

Ord-Mal'tsev categories

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17th July 2025

[Clementino, Rodelo, *Enriched aspects of calculus of relations and 2-permutability*, J. Algebra and its Appl., 2025, 2650233]

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- **Aim:** Study an **Ord**-enriched version of the (1-dimensional) Mal'tsev property and its well-known characterisations through properties on (internal) relations; we call them here **ordinary relations**

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 - Show that every Mal'tsev category is an **Ord-Mal'tsev category**, for any **Ord**-enrichment
 - Give examples of **Ord-Mal'tsev categories** which are not Mal'tsev categories

Mal'tsev varieties

[Smith, *Mal'cev varieties*, Lecture Notes in Math., vol. 554, Springer, 1976]

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- **Def:** $\mathcal{V}$ *Mal'tsev variety* if its theory admits ternary operation p sth

$$(M) \quad \begin{cases} p(x, z, z) = x, \\ p(x, x, z) = z; \end{cases}$$

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$\rightsquigarrow$ **Mal'tsev category**

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• **Def:** **C** *Mal'tsev category*: every ordinary relation $X \xleftarrow{d_1} D \xrightarrow{d_2} Y$ is **difunctional**

i.e.
$$\mathbf{C}(W, X) \xleftarrow{\mathbf{C}(W, d_1)} \mathbf{C}(W, D) \xrightarrow{\mathbf{C}(W, d_2)} \mathbf{C}(W, Y)$$

in **Set** is difunctional, for every object W of **C**

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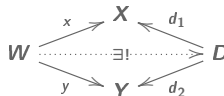
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$x D y$ or $(x, y) \in_W D$ means that
(x, y seen as “generalised elements”)



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- **Exs:** - Mal'tsev varieties
- $(\mathbf{Topos})^{\text{op}}$, $(\mathbf{Set}_*)^{\text{op}}$ and $\mathbf{TopGrp}$ are non-varietal Mal'tsev categories
- protomodular ($\Rightarrow$ semi-abelian) categories are Mal'tsev categories
- $\mathbf{C}$ Mal'tsev category $\Rightarrow \mathbf{C}/X, X/\mathbf{C}$ are Mal'tsev categories ($\forall X$)

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• **Thm 1:** **C** regular category. TFAE:

(i) $\forall$ ordinary equiv relations $R, S: X \rightrightarrows X$, $RS: X \rightrightarrows X$ ordinary equiv relation

(ii) $\forall$ ordinary equiv relations $R, S: X \rightrightarrows X$, **(CP)** $RS \cong SR$

(iii) $\forall$ ordinary **effective** equiv relations $R, S: X \rightrightarrows X$, **(CP)** $RS \cong SR$

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lex + pb-stable (regular epi,mono) f.s.

possible to compose ordinary relations; comp. is associative

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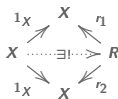
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• **Rem:** Mal'tsev categories have been widely studied; they **capture many group-like features**: homological diagram lemmas, central extensions, commutator theory, etc.

Well-known characterisations of Mal'tsev categories

[Carboni, Lambek, Pedicchio, *Diagram chasing in Mal'cev categories*, J. Pure Appl. Algebra **69** (1991) 271–284]

[Carboni, Pedicchio, Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, Am. Math. Soc. . . . , 1992, 97–109]

[Carboni, Kelly, Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Struct. **14** (1993) 385–421]

• **Thm 1:** **C** regular category. TFAE:

(i) $\forall$ ordinary equiv relations $R, S: X \rightrightarrows X$, $RS: X \rightrightarrows X$ ordinary equiv relation

(ii) $\forall$ ordinary equiv relations $R, S: X \rightrightarrows X$, **(CP)** $RS \cong SR$

(iii) $\forall$ ordinary **effective** equiv relations $R, S: X \rightrightarrows X$, **(CP)** $RS \cong SR$

(iv) $\forall$ ordinary relation $D: X \rightrightarrows Y$ is **difunctional**, i.e. $DD^\circ D \cong D$

(v) $\forall$ ordinary reflexive relation $R: X \rightrightarrows X$ is equivalence relation

(vi) $\forall$ ordinary reflexive relation $R: X \rightrightarrows X$ is transitive

(vii) $\forall$ ordinary reflexive relation $R: X \rightrightarrows X$ is symmetric

Reflexive: $\Delta_X \subseteq R$

Symmetric: $R^\circ \subseteq R$

Transitive: $RR \subseteq R$

• **Rem:** Mal'tsev categories have been widely studied; they **capture many group-like features**: homological diagram lemmas, central extensions, commutator theory, etc.

