Stability from the categorical point of view

J. Rosický

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This concept is syntactic. In abstract elementary classes, types over M are pairs (f, a) where $f : M \to N$ and $a \in N$. Two types are equivalent if they can be amalgamated. Equivalence classes of types over M are called *Galois types* because they correspond to orbits of automorphisms of a monster model fixing M.

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Let $\mathcal{L}$ be a locally finitely presentable category and $\mathcal{M}$ a class of monomorphisms in $\mathcal{L}$ such that

1. $\mathcal{M}$ closed under pushouts, compositions and contains all isomorphisms,
2. coherent, i.e., $gf \in \mathcal{M}$ and $g \in \mathcal{M}$ then $f \in \mathcal{M}$,
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Let $\mathcal{K} = \mathcal{L}_{\mathcal{M}}$ have the same objects as $\mathcal{L}$ whose morphisms are precisely those of $\mathcal{M}$. It will be our typical example of an abstract elementary class.

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Theorem 1. (Mazari-Armida, JR) Assume that $\mathcal{M}$ is cofibrantly generated by a set $\mathcal{X}$. Let $\lambda \geq \gamma$ be an infinite cardinal such that

1. domains and codomains of morphisms from $\mathcal{X}$ are λ-presentable,
2. for every object M of size λ, the number of morphisms from domains of morphisms from $\mathcal{X}$ to M is $\leq \lambda$.

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Then $\mathcal{L}_\mathcal{M}$ is λ-stable.

Proof. The universal object N over M is given by a small object argument. Our assumptions ensure that it does not increase sizes. A morphism $M \to K$ from M is given by a transfinite composition of pushouts of morphisms from $\mathcal{X}$. The induced morphisms $K_i \to N_i$ are given by pushouts, hence they are in $\mathcal{M}$.

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Theorem 2. (Mazari-Armida,JR) $\mathbf{Set}^{\mathcal{C}}_{\text{Mono}}$ is superstable iff $\mathcal{C}$ is weakly noetherian.

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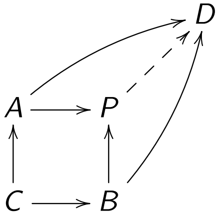
It might be difficult to decide whether $\mathcal{M}$ is cofibrantly generated. In this case, one has enough $\mathcal{M}$-injectives, i.e., every object has an $\mathcal{M}$-morphism to an $\mathcal{M}$-injective object. Moreover, the category of $\mathcal{M}$-injective objects is accessible. This excludes embeddings in posets because injectives are complete lattices. On the other hand, every poset is injective w.r.t. split monomorphisms but they are not cofibrantly generated.

A commuting square in $\mathcal{L}_{\mathcal{M}}$ is $\mathcal{M}$-*effective* if the induced arrow from the pushout in $\mathcal{L}$ is in $\mathcal{M}$.



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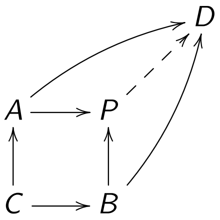

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The *independence category* $\mathsf{Idp}_{(\mathcal{L},\mathcal{M})}$ is a subcategory of $(\mathcal{L}_{\mathcal{M}})^2$ whose objects are $\mathcal{M}$-morphisms and whose morphisms are $\mathcal{M}$-effective squares.

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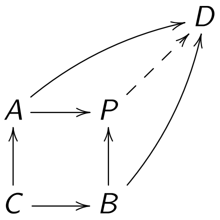

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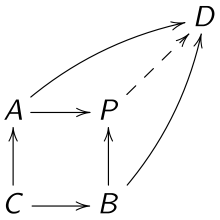

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Moreover, $\mathcal{M}$-effective squares form a stable independence in $\mathcal{L}_{\mathcal{M}}$.

We call a category $\mathcal{C}$ *locally linearly preordered* if for any span of arrows $Y \xleftarrow{f} X \xrightarrow{g} Z$ in $\mathcal{C}$ there is either $h : Y \to Z$ such that $hf = g$ or there is $h' : Z \to Y$ such that $f = h'g$.

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Corollary 1. If $\mathcal{C}$ is locally linearly preordered then $\mathbf{Set}^{\mathcal{C}}$ has enough pure injectives.

Corollary 2. (Banaschewski 1974) The category G-**Set** of acts over a group G has enough pure injectives.

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The same argument works for fields but not for algebraically closed ones. Here, algebraic independence is a stable independence.

Recall that

$$\begin{array}{ccc} A & \longrightarrow & D \\ \uparrow & & \uparrow \\ C & \longrightarrow & B \end{array}$$

is algebraically independent if for every finite tuple $\bar{a}$ in A, the transcendence degree of $\bar{a}$ over C is the same as the transcendence degree of $\bar{a}$ over B.

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This indicates that $\mathcal{M}$-effectivity is important in a more general situation where $\mathcal{A}$ is a finitely accessible full subcategory of a locally finitely presentable category $\mathcal{L}$ and $\mathcal{A}_{\mathcal{M}} = \mathcal{L}_{\mathcal{M}} \cap \mathcal{A}$.

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