Ord-Mal'tsev categories

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Aim

[Clementino, Rodelo, *Enriched aspects of calculus of relations and* **2***-permutability*, J. Algebra and its Appl., 2025, 2650233]

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  - Show that every Mal'tsev category is an **Ord**-Mal'tsev category, for any **Ord**-enrichment
  - Give examples of **Ord**-Mal'tsev categories which are not Mal'tsev categories

Mal’tsev varieties

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- **Def**: $\mathcal{V}$ *Mal'tsev variety* if its theory admits ternary operation p sth

$$\textbf{(M)} \quad \begin{cases} p(x,z,z) = x, \\ p(x,x,z) = z; \end{cases}$$

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- **Def**: $\mathbf{C}$ *Mal'tsev category*: every ordinary relation $X \xleftarrow{d_1} D \xrightarrow{d_2} Y$ is **difunctional**

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xDy or $(x, y) \in_W D$ means that

(x, y seen as "generalised elements")

$$\begin{tikzcd} & X & \\ W \arrow[ur, "x"] \arrow[dr, "y"'] \arrow[rr, dotted, "\exists!"] & & D \arrow[ul, "d_1"'] \arrow[dl, "d_2"] \\ & Y & \end{tikzcd}$$

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- **Exs**: - Mal'tsev varieties
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  - protomodular ($\Rightarrow$ semi-abelian) categories are Mal'tsev categories
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  - (ii) ∀ ordinary equiv relations $R, S \colon X \relbar\joinrel\mapstochar\joinrel\rightarrow X$, **(CP)** $RS \cong SR$
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Reflexive: $\Delta_X \subseteq R$ Symmetric: $R^{\circ} \subseteq R$ Transitive: $RR \subseteq R$

$$\begin{tikzcd}
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Well-known characterisations of Mal'tsev categories

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lex + pb-stable (regular epi,mono) f.s.
possible to compose ordinary relations; comp. is associative

- **Thm 1**: **C** regular category. TFAE:

  (i) ∀ ordinary equiv relations $R, S\colon X \rightarrow\!\!\!\!\!+\, X$, $RS\colon X \rightarrow\!\!\!\!\!+\, X$ ordinary equiv relation

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- **Rem**: Mal'tsev categories have been widely studied; they **capture many group-like features**: homological diagram lemmas, central extensions, commutator theory, etc.

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Ord-enriched cats, ff-(mono)morphisms and so-morphisms

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Quasi-regular **Ord**-categories

[Bourke, Garner, *Two-dimensional regularity and exactness*, J. Pure Appl. Algebra **218** (2014) 1346–1371]

- **Def**: An **Ord**-category $\mathbb{C}$ is called quasi-regular when:

  (R1) $\mathbb{C}$ has finite (weighted) limits

  (R2) $\mathbb{C}$ admits an (so-morphism, ff-monomorphism) factorisation system

  (R3) so-morphisms are stable under **2**-pullbacks in $\mathbb{C}$

- **Rem**: - $\mathbb{C}$ is called regular when (R1)-(R3) + (R4) every so-morphism is bicoinserter
  - Is (R4) a consequence of (R1)-(R3) as for **Pos**-categories?
  - Only need (R1)-(R3) to develop convenient calculus of relations for **Ord**-cats

- **Exs**: The following are (quasi-)regular **Ord**-categories:
  - **Ord** is regular
  - Ordinary regular categories with discrete (pre)order are regular
  - Quasivarieties of (pre)ordered algebras are regular
  - The category of internal (pre)orders in an ordinary regular cat is quasi-regular

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$$\big(x' \preccurlyeq x,\ y \preccurlyeq y' \ \Rightarrow\ fx' \preccurlyeq fx \preccurlyeq gy \preccurlyeq gy' \ \Rightarrow\ (x', y') \in_A f/g\big)$$

$$\begin{tikzcd}
A \arrow[rrd, bend left, "y"] \arrow[rdd, bend right, "x"'] \arrow[rd, dotted, "\langle x,y \rangle"] & & \\
& f/g \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[rd, phantom, "\preccurlyeq"] & Y \arrow[d, "g"] \\
& X \arrow[r, "f"'] & Z
\end{tikzcd}$$

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  [The proof follows that of Thm 1, with the adapted calculus of relations for quasi-regular **Ord**-categories]

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Given $\langle d_1, d_2\rangle \colon D \rightarrowtail X \times Y$ ideal, $f, h, h' \colon A \to X$ and $g, g', k \colon A \to Y$ sth

$$\begin{array}{ccc} f & D & g \\ & & \rotatebox{-90}{$\preccurlyeq$} \\ h & D & g' \\ \rotatebox{-90}{$\preccurlyeq$} & & \\ h' & D & k \end{array}$$

$$\begin{tikzcd}
& X & \\
A \arrow[ur, bend left, "0"] \arrow[ur, "f"'] \arrow[ur, phantom, "\preccurlyeq"] \arrow[rr, dotted, bend left=15, "\langle 0,g\rangle"] \arrow[rr, phantom, "\preccurlyeq"] \arrow[rr, dotted, bend right=15, "\langle f,g\rangle"'] \arrow[dr, "g"'] & & D \arrow[ul, "d_1"'] \arrow[dl, "d_2"] \\
& Y &
\end{tikzcd}$$

