

Extending strong conceptual completeness via virtual ultracategories

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An overview

How to categorify topological spaces?

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lattice of opens
topoi

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axiomatizing the convergence of ultrafilters instead of opens subsets.

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The ultraconvergence relation is an extension of the specialization order,
that is strong enough to recover the topology.

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A **coherent topos** (= coherent FO-theory)
can be recovered from its **ultracategory of points** (= models).

We extend this result: a **topos with enough points** can be recovered
from its **virtual ultracategory of points**.

1) Ultracategories

2) Virtual ultracategories

3) The proof

The usual Stone duality

Classical propositional theories = Boolean Algebras.

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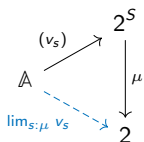
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In particular, $\text{Mod} : \text{BoolAlg}^{\text{op}} \hookrightarrow \text{TopSp}$ is fully faithful.

The Stone topology on $\text{Mod}(\mathbb{A})$ arises from the $\mu \in \text{BoolAlg}(2^S, 2)$



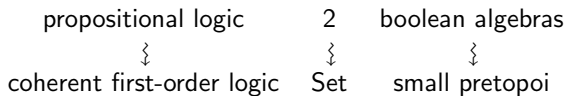
$$\begin{array}{ccc} \text{Mod}(\mathbb{A})^S & \longrightarrow & \text{Mod}(\mathbb{A}) \\ (v_s)_{s:S} & \longmapsto & \lim_{s:\mu} v_s \end{array}$$

$$\beta(S) := \text{BoolAlg}(2^S, 2) = \{\text{ultrafilters on } S\}$$

The $\lim_{s:\mu}(-)$ determine the topology on $\text{Mod}(\mathbb{A})$ by ultraconvergence.

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\Downarrow	\Downarrow	\Downarrow
coherent first-order logic	Set	small pretopoi

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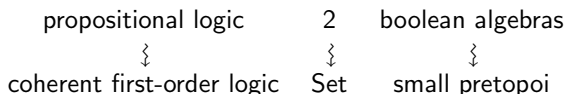
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The Stone topology on $\text{Mod}(\mathbb{A})$ comes from $\beta(S) = \text{BoolAlg}(2^S, 2)$

\rightsquigarrow we should look at $\text{Pretop}(\text{Set}^S, \text{Set})$.

Ultraproducts

Łoś's theorem: the **ultraproduct functor**

$$\begin{aligned} \int_{\mu} : \text{Set}^S &\longrightarrow \text{Set} \\ (A_s)_{s:S} &\longmapsto \int_{s:\mu} A_s := \text{colim}_{\mu(S_0)=1} \prod_{s:S_0} A_s \end{aligned}$$

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induces an operation $\int_{\mu} : \mathbf{Mod}(\mathbb{T})^S \longrightarrow \mathbf{Mod}(\mathbb{T})$.

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Actually, ultraproducts are enough!

$$\mathbf{Pretop}(\mathbf{Set}^S, \mathbf{Set}) \simeq \operatorname{Ind}(\mathcal{UF}_{\mathbf{Set}^S}^{\operatorname{op}}) \quad \underline{\text{Joyal 1971}}$$

i.e. a pretopos functor $\mathbf{Set}^S \rightarrow \mathbf{Set}$ is a filtered colimit of ultraproducts.

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An **ultracategory** is a category \mathcal{M} together with:

- ultraproduct functors $\int_{\mu} : \mathcal{M}^{\mu} \rightarrow \mathcal{M}$ ($\mathcal{M}^{\mu} := \text{colim}_{\mu(S_0)=1} \mathcal{M}^{S_0}$)

- functorial reindexing

$f^{\#} : \int_{s:\mu} A_s \rightarrow \int_{t:\nu} A_{f(t)}$, for $f : \nu \rightarrow \mu$

A commutative triangle diagram illustrating the relationship between ultraproduct functors and reindexing functors. The top-left node is \mathcal{M}^{μ} , the top-right node is \mathcal{M}^{ν} , and the bottom node is \mathcal{M} . A horizontal arrow labeled f^* points from \mathcal{M}^{μ} to \mathcal{M}^{ν} . A diagonal arrow labeled \int_{μ} points from \mathcal{M}^{μ} down to \mathcal{M} . A diagonal arrow labeled \int_{ν} points from \mathcal{M}^{ν} down to \mathcal{M} . A blue double-lined arrow labeled $f^{\#}$ points from the \int_{μ} arrow to the \int_{ν} arrow, indicating a natural transformation between the functors.

