Grothendieck coverages on free monoids

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Overview

1. Toposes of monoid actions
2. Grothendieck coverages on monoids
3. Free monoids
4. Exploiting étendues
5. Constructing the lattice of coverages

A monoid acting on sets

Let M be a monoid. A *right M-set* is a set X equipped with a function $\alpha : X \times M \to X$ compatible with multiplication, meaning

$$\alpha(x, mn) = \alpha(\alpha(x, m), n), \quad \text{also written} \quad x \cdot mn = (x \cdot m) \cdot n.$$

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The class of M-sets forms a category, with morphisms the M-equivariant maps.

Lemma

The category of right M-sets is equivalent to the category $\mathrm{PSh}(M)$ of presheaves on M, where M is viewed as a one-object category.

In particular, it is a (Grothendieck) topos.

Continuous actions

For a topology τ on M, we can consider the subcategory $\mathrm{Cont}(M, \tau)$ of $\mathrm{PSh}(M)$ on those actions (X, α) such that α is continuous.

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Proposition [Rog]

$\mathrm{Cont}(M, \tau)$ is a (full, lex) coreflective subcategory of $\mathrm{PSh}(M)$.
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Subtoposes

Usually one is interested in (full, lex) *reflective* subcategories of toposes: subtoposes.

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These correspond to (Grothendieck) coverages on M. Hence we ask...

Q: What are the coverages on M and the corresponding subtoposes of $\mathrm{PSh}(M)$?

In particular, how precisely can we understand the lattice of coverages?

This is particularly interesting to contrast with subtoposes of *localic toposes*, which are extensively studied. (More on those later!)

A sieve is a right ideal

Definition

Let $\mathcal{C}$ be a small category and c an object of $\mathcal{C}$. A *sieve* over c is a collection of morphisms with codomain c closed under precomposition.

When $\mathcal{C}$ is a monoid M (viewed as a one-object category), all morphisms are composable, so a sieve is a right ideal in M: a collection $I \subseteq M$ such that $m \in I$ and $n \in M$ implies $mn \in I$.[1]

[1] Note that we allow the empty set and M as ideals!

A coverage is a collection of right ideals

Definition

A *coverage* J on a small category $\mathcal{C}$ consists of a collection $J(c)$ of sieves over each object c satisfying:

(M) The maximal sieve of all morphisms with codomain c belongs to $J(c)$.

(S) If $S \in J(c)$ and $f : d \to c$ then $f^*(S) \in J(d)$.

(T) If $f^*(S') \in J(d_f)$ for each $f : d_f \to c$ in $S \in J(c)$, then $S' \in J(c)$.

Here $f^*(S) := \{g : e \to d \mid f \circ g \in I\}$.

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Here $f^*(S) := \{g : e \to d \mid f \circ g \in I\}$.

Thus a *coverage* on M consists of a collection J of ideals satisfying:

(M) The ideal of all elements of M belongs to J.

(S) If $I \in J$ then $m^*(I) := \{n \mid mn \in I\} \in J$ for each $m \in M$.

(T) If $m^*(I') \in J$ for each m in some fixed $I \in J$, then $I' \in J$.

The collection of covering ideals is upward-closed.

Extremal examples

Some examples valid for all toposes are the following.

- The *trivial coverage* J_{triv} has only M covering.
- The *degenerate coverage* J_{deg} has all ideals covering.
- The *double-negation coverage* $J_{\neg\neg}$ has an ideal I covering if and only if for all $m \in M$ there exists n with $mn \in I$.

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Lemma*

For any monoid M there is a unique maximal ideal I^* that does not contain the identity.

I^* consists of all elements which do not have a right inverse. Thus we have a coverage J_{min} generated by M and I^*. I is covering for J_{min} iff for every sequence $m_0, m_1, \dots$ of elements of I^*, there exists $n \in \mathbb{N}$ such that $m_0 m_1 \cdots m_n \in I$.

