pretopos, we consider disjoint unions in **Pos**-coherent categories.

The disjoint union of A and B is an object $A + B$, equipped with embeddings $A \hookrightarrow A + B$ and $B \hookrightarrow A + B$, s.t. A and B cover $A + B$ and are incomparable:

$$\vdash A(x) \vee B(x), \quad A(x) \wedge B(y) \wedge (x \leq y) \vdash \bot, \quad A(x) \wedge B(y) \wedge (y \leq x) \vdash \bot.$$

Exact categories and pretoposes

Let **C** be a poset-enriched regular category, and $X \in \mathbf{C}$.

- A congruence on X is a relation $R \subseteq X^2$ that is transitive and satisfies $x \leq y \vdash R(x, y)$.
- A quotient of X by R is a surjection $X \twoheadrightarrow X/R$ whose lax kernel is R.

C is exact if it is regular and every congruence has a quotient.

To define a pretopos, we consider disjoint unions in **Pos**-coherent categories.

The disjoint union of A and B is an object $A + B$, equipped with embeddings $A \hookrightarrow A + B$ and $B \hookrightarrow A + B$, s.t. A and B cover $A + B$ and are incomparable:

$$\vdash A(x) \vee B(x), \qquad A(x) \wedge B(y) \wedge (x \leq y) \vdash \bot, \qquad A(x) \wedge B(y) \wedge (y \leq x) \vdash \bot.$$

A poset-enriched category is a pretopos if it is coherent, exact, and has disjoint unions.

Examples: **Pos** and **KOrd**, but not **Set** nor **KH**.

Projective covers and generators

Projective covers

Let **C** be a poset-enriched regular category.

- $X \in \mathbf{C}$ is projective if $\mathbf{C}(X, -)\colon \mathbf{C} \to \mathbf{Pos}$ preserves surjections.
- A full subcategory $\mathcal{P} \subseteq \mathbf{C}$ is a projective cover of **C** if every object in $\mathcal{P}$ is projective and each object of **C** is covered by an object in $\mathcal{P}$.

Projective covers

Let $\mathbf{C}$ be a poset-enriched regular category.

- $X \in \mathbf{C}$ is projective if $\mathbf{C}(X, -) \colon \mathbf{C} \to \mathbf{Pos}$ preserves surjections.
- A full subcategory $\mathcal{P} \subseteq \mathbf{C}$ is a projective cover of $\mathbf{C}$ if every object in $\mathcal{P}$ is projective and each object of $\mathbf{C}$ is covered by an object in $\mathcal{P}$.

The following is a variant, for poset-enriched regular categories with enough projectives, of Barr's embedding theorem for ordinary regular categories:

Theorem (Marquès, LR)

Let $\mathcal{P} \subseteq \mathbf{C}$ be a projective cover. The nerve $N_{\mathcal{P}} \colon \mathbf{C} \to [\mathcal{P}^{\mathrm{op}}, \mathbf{Pos}]$ is regular and fully faithful.

If $\mathbf{C}$ is exact, the essential image of $N_{\mathcal{P}}$ consists of the presheaves obtained as the quotient of a representable presheaf by a congruence that is covered by another representable presheaf.

Projective covers

By a result of Vitale, two ordinary exact categories are equivalent if they have equivalent projective covers. The previous thm yields a similar result for poset-enriched categories:

Corollary

Let $\mathbf{C}, \mathbf{D}$ *be poset-enriched exact categories with equivalent projective covers. Then* $\mathbf{C} \simeq \mathbf{D}$.

Projective covers

By a result of Vitale, two ordinary exact categories are equivalent if they have equivalent projective covers. The previous thm yields a similar result for poset-enriched categories:

Corollary

Let $\mathbf{C}, \mathbf{D}$ *be poset-enriched exact categories with equivalent projective covers. Then* $\mathbf{C} \simeq \mathbf{D}$.

- In **Pos**, a projective cover is given by sets (i.e., the order-discrete posets).
- In **KOrd**, the full subcategory $\{\beta S \mid S \text{ a set}\}$ is a projective cover.

The two examples above are both instances of the same phenomenon. Start with a projective *generator* — in this case, the terminal object — and close under copowers.

Generators

An object G of a poset-enriched regular category **C** is a discrete generator if it satisfies:

1. for every set S, the copower $S \cdot G$ exists in **C**;
2. for every $X \in \mathbf{C}$, the canonical arrow $|\mathbf{C}(G, X)| \cdot G \to X$ is a surjection.