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- coherent unitor and associators
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$$\int_{\star} \xrightarrow{\sim} \text{Id} , \quad \int_{\sum_{s:\mu} \nu_s} \xrightarrow{\sim} \int_{s:\mu} \int_{t:\nu_s}$$

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e.g. Set , $\text{Mod}(\mathbb{T})$, Compact Hausdorff spaces (= β -algebras).

Conceptual completeness

Makkai 1987: \mathbb{T} can be recovered from its ultracategory of models:

$\text{ev} : \mathbb{T} \xrightarrow{\sim} \text{Ult}(\text{Mod}(\mathbb{T}), \text{Set})$ is an equivalence.

\rightsquigarrow conceptual completeness for coherent logic:

$\text{Mod} : \text{Pretop}^{\text{op}} \hookrightarrow \text{Ult}$ is fully faithful.

Whereas the category $\text{Mod}(\mathbb{T}) \in \text{CAT}$ is **not enough** to recover \mathbb{T} ,
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a topological answer:

ultraproducts \sim categorified ultraconvergence.

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Lurie's reconstruction theorem:

A coherent topos \mathcal{E} can be recovered from its ultracategory of points

$$\mathrm{ev} : \mathcal{E} \xrightarrow{\sim} \mathrm{Ult}^{\mathrm{L}}(\mathrm{pt}(\mathcal{E}), \mathbf{Set}) \text{ is an equivalence}$$

\leadsto reconstruction theorem for coherent topoi:

$$\mathrm{pt} : \mathbf{CohTop} \hookrightarrow \mathrm{Ult}^{\mathrm{L}} \text{ is } \underline{\text{fully faithful}}.$$

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We will generalize these both results even further.

We can recover a topos with enough points (= theory)
from the **virtual ultracategory** structure over its points (= models).

\leadsto this is a reconstruction theorem for topoi with enough points:

$\text{pt} : \text{GTop}_{\text{wep}} \hookrightarrow \text{vUlt}$ is fully faithful.

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\mathcal{E} = sheaves over a compact Hausdorff space

X compact Hausdorff $\rightsquigarrow \mathbf{pt}(X)$ the **ultracategory of its points**.

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$$\begin{aligned} \mathbf{Sh}(X) &\xrightarrow{\sim} \mathbf{Ult}^{\mathbf{L}}(\mathbf{pt}(X), \mathbf{Set}) \\ E &\longmapsto \begin{cases} x \mapsto E_x \\ + \text{ coherent maps } \sigma : E_a \rightarrow \int_{x:\mu} E_x, \text{ for } a = \lim \mu. \end{cases} \end{aligned}$$

A sheaf can be reconstructed from its stalks + some data (the σ 's).

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We generalize from X compact Hausdorff to T topological space.

We still have the construction,

$$E \in \mathbf{Sh}(T) \longmapsto \begin{cases} x \mapsto E_x \\ + \text{ coherent maps } \sigma_{a,\mu} : E_a \rightarrow \int_{x:\mu} E_x, \text{ for } a \preccurlyeq \mu. \end{cases}$$

Can we reconstruct the sheaf E from this data?

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For the reverse construction, the étale space is given by:

$$E := \bigsqcup_{x:T} E_x, \text{ opens} := \text{subsets stable by the } \sigma\text{'s.}$$

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To show it is étale, we use the following characterization.

Lemma (S.) : For $p : E \rightarrow T$ continuous,

$$p \text{ is étale} \quad \text{iff} \quad \forall \mu \succcurlyeq p(e), \exists! \nu \succcurlyeq e, p_*(\nu) = \mu$$

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$$p \text{ is étale} \quad \text{iff} \quad \forall \mu \succcurlyeq p(e), \exists! \nu \succcurlyeq e, p_*(\nu) = \mu \\ \text{and this } \nu \text{ is isomorphic to } \mu \text{ via } p.$$

Categorifying ultraconvergence

$a \preccurlyeq (b_s)_{s:\mu}$ means $\forall U \text{ open}, U \ni a \rightarrow (\forall s : \mu, U \ni b_s)$

For $a, (b_s)_{s:\mu}$ points of \mathcal{E} topos, we denote by $\alpha : a \multimap (b_s)_{s:\mu}$ for

$$\begin{array}{ccc}
 & \text{Set}^S & \\
 (b_s^*) \nearrow & \downarrow \int_\mu & \\
 \mathcal{E} & & \text{Set} \\
 \searrow a^* & & \\
 & \uparrow \alpha &
 \end{array}
 \quad \alpha \in \text{Nat}_{E:\mathcal{E}}(E_a, \int_{s:\mu} E_{b_s})$$

it is a **proof-relevant** version of “ $a \preccurlyeq (b_s)_{s:\mu}$ ”.

