An intrinsic approach to kernels in general categories

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Kernels and cokernels in a pointed category

Recall that a category is said to be **pointed** if it has an object $\mathbf{0}$ which is both initial and terminal.

In a pointed category, the kernel of a morphism $f\colon A \to B$ can be defined as the pullback of the unique morphism $\mathbf{0} \to B$ along f.

$$\begin{array}{ccc} \ker(f) & \longrightarrow & \mathbf{0} \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

Similarly, the cokernel of $g\colon A \to B$ can be defined as the pushout

$$\begin{array}{ccc} A & \longrightarrow\!\!\!\!\!\rightarrow & \mathbf{0} \\ {\scriptstyle g}\downarrow & & \downarrow \\ B & \longrightarrow\!\!\!\!\!\rightarrow & \ulcorner\ \operatorname{coker}(g)\,. \end{array}$$

The usefulness of kernels

(Co)kernel calculus works better in some categories than others.

For instance, it is widely used when working with groups, but not as useful when working with monoids.

Goal

Modify the definition so that it works well in any nice category.

First, given a coreflective subcategory

$$\mathcal{Z} \underset{d}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathcal{C}$$

and a morphism $f\colon A \to B$ in $\mathcal{C}$, one can consider the pullback

$$\begin{array}{ccc} K & \longrightarrow & d(B) \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \varepsilon_B} \\ A & \xrightarrow[f]{} & B. \end{array}$$

When $\mathcal{Z} = \{\mathbf{0}\}$, then this is just the kernel of f in the usual sense.

Examples of choices for $\mathcal{Z}$

Example

Take $\mathcal{C}$ to be a pointed category and $\mathcal{Z} = \{\mathbf{0}\}$ the zero subcategory. Then one gets kernels in the usual sense.

Example

Take $\mathcal{C}$ to be any category and $\mathcal{Z} = \mathcal{C}$, so

$$\mathcal{Z} \underset{d}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathcal{C}$$

becomes the identity adjunction with $d(B) = B$.

Example

Take $\mathcal{C} = \mathsf{Gpd}$ to be the category of groupoids and $\mathcal{Z} = \mathsf{Set}$, the subcategory of discrete groupoids. Then the inclusion of the largest discrete subgroupoid $d(B) \to B$ is the counit of the adjunction

$$\mathsf{Set} \underset{d}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathsf{Gpd}\,.$$

Kernels with respect to $\mathcal{Z}$

In the pullback

$$\begin{array}{ccc} K & \longrightarrow & d(B) \\ \big\downarrow & \lrcorner & \big\downarrow \varepsilon_B \\ A & \xrightarrow[f]{} & B, \end{array}$$

we interpret the span $d(B) \leftarrow K \rightarrow A$ as the kernel of $f\colon A \rightarrow B$.

Note that if $d(B) = \mathbf{1}$ is the terminal object, then the left leg $d(B) \leftarrow K$ of the span carries no information.

We need to choose a coreflective subcategory $\mathcal{Z}$ if we want to take kernels, but which one to choose?

Taking $\mathcal{Z} = \{\mathbf{0}\}$ when $\mathcal{C}$ is the category of monoids would not give a very useful notion of kernel, so we would like to have a notion a choice of $\mathcal{Z}$ being **good** that would exclude this example.

Short exact sequences by taking kernels

A short exact sequence in a pointed category can be defined as a square of the form

$$\begin{array}{ccc} A & \twoheadrightarrow & \mathbf{0} \\ {\scriptstyle f}\downarrow & & \downarrow \\ B & \xrightarrow[g]{} & C, \end{array}$$

which is both a pullback and a pushout.

In the category of groups, any surjective morphism will be the cokernel of its kernel, meaning any pullback square

$$\begin{array}{ccc} \ker(f) & \longrightarrow & \mathbf{0} \\ \downarrow & & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

with f surjective will also be a pushout square. This would not hold in the category of monoids.