There is a more standard notion of generator, defined in terms of tensors $P \cdot G$ rather than copowers. Every generator is a discrete generator, and the converse holds if **C** is exact.

Generators

An object G of a poset-enriched regular category $\mathbf{C}$ is a discrete generator if it satisfies:

1. for every set S, the copower $S \cdot G$ exists in $\mathbf{C}$;
2. for every $X \in \mathbf{C}$, the canonical arrow $|\mathbf{C}(G, X)| \cdot G \to X$ is a surjection.

There is a more standard notion of generator, defined in terms of tensors $P \cdot G$ rather than copowers. Every generator is a discrete generator, and the converse holds if $\mathbf{C}$ is exact.

Lemma

If G is a projective discrete generator, $\{S \cdot G \mid S \text{ a set}\}$ is a projective cover of $\mathbf{C}$.

Generators

An object G of a poset-enriched regular category $\mathbf{C}$ is a discrete generator if it satisfies:

1. for every set S, the copower $S \cdot G$ exists in $\mathbf{C}$;
2. for every $X \in \mathbf{C}$, the canonical arrow $|\mathbf{C}(G, X)| \cdot G \to X$ is a surjection.

There is a more standard notion of generator, defined in terms of tensors $P \cdot G$ rather than copowers. Every generator is a discrete generator, and the converse holds if $\mathbf{C}$ is exact.

Lemma

If G is a projective discrete generator, $\{S \cdot G \mid S \text{ a set}\}$ is a projective cover of $\mathbf{C}$.

Theorem (Cf. also Kurz-Velebil 2017)

If $\mathbf{C}$ is exact and has a projective (discrete) generator G, then it is equivalent to $\mathrm{EM}(\mathbf{T})$, where $\mathbf{T}$ is the monad on $\mathbf{Pos}$ induced by the adjunction

$$\mathbf{Pos} \underset{- \cdot G}{\overset{\mathbf{C}(G,-)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\top}}}} \mathbf{C} \, .$$

A characterisation of **KOrd**

Order-filtrality

An object X of a **Pos**-coherent category is order-filtral if the monotone map

$$\mathrm{Sub}^{\uparrow}(X) \to \mathcal{F}(\mathrm{Sub}^{\uparrow}_{\neg}(X)), \quad U \mapsto \{V \in \mathrm{Sub}^{\uparrow}_{\neg}(L) \mid U \subseteq V\}$$

is an isomorphism.

- The order-filtral objects in **Pos** are the finite posets.
- The order-filtral objects in **KOrd** are the Priestley spaces.

Order-filtrality

An object X of a **Pos**-coherent category is order-filtral if the monotone map

$$\mathrm{Sub}^{\uparrow}(X) \to \mathcal{F}(\mathrm{Sub}^{\uparrow}_{\neg}(X)), \quad U \mapsto \{V \in \mathrm{Sub}^{\uparrow}_{\neg}(L) \mid U \subseteq V\}$$

is an isomorphism.

- The order-filtral objects in **Pos** are the finite posets.
- The order-filtral objects in **KOrd** are the Priestley spaces.

Theorem (Marquès & LR)

Up to equivalence, **KOrd** *is the unique non-trivial* **Pos***-pretopos such that:*

1. *the terminal object* $\mathbf{1}$ *is a discrete generator;*
2. *every object is covered by an order-filtral object.*

A sketch of the proof

Let **C** be a non-trivial **Pos**-pretopos such that **1** is a discrete generator, and each object of **C** is covered by an order-filtral object.

A sketch of the proof

Let **C** be a non-trivial **Pos**-pretopos such that **1** is a discrete generator, and each object of **C** is covered by an order-filtral object.

- **1** is projective, hence $\{S \cdot \mathbf{1} \mid S \text{ a set}\}$ is a projective cover of **C**.

A sketch of the proof

Let **C** be a non-trivial **Pos**-pretopos such that **1** is a discrete generator, and each object of **C** is covered by an order-filtral object.

- **1** is projective, hence $\{S \cdot \mathbf{1} \mid S \text{ a set}\}$ is a projective cover of **C**.
- The (enriched) presheaf $\mathrm{Sub}_{\neg}^{\uparrow} \colon \mathbf{C}^{\mathrm{op}} \to \mathsf{DL}$ is represented by $2 \cdot \mathbf{1}$, where $2 = \{0 < 1\}$.