Ord-enriched cats, ff-(mono)morphisms and so-morphisms

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[Aravatinos-Sotiropoulos, *The exact completion for regular categories enriched in posets*, JPAA 226(7) (2022) 106885]

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- **Exs:** The following are (quasi-)regular **Ord**-categories:
 - **Ord** is regular
 - Ordinary regular categories with discrete (pre)order are regular
 - Quasivarieties of (pre)ordered algebras are regular
 - The category of internal (pre)orders in an ordinary regular cat is quasi-regular

Relations in **Ord**-categories

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- **Ex:** - Any **comma object** in $\mathbb{C}$ gives a relation

$$X \xleftarrow{\pi_1} f/g \xrightarrow{\pi_2} Y$$

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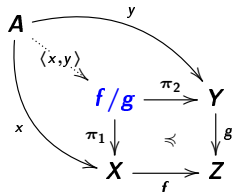
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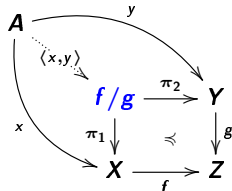
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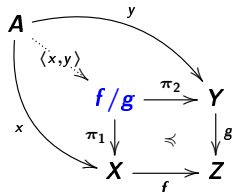
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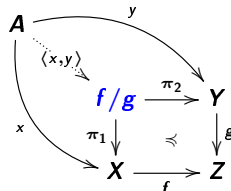
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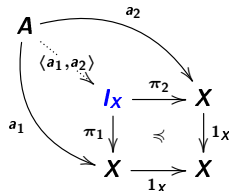
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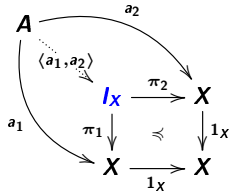
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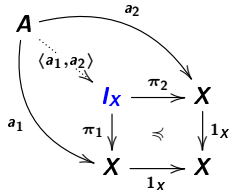
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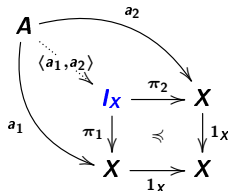
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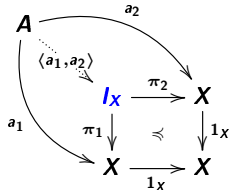
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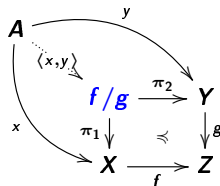
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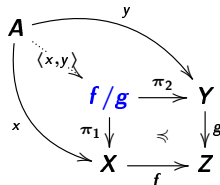
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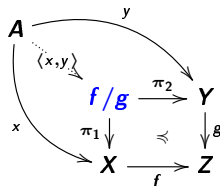


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THIS IS THE SETTING!

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 & & \wr \\
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$$\text{(OrdD)} \quad \text{iff} \quad D \cong DD^*D \cong DD^\circ D \quad \left\{ \begin{array}{l} D \text{ ideal} \not\Rightarrow D^\circ \text{ ideal} \\ D^* \text{ smallest ideal s.t. } D^\circ \subseteq D^* \\ D \text{ ideal} \Rightarrow DD^\circ D \text{ ideal} \\ D \cong DD^\circ D \text{ is 1-dimensional difunctionality} \end{array} \right.$$

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[The proof follows that of Thm 1, with the adapted calculus of relations for quasi-regular **Ord**-categories]

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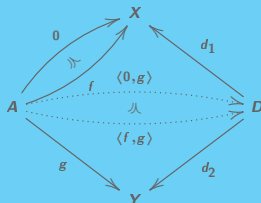
Note that: - y_x is uniquely determined because of left cancellation

- $0 \preceq f$, for any $f : X \rightarrow Y$ (each $y_x = f(x)$)