- $(f, g) \in_A D$ and $0 \preccurlyeq f \Rightarrow (0, g) \in_A D$
- $\langle 0, g\rangle \preccurlyeq \langle f, g\rangle \;\Rightarrow\; \forall a \in A,\; \exists(!)(\gamma_a, \delta_a) \in D$:

$$\begin{cases} 0 + \gamma_a = f(a) \\ g(a) + \delta_a = g(a) \end{cases} \underset{\text{(lc)}}{\Rightarrow} \begin{cases} \gamma_a = f(a) \\ \delta_a = 0 \end{cases} \Rightarrow \boxed{(f(a), 0) \in D},\ \forall a \in A$$

- $(h', k) \in_A D$ and $0 \preccurlyeq h' \;\Rightarrow\; (0, k) \in_A D \;\Rightarrow\; (0, k(a)) \in D,\ \forall a \in A$

$\therefore\; (f(a), k(a)) \in D, \forall a \in A \;\Rightarrow\; \boxed{f\ D\ k}$

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Variations: $f \preccurlyeq g$ iff $\cdots$ $f(x) = g(x) + y_x$; $\mathbb{M}$**on$_{rc}$** monoids with right cancellation

Examples of **Ord**-Mal'tsev categories II

- **Ex 3**: **GMon** - category of *gregarious monoids*, i.e monoids $(X, +, 0)$ sth $\forall x \in X, \exists u_x, v_x \in X : u_x + x + v_x = 0$

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- $(f,g) \in_A D \Rightarrow (f(a),g(a)) \in D$, $\forall a \in A$
- $\exists u_a, v_a \in A : u_a + a + v_a = 0$
- $(f,g) \in_A D$ and $0 \preccurlyeq f \Rightarrow (0,g) \in_A D \Rightarrow$

  $\Rightarrow (0,g(u_a)),(0,g(v_a)) \in D$, $\forall a \in A$
- $(0,g(u_a)) + (f(a),g(a)) + (0,g(v_a)) = \boxed{(f(a),0) \in D}$, $\forall a \in D$
- Similar proof as before $\rightsquigarrow$ $\boxed{f \ D \ k}$

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$$\begin{tikzcd}[row sep=small]
f & D & g \arrow[d, phantom, sloped, "\preccurlyeq"] \\
h \arrow[d, phantom, sloped, "\preccurlyeq"] & D & g' \\
h' & D & k
\end{tikzcd}$$

$$\begin{tikzcd}
& (X, PX) & \\
(A, P_A) \arrow[ur, bend left, "0"] \arrow[ur, "f"'] \arrow[ur, phantom, shift left=2, "\rotatebox{-45}{$\preccurlyeq$}"] \arrow[rr, dotted, bend left=10, "\langle 0,g\rangle"] \arrow[rr, dotted, bend right=10, "\langle f,g\rangle"'] \arrow[rr, phantom, sloped, "\rotatebox{-90}{$\preccurlyeq$}"] \arrow[dr, "g"'] & & (D, P_D) \arrow[ul, "d_1"'] \arrow[dl, "d_2"] \\
& (Y, P_Y) &
\end{tikzcd}$$

- D ideal $\Rightarrow D \cong X \times Y$, as a group
- Group homomorphism $\langle f, k\rangle \colon A \to D \cong X \times Y$ always exists
- Is $\langle f, k\rangle \colon (A, P_A) \to (D, P_D)$ monotone?
- $\langle 0, g\rangle \preccurlyeq \langle f, g\rangle \Rightarrow \forall a \in P_A, (0, g(a)) \leqslant (f(a), g(a))$

$$\Rightarrow \boxed{(f(a), 0) \in P_D}, \forall a \in P_A$$

- $(0, k) \in_{(A, P_A)} D \Rightarrow (0, k(a)) \in P_D, \forall a \in P_A$

$\therefore\ (f(a), k(a)) \in P_D, \forall a \in P_A \ \Rightarrow \boxed{f \;\; D \;\; k}$

Examples of **Ord**-Mal'tsev categories III

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OrdGrp is **not** a Mal'tsev category ([Clementino, Martins-Ferreira, Montoli, *On the categorical behaviour of preordered groups*, J. Pure Appl. Algebra **223** (2019) 4226–4245])

Define $f \preccurlyeq g$ iff $\forall x \in P_X,\ f(x) \leqslant g(x)$ ($\Rightarrow$ $0 \preccurlyeq f$); denote $\mathbb{O}$**rdGrp**

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