A sketch of the proof

Let **C** be a non-trivial **Pos**-pretopos such that **1** is a discrete generator, and each object of **C** is covered by an order-filtral object.

- **1** is projective, hence $\{S \cdot \mathbf{1} \mid S \text{ a set}\}$ is a projective cover of **C**.
- The (enriched) presheaf $\mathrm{Sub}^{\uparrow}_{\neg} \colon \mathbf{C}^{\mathrm{op}} \to \mathsf{DL}$ is represented by $2 \cdot \mathbf{1}$, where $2 = \{0 < 1\}$.
- Composing $\mathrm{Sub}^{\uparrow}_{\neg}$ with $\mathsf{DL}^{\mathrm{op}} \to \mathbf{KOrd}$, we get $F \colon \mathbf{C} \to \mathbf{KOrd}$ that sends $S \cdot \mathbf{1}$ to βS.

A sketch of the proof

Let **C** be a non-trivial **Pos**-pretopos such that **1** is a discrete generator, and each object of **C** is covered by an order-filtral object.

- **1** is projective, hence $\{S \cdot \mathbf{1} \mid S \text{ a set}\}$ is a projective cover of **C**.
- The (enriched) presheaf $\mathrm{Sub}^{\uparrow}_{\neg} \colon \mathbf{C}^{\mathrm{op}} \to \mathrm{DL}$ is represented by $2 \cdot \mathbf{1}$, where $2 = \{0 < 1\}$.
- Composing $\mathrm{Sub}^{\uparrow}_{\neg}$ with $\mathrm{DL}^{\mathrm{op}} \to \mathbf{KOrd}$, we get $F \colon \mathbf{C} \to \mathbf{KOrd}$ that sends $S \cdot \mathbf{1}$ to βS.
- $S \cdot \mathbf{1}$ covered by a filtral object $\Rightarrow S \cdot \mathbf{1}$ *compact* and *separated* $\Rightarrow \mathbf{C}(\mathbf{1}, S \cdot \mathbf{1}) \cong \beta S$.

A sketch of the proof

Let $\mathbf{C}$ be a non-trivial **Pos**-pretopos such that $\mathbf{1}$ is a discrete generator, and each object of $\mathbf{C}$ is covered by an order-filtral object.

- $\mathbf{1}$ is projective, hence $\{S \cdot \mathbf{1} \mid S \text{ a set}\}$ is a projective cover of $\mathbf{C}$.
- The (enriched) presheaf $\mathrm{Sub}^{\uparrow}_{\neg} \colon \mathbf{C}^{\mathrm{op}} \to \mathsf{DL}$ is represented by $2 \cdot \mathbf{1}$, where $2 = \{0 < 1\}$.
- Composing $\mathrm{Sub}^{\uparrow}_{\neg}$ with $\mathsf{DL}^{\mathrm{op}} \to \mathbf{KOrd}$, we get $F \colon \mathbf{C} \to \mathbf{KOrd}$ that sends $S \cdot \mathbf{1}$ to βS.
- $S \cdot \mathbf{1}$ covered by a filtral object $\Rightarrow$ $S \cdot \mathbf{1}$ *compact* and *separated* $\Rightarrow$ $\mathbf{C}(\mathbf{1}, S \cdot \mathbf{1}) \cong \beta S$.
- $\mathbf{C}(S \cdot \mathbf{1}, T \cdot \mathbf{1}) \cong \mathbf{KOrd}(\beta S, \beta T) \Rightarrow F \colon \{S \cdot \mathbf{1} \mid S \text{ a set}\} \simeq \{\beta S \mid S \text{ a set}\}$.

A sketch of the proof

Let **C** be a non-trivial **Pos**-pretopos such that **1** is a discrete generator, and each object of **C** is covered by an order-filtral object.