Mon_{lc} is an **Ord**-Mal'tsev category

Given $\langle d_1, d_2 \rangle : D \rightarrow X \times Y$ ideal, $f, h, h' : A \rightarrow X$ and $g, g', k : A \rightarrow Y$ sth

$$\begin{array}{ccc} f & D & g \\ \wr & & \wr \\ h & D & g' \\ \wr & & \wr \\ h' & D & k \end{array}$$



• $(f, g) \in_A D$ and $0 \preccurlyeq f \Rightarrow (0, g) \in_A D$

• $\langle 0, g \rangle \preccurlyeq \langle f, g \rangle \Rightarrow \forall a \in A, \exists (!)(\gamma_a, \delta_a) \in D :$

$$\begin{cases} 0 + \gamma_a = f(a) \\ g(a) + \delta_a = g(a) \end{cases} \Rightarrow \begin{cases} \gamma_a = f(a) \\ \delta_a = 0 \end{cases} \Rightarrow \boxed{(f(a), 0) \in D}, \forall a \in A$$

• $(h', k) \in_A D$ and $0 \preccurlyeq h' \Rightarrow (0, k) \in_A D \Rightarrow (0, k(a)) \in D, \forall a \in A$

$\therefore (f(a), k(a)) \in D, \forall a \in A \Rightarrow \boxed{f D k}$

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Examples of **Ord**-Mal'tsev categories I

- **Ex 1:** Any **Ord**-enrichment of a Mal'tsev category is an **Ord**-Mal'tsev category
- **Ex 2:** **Mon**_{lc} - category of monoids with *left cancellation* ($a + b = a + c \Rightarrow b = c$)

Mon_{lc} is not a Mal'tsev category: $\leq$ in $\mathbb{N}_0$ is not difunctional

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Variations: $f \preceq g$ iff $\dots f(x) = g(x) + y_x$; **Mon**_{rc} monoids with right cancellation

Examples of Ord-Mal'tsev categories II

- Ex 3: **GMon** - category of *gregarious monoids*, i.e monoids $(X, +, 0)$ sth
 $\forall x \in X, \exists u_x, v_x \in X : u_x + x + v_x = 0$

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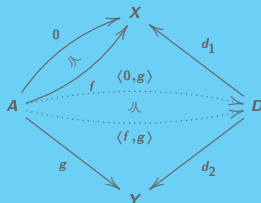
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$$\bullet (f, g) \in {}_A D \Rightarrow (f(a), g(a)) \in D, \forall a \in A$$

$$\bullet \exists u_a, v_a \in A : u_a + a + v_a = 0$$

$$\bullet (f, g) \in {}_A D \text{ and } 0 \preccurlyeq f \Rightarrow (0, g) \in {}_A D \Rightarrow \\ \Rightarrow (0, g(u_a)), (0, g(v_a)) \in D, \forall a \in A$$

$$\bullet (0, g(u_a)) + (f(a), g(a)) + (0, g(v_a)) = (f(a), 0) \in D, \forall a \in D$$

$$\bullet \text{ Similar proof as before } \rightsquigarrow f D k$$

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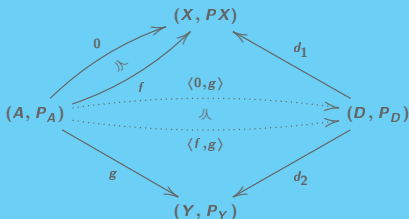
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OrdGrp is

- D ideal $\Rightarrow D \cong X \times Y$, as a group
- Group homomorphism $\langle f, k \rangle: A \rightarrow D \cong X \times Y$ always exists
- Is $\langle f, k \rangle: (A, P_A) \rightarrow (D, P_D)$ monotone?
- $\langle 0, g \rangle \preceq \langle f, g \rangle \Rightarrow \forall a \in P_A, (0, g(a)) \leq (f(a), g(a))$
 $\Rightarrow (f(a), 0) \in P_D, \forall a \in P_A$
- $(0, k) \in_{(A, P_A)} D \Rightarrow (0, k(a)) \in P_D, \forall a \in P_A$
- $\therefore (f(a), k(a)) \in P_D, \forall a \in P_A \Rightarrow \boxed{f D k}$

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Fixed unital and integral quantale
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