Arranging the extremal examples

Lemma

Any subtopos of $\mathrm{PSh}(M)$ is either degenerate or *dense*, so contains $\mathrm{Sh}(M, J_{\neg\neg})$. In the latter case, it is *two-valued* (equivalently, *hyperconnected* over **Set**).

Arranging the extremal examples

Lemma

Any subtopos of PSh(M) is either degenerate or *dense*, so contains Sh($M, J_{\neg\neg}$). In the latter case, it is *two-valued* (equivalently, *hyperconnected* over **Set**).

M is a group if and only if $J_{\neg\neg}$ coincides with the trivial topology. In this case, $I^* = \emptyset$, so there are only two coverages:

$$J_{triv} = J_{\neg\neg} \subsetneq J_{min} = J_{deg}.$$

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$$J_{triv} = J_{\neg\neg} \subsetneq J_{min} = J_{deg}.$$

Otherwise, I^* is non-empty and $J_{min} \subseteq J_{\neg\neg}$, so the lattice of coverages looks like:

$$\begin{tikzcd} J_{triv} \arrow[r, dash] & J_{min} \arrow[r, dash, bend left=60] \arrow[r, dash, bend right=60] & J_{\neg\neg} \arrow[r, dash] & J_{deg} \end{tikzcd}$$

Little free monoids

From now on, we let $M = \Sigma^*$ for some alphabet Σ.
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Small Σ are easy!

When $\Sigma = \emptyset$, Σ^* is the trivial monoid (indeed, the trivial group):

$$\mathrm{PSh}(*) \simeq \mathbf{Set} \supsetneq \mathrm{Sh}(*, J_{deg}) \simeq 1$$

When $\Sigma = \{*\}$, $\Sigma^* \cong \mathbb{N}$, and we have $J_{min} = J_{\neg\neg}$:

$$\mathrm{PSh}(\mathbb{N}) \supsetneq \mathrm{Sh}(\mathbb{N}, J_{\neg\neg}) \simeq \mathrm{PSh}(\mathbb{Z}) \supsetneq \mathrm{Sh}(\mathbb{N}, J_{deg}) \simeq 1$$

Greater freedom

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Lemma (spoiler)

When $|\Sigma| \geq 2$, $\mathrm{PSh}(\Sigma^*)$ has uncountably many subtoposes.

$$\begin{tikzcd}
J_{triv} \arrow[r, dash] & J_{min} \arrow[rr, dash, bend left] \arrow[rr, dash, bend right] \arrow[rr, phantom, "!"] & & J_{\neg\neg} \arrow[r, dash] & J_{deg}
\end{tikzcd}$$

What tools can we use to understand the intermediate coverages?

A fateful slice

Σ^* acts on itself by right multiplication; this is the canonical action.
We can consider it as an object of $\mathrm{PSh}(\Sigma^*)$.

Lemma

We have $\mathrm{PSh}(\Sigma^*)/\Sigma^* \simeq \mathrm{PSh}(G^*_\Sigma)$, where G^*_Σ is the category of elements of the canonical action.

G^*_Σ is also the free category on the Cayley graph for Σ^*. For $\Sigma = \{a, b\}$:

$$
\begin{tikzcd}
\vdots & & & \vdots & & \vdots & & & \vdots \\
aaa \arrow[dr] & aab \arrow[d] & aba \arrow[d] & abb \arrow[dl] & & baa \arrow[dr] & bab \arrow[d] & bba \arrow[d] & bbb \arrow[dl] \\
 & aa \arrow[dr] & ab \arrow[d] & & & & ba \arrow[d] & bb \arrow[dl] & \\
 & & a \arrow[drr] & & & & b \arrow[dll] & & \\
 & & & & \epsilon & & & &
\end{tikzcd}
$$

L'étendue très attendue

Does this actually make the problem easier?

Definition

A topos $\mathcal{E}$ is an *étendue* if there is some $X \twoheadrightarrow 1$ with $\mathcal{E}/X$ localic: equivalent to sheaves on some locale.

These were studied extensively by Rosenthal, [Ros81].