- **1** is projective, hence $\{S \cdot \mathbf{1} \mid S \text{ a set}\}$ is a projective cover of **C**.
- The (enriched) presheaf $\mathrm{Sub}^{\uparrow}_{\neg} \colon \mathbf{C}^{\mathrm{op}} \to \mathrm{DL}$ is represented by $2 \cdot \mathbf{1}$, where $2 = \{0 < 1\}$.
- Composing $\mathrm{Sub}^{\uparrow}_{\neg}$ with $\mathrm{DL}^{\mathrm{op}} \to \mathbf{KOrd}$, we get $F \colon \mathbf{C} \to \mathbf{KOrd}$ that sends $S \cdot \mathbf{1}$ to βS.
- $S \cdot \mathbf{1}$ covered by a filtral object $\Rightarrow$ $S \cdot \mathbf{1}$ *compact* and *separated* $\Rightarrow$ $\mathbf{C}(\mathbf{1}, S \cdot \mathbf{1}) \cong \beta S$.
- $\mathbf{C}(S \cdot \mathbf{1}, T \cdot \mathbf{1}) \cong \mathbf{KOrd}(\beta S, \beta T) \Rightarrow F \colon \{S \cdot \mathbf{1} \mid S \text{ a set}\} \simeq \{\beta S \mid S \text{ a set}\}$.
- Since **C** and **KOrd** have equivalent projective covers, they are equivalent.

Epilogue: future directions

- Beyond **KH** and **KOrd**: *Weaken* these axiomatisations to capture categories that "behave like compact ordered spaces" (e.g. sheaves of compact ordered spaces).
- Extensivity for **Pos**-categories as a two-dimensional exactness condition (cf. Bourke and Garner, 2014)?
- Regular/coherent **Pos**-categories correspond to regular/coherent monotone theories: for each sort X there is a binary relation $\leq_X : X \times X$ such that T proves that
  1. $\leq_X$ is a partial order;
  2. every function symbol is monotone with respect to these orders.
- Strong Conceptual completeness for monotone coherent theories (generalising the work of Makkai and Lurie to the poset-enriched setting)?
- Explore the use of Conceptual completeness to characterise pretoposes or coherent categories (up to Morita equivalence).

- Beyond **KH** and **KOrd**: *Weaken* these axiomatisations to capture categories that "behave like compact ordered spaces" (e.g. sheaves of compact ordered spaces).

- Extensivity for **Pos**-categories as a two-dimensional exactness condition (cf. Bourke and Garner, 2014)?

- Regular/coherent **Pos**-categories correspond to regular/coherent monotone theories: for each sort X there is a binary relation $\leq_X : X \times X$ such that T proves that
  1. $\leq_X$ is a partial order;
  2. every function symbol is monotone with respect to these orders.

- Strong Conceptual completeness for monotone coherent theories (generalising the work of Makkai and Lurie to the poset-enriched setting)?

- Explore the use of Conceptual completeness to characterise pretoposes or coherent categories (up to Morita equivalence).

Thank you!

Pos-pretoposes and compact ordered spaces

Luca Reggio

joint work with

Jérémie Marquès

Dipartimento di Matematica "Federigo Enriques", Università di Milano, Italy

International Category Theory Conference
Brno, 15 July, 2025

References I

Ordered algebras

S. L. Bloom (1976)
Varieties of ordered algebras
Journal of Computer and System Sciences.

S. L. Bloom & J. B. Wright (1983)
P-varieties — A signature independent characterization of varieties of ordered algebras
Journal of Pure and Applied Algebra.

A. Kurz & J. Velebil (2017)
Quasivarieties and varieties of ordered algebras: regularity and exactness
Math. Struct. in Comp. Science.

J. Adámek and J. Rosický (2023)
Varieties of ordered algebras as categories
Algebra Universalis.

J. Adámek (2025)
Categories which are varieties of classical or ordered algebras
Theory and Applications of Categories.

Regularity and exactness

R. Street (1982)
Two-dimensional sheaf theory
Journal of Pure and Applied Algebra.

J. Bourke & R. Garner (2014)
Two-dimensional regularity and exactness
Journal of Pure and Applied Algebra.

V. Aravantinos-Sotiropoulos (2022)
The exact completion for regular categories enriched in posets
Journal of Pure and Applied Algebra.

References II

Enriched internal languages

S. Lack & G. Tendas (2020)
Enriched regular theories
Journal of Pure and Applied Algebra.

J. Rosický & G. Tendas (2023)
Towards enriched universal algebra
arXiv:2310.11972.

J. Rosický & G. Tendas (2025)
Enriched concepts of regular logic
To appear in *Journal of Symbolic Logic.*

Projective covers

E. M. Vitale (1994)
On the characterization of monadic categories over SET
Cahiers Topologie Géom. Différentielle Catég.