Good choice = nice epi is the cokernel of its kernel

We start by fixing a pullback stable class $\mathcal{E}$ of nice epimorphisms in the category $\mathcal{C}$. For instance $\mathcal{E}$ can be split epimorphisms or strong epimorphisms, if they are pullback stable.

A choice of a coreflective subcategory $\mathcal{Z}$ is **good for kernels**, if for any morphism $f\colon A \to B$ in $\mathcal{E}$, the pullback square

$$\begin{array}{ccc} K & \longrightarrow & d(B) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \varepsilon_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

is also a pushout square.

One interpretation of this would be that we are viewing the span-cospan correspondence through pullbacks and pushouts as fundamental and the counit $d(B) \to B$ provides us with a method of turning morphisms $A \to B$ into cospans $A \to B \leftarrow d(B)$.

"Intrinisic kernels" $=$ take $\mathcal{Z}$ to be good and minimal

Any category $\mathcal{C}$ admits the largest good coreflective subcategory $\mathcal{Z} = \mathcal{C}$.

Indeed, then the counit $d(B) \to B$ is always the identity morphism and for any morphism f the square

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle \mathrm{id}_A}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} \\ A & \xrightarrow[f]{} & B \end{array}$$

is both a pullback and a pushout.

We would like to minimize $\mathcal{Z}$, so ideally we take $\mathcal{Z}$ to be the smallest good coreflective subcategory, if it exists.

If $\mathcal{C}$ is regular and protomodular, we can take $\mathcal{Z} = \{\mathbf{0}\}$ as the singleton subcategory consisting of the initial object.

Monic counit

One may wish to put additional conditions on $\mathcal{Z}$ to improve the kernel-cokernel calculus in various ways to suit the needs of their situation.

For instance, one can ask counit $\varepsilon_B \colon d(B) \to B$ to belong to a class $\mathcal{M}$ of monics, so that the $K \to A$ component of the kernel becomes a subobject.

In that case, the $K \to A$ part of the kernel will satisfy the traditional universal property, in that it is universal with respect to making the composite

$$K \longrightarrow A \xrightarrow{f} B$$

trivial in the sense of factoring through some object of $\mathcal{Z}$.

Monic replacement

If the classes $\mathcal{E}$ and $\mathcal{M}$ form an orthogonal factorization system, then factoring the counit

$$\begin{array}{ccc} d(A) & \xrightarrow{\quad\varepsilon_A\quad} & A \\ & \searrow\!\!\!\!\searrow \qquad \nearrow_{\varepsilon'_A} & \\ & d'(A) & \end{array}$$

of the coreflection of $\mathcal{Z}$ yields a new coreflective subcategory

$$\mathcal{Z}' \underset{d'}{\overset{\hookrightarrow}{\underset{\longleftarrow}{\perp}}} \mathcal{C}\,,$$

with the counit now having components $\varepsilon'_A \colon d'(A) \to A$ in the class $\mathcal{M}$.

If $\mathcal{Z}$ was good for kernels, then $\mathcal{Z}'$ is good as well.

Examples of good coreflective subcategories

Example

For $\mathcal{C} = \mathsf{Set}$, we can't do better than $\mathcal{Z} = \mathsf{Set}$.

Example

For $\mathcal{C} = \mathsf{Gpd}$, the category of (small) groupoids, $\mathcal{Z} = \mathsf{Set}$ is good.

Example

For $\mathcal{C} = \mathsf{InvMon}$, the category of inverse monoids, $\mathcal{Z} = \{\mathbf{0}\}$ is not good, but $\mathcal{Z} = \mathsf{SemLat}$, the subcategory of semilattices with a top element, is good.

Example

For $\mathcal{C} = \mathsf{Top}$, the subcategory $\mathcal{Z} = \mathsf{Set}$ of discrete spaces is good.

Example

In a regular protomodular category, $\mathcal{Z} = \{\mathbf{0}\}$ is good.

Thank you!