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Proposition

$\mathrm{PSh}(G_{\Sigma}^{*})$ is localic: it is equivalent to $\mathrm{Sh}(\Sigma^{\leq\omega})$, for $\Sigma^{\leq\omega}$ the space of finite and infinite sequences with the *pointwise convergence* topology, having basic opens,

$$\hat{U}(v) := \{w \in \Sigma^{\leq\omega} \mid v \trianglelefteq w\}$$

for $v \in \Sigma^*$, where $\trianglelefteq$ means 'is a prefix of'.

That is, $\mathrm{PSh}(\Sigma^*)$ *is an étendue*.

Slicing for subtoposes

Why does that matter?

Lemma

For $X \twoheadrightarrow 1$ in a topos $\mathcal{E}$, pulling back induces an injective map from the subtoposes of $\mathcal{E}$ to those of $\mathcal{E}/X$:

$$\begin{tikzcd}
\mathcal{F}/i^*(X) \arrow[r, hook, "\pi^*(i)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}/X \arrow[d, "\pi"] \\
\mathcal{F} \arrow[r, hook, "i"'] & \mathcal{E}
\end{tikzcd}$$

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For any locale L, subtoposes of $\mathrm{Sh}(L)$ correspond to *sublocales* of L, so we can leverage locale theory!

Self-similar subtoposes

It remains to identify *which* sublocales of $\Sigma^{\leq\omega}$ are relevant.
An endomorphism $m : X \to X$ induces one of $\mathcal{E}/X$, which we also call m.

Lemma

Suppose that the joint coequalizer of endomorphisms of X is 1. Then the subtoposes of $\mathcal{E}/X$ of the form $\pi^*(i)$ are the *self-similar* ones, meaning that for each $m : X \to X$, we have a pullback square:

$$\begin{tikzcd} \mathcal{F} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}/X \arrow[d, "m"] \\ \mathcal{F} \arrow[r, hook] & \mathcal{E}/X \end{tikzcd}$$

Self-similar sublocales

When $\mathcal{E}/X \simeq \mathrm{Sh}(L)$, m corresponds to a unique endomorphism $L \to L$.

Theorem

Subtoposes of $\mathrm{PSh}(\Sigma^*)$ correspond to *self-similar sublocales* of $\Sigma^{\leq\omega}$, meaning sublocales L' such that for each word $w \in \Sigma^*$, the inclusion fits into a pullback square:

$$\begin{tikzcd} L' \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Sigma^{\leq\omega} \arrow[d, "w\cdot"] \\ L' \arrow[r, hook] & \Sigma^{\leq\omega} \end{tikzcd}$$

Extremal cases again

It is convenient to first consider the subtoposes established earlier.

$$\begin{tikzcd}
\mathrm{Sh}(\Sigma^{\leq\omega}) \arrow[d] & \mathrm{Sh}(\Sigma^{\omega}) \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[rr, bend left, no head] \arrow[rr, bend right, no head] & & \mathrm{Sh}(\Sigma^{\omega})_{\neg\neg} \arrow[d] & 1 \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
\mathrm{Sh}(\Sigma^*, J_{triv}) \arrow[d, phantom, "\wr" description] & \mathrm{Sh}(\Sigma^*, J_{min}) \arrow[l] \arrow[d, phantom, "\wr" description] \arrow[rr, bend left, no head] \arrow[rr, bend right, no head] & & \mathrm{Sh}(\Sigma^*, J_{\neg\neg}) & 1 \arrow[l] \\
\mathrm{PSh}(\Sigma^*) & \color{red}{\mathcal{JT}_{|\Sigma|}} & & &
\end{tikzcd}$$

Example: the Jónsson-Tarski topos

The minimal coverage yields 'the' Jonsson-Tarski topos described by Johnstone [Joh85] (attributed to Freyd), a well-known étendue.

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A right Σ^*-act X is a sheaf for J_{min} if and only if the canonical map,

$$X \to \prod_{a \in \Sigma} X, \quad x \mapsto (x \cdot a)_{a \in \Sigma}$$

is an isomorphism.

This characterization is the one generalized by Leinster [Lei07].

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We will actually organize subtoposes of the Jónsson-Tarski topos and hence sublocales of 'sequence space' Σ^ω, using [PP12].

Sequence spaces

When $\Sigma = \{0, 1\}$, Σ^{ω} is Cantor space.

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Lemma

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Lemma

Σ^ω is a (compact) Hausdorff locale with no isolated points.

These properties guarantee a particularly nice relationship between sublocales and sub*spaces*:

$$\begin{tikzcd} \mathsf{Sub}_{\mathbf{Loc}}(L) \arrow[rr, "\mathrm{pt}" description] & & \mathsf{Sub}_{\mathbf{Top}}(\mathsf{pt}(L)) \arrow[ll, bend right, "\mathrm{Loc}"', "\perp"] \arrow[ll, bend left, "\mathrm{Max}", "\perp"'] \end{tikzcd}$$

where

$$\mathrm{Max}(S) := \bigcap_{s \in \mathsf{pt}(L) \setminus S} L \setminus \{s\}.$$

Self-similarity

Lemma

Under an action by local homeomorphisms, the three adjoint functors preserve self-similarity.

A subspace of $\mathrm{pt}(\Sigma^\omega)$ being self-similar means $w \in \mathrm{pt}(L')$ if and only if $m \cdot w \in \mathrm{pt}(L')$ for each finite word m and sequence w.

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Lemma

Under an action by local homeomorphisms, if L has no isolated points, $L_{\neg\neg}$ is self-similar in L.

Thus we can upgrade the adjunction with self-similarity and denseness as follows:

$$\begin{tikzcd} \mathrm{dSub}^M_{\mathbf{Loc}}(L) \arrow[r, "\mathrm{pt}" description] \arrow[r, phantom, shift left=2.5, "\perp"] \arrow[r, phantom, shift right=2.5, "\perp"] & \mathrm{Sub}^M_{\mathbf{Top}}(\mathrm{pt}(L)) \arrow[l, shift right=5, "\mathrm{Loc} \vee L_{\neg\neg}"'] \arrow[l, shift left=5, "\mathrm{Max}"] \end{tikzcd}$$

Countable and uncountable

We partition $\mathrm{pt}(\Sigma^\omega)$ into equivalence classes via $w \sim m \cdot w$ for finite words m.

Each class is countable and dense in Σ^ω, so there are uncountably many classes. The Boolean algebra $\mathcal{B}$ of unions of equivalence classes coincides with that of subspaces of $\mathrm{pt}(\Sigma^\omega)$.

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Proposition [PP12, Chapter VII]

Suppose that L is a spatial locale such that $\mathrm{pt}(L)$ is a compact Hausdorff space. Then a countable intersection of dense open sublocales of L is spatial.

In particular, $\mathrm{Max}(S) = \mathrm{Loc}(S)$ whenever $S \subseteq \mathrm{pt}(L)$ has countable complement.

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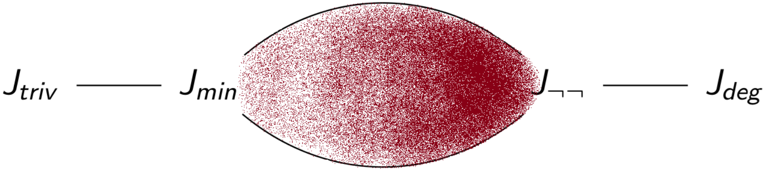
Corollary

For each element S of $\mathcal{B}$, there is a bounded lattice of self-similar sublocales of Σ^ω having S as its set of points. When S has countable complement, this lattice has a unique element.

In terms of coverings

Translating back to coverings, we arrive at the following sketch.



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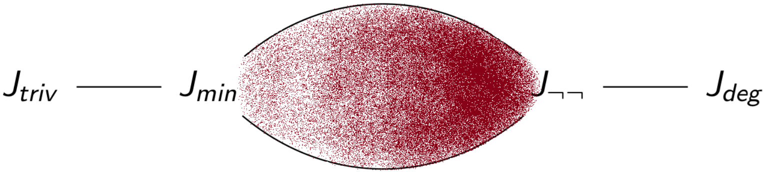

How do we actually know that the lattices get bigger as we reduce the size of S?

An ideal belongs to the coverage $J_{\top}(S)$ corresponding to $\mathrm{Loc}(S)$ iff every finite word is a prefix of an element of I and each infinite word $w \in S$ has a prefix in I.

An ideal belongs to the coverage $J_{\perp}(S)$ corresponding to $\mathrm{Max}(S)$ iff it is in $J_{\top}(S)$ *and* the number of w (outside S) which do not have a prefix in I is at most $|\Sigma^*|$.

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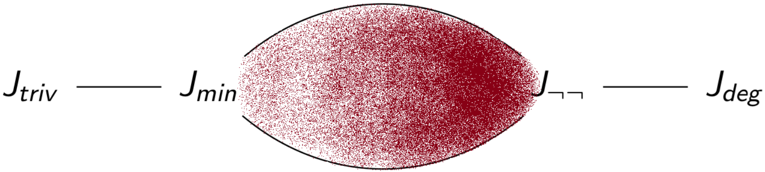

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For S uncountable and Σ countable, this is clearly a non-trivial condition distinguishing the coverages!

Continuum hypothesis implications

The gap between the extremes corresponds to the cardinality gap $|\Sigma^*| < |\Sigma^\omega|$.

Example

Consider the endomorphism $\Sigma^\omega \to \Sigma^\omega$ which duplicates every element of a sequence, so $01001011\ldots \mapsto 0011000011001111\ldots$
The complement I of the image is a right ideal covering in $J_{\neg\neg}$ such that there are $|\Sigma^\omega|$ words w with no prefix in I.

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Let $S \in \mathcal{B}$. For each cardinal $|\Sigma^*| \leq \kappa \leq |\Sigma^\omega \backslash S|$, we have an intermediate coverage $J_\kappa(S)$ consisting of those $J_\top(S)$-covering sieves such that the number of w which do not have a prefix in I is at most κ.

The non-existence of a κ strictly between these extremes is precisely the continuum hypothesis (CH). But we don't normally impose CH *or* its negation in topos theory!

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With or without the intermediate coverages $J_\kappa(S)$, for each (uncountable) $S' \subseteq S$, we have $J_\bot S \subseteq J_\top(S') \vee J_\bot(S) \subseteq J_\top(S)$ between the extremes.

References I

Peter T. Johnstone, *When is a variety a topos?*, Algebra Universalis **21** (1985), 198–212.

T. Leinster, *Jónsson-tarski toposes*, 85th Peripatetic Seminar on Sheaves and Logic, 2007.

J. Picado and A. Pultr, *Frames and locales*, Birkhäuser Basel, 2012.

M. Rogers, *Toposes of Topological Monoid Actions*.

K. I. Rosenthal, *Etendues and categories with monic maps*, Journal of Pure and Applied Algebra **22** (1981), no. 2, 193–212.

Fin

Thank you! Questions?

Back to the general case

Whatever we understand about these lattices will de transferable to general monoids as follows:

$$\begin{tikzcd}
\mathcal{F}'' \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F}' \arrow[r, two heads] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d, hook] \\
\mathrm{PSh}(\Sigma^*) \arrow[r, two heads] & \mathrm{PSh}(M) \arrow[r, two heads] & \mathrm{Cont}(M, \tau)
\end{tikzcd}$$

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\mathrm{PSh}(\Sigma^*) \arrow[r, two heads] & \mathrm{PSh}(M) \arrow[r, two heads] & \mathrm{Cont}(M, \tau)
\end{tikzcd}$$

Example

Any monoid with a single generator has either exactly 2 or 3 subtoposes, the former if and only if it is a group.