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it is a **proof-relevant** version of “ $a \preceq (b_s)_{s:\mu}$ ”.

$a \multimap (b_s)_{s:\mu}$ is an **ultraarrow** with codomain an ultrafamily of objects, these ultraarrows form a **generalized multicategorical structure** on the points of \mathcal{E} that we denote $\text{pt}(\mathcal{E}) \in \mathbf{vUlt}$.

\rightsquigarrow the notion of **virtual ultracategory**.

(generalized multicategories are introduced in Cruttwell and Shulman 2010)

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The **virtual ultracategory of points** of a topos is defined as above

$$\text{pt} : \text{GTop} \longrightarrow \text{vUlt}$$

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Ultracategories are recovered as the **representable** virtual ultracategories.

Statement of the theorem

Theorem (S.): Let \mathcal{E} be a topos with enough points, the functor

$ev : \mathcal{E} \xrightarrow{\sim} \mathbf{vUlt}(\mathbf{pt}(\mathcal{E}), \mathbf{Set})$ is an equivalence.

Some particular cases:

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This theorem generalizes, and gives a new proof, to Lurie's result.

- 1) Ultracategories
- 2) Virtual ultracategories
- 3) The proof

Strategy

We already proved the case $\mathcal{E} = \text{Sh}(\text{topological space})$; to extend from the topological case to the general case, we use **representation of topoi by topological groupoids**.

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Butz and Moerdijk 1998: any topos with enough points \mathcal{E} can be represented by a **topological groupoid** (T_\bullet) ,

i.e. there is a **universal descent cocone**:

$$\text{Sh}(T_1 \times_{T_0} T_1) \xrightarrow{-m \rightarrow} \text{Sh}(T_1) \begin{array}{c} \xrightarrow{s} \\ \xleftarrow{u} \\ \xrightarrow{t} \end{array} \text{Sh}(T_0) \quad \begin{array}{c} \xrightarrow{\pi} \\ \searrow \end{array} \mathcal{E}$$

and so, $\mathcal{E} = \text{Sh}_{\text{eq}}(T_\bullet)$ the topos of equivariant sheaves over (T_\bullet) .

A problem!

$$\mathrm{Sh}(T_1 \times_{T_0} T_1) \longrightarrow \mathrm{Sh}(T_1) \begin{matrix} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{matrix} \mathrm{Sh}(T_0) \begin{matrix} \xrightarrow{\quad \pi \quad} \\ \searrow \end{matrix} \mathcal{E}$$

GTop

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(X is the image of the points of T_0 by π)

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Problem: $\mathrm{pt} : \mathrm{GTop} \rightarrow \mathrm{vUlt}$ does not preserve colimits (right adjoint!)

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- 1) We prove a proposition ensuring a functor of virtual ultracategories to be effective descent.

Proposition: A functor of v -ultracats $F : X \rightarrow Y$ surjective on objects and such that any $F(x) \multimap (b_s)_{s:\mu}$ can be lift to some $x \multimap (y_s)_{s:\mu}$ is effective descent.

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Definition: An indexing $\alpha : \kappa \rightrightarrows M$ is ample if $|\kappa \setminus \text{dom}(\alpha)| = |\kappa|$.

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- 3) A theorem of **Wrigley 2023** ensures that the ample condition is not too strong, i.e., that $(T_{\bullet}^{\text{amp}})$ represents \mathcal{E} .

The duality

We have shown:

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Proof: note that $\mathbf{pt}(\mathcal{E}) \simeq \operatorname{colim}_A \mathbf{pt}(\mathcal{E})_{\upharpoonright A}$, where A range over small separating sets of points of \mathcal{E} , to reduce to the above case. \square

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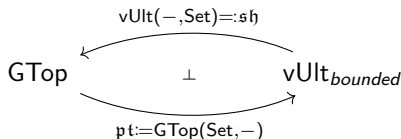
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We get a **pseudoidempotent 2-adjunction**



inducing a fully faithful embedding $\mathbf{pt} : \mathbf{GTop}_{\text{wep}} \hookrightarrow \mathbf{vUlt}$.

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This adjunction induces a **fully faithful embedding** $\text{pt} : \text{GTop}_{\text{wep}} \hookrightarrow \text{vUlt}$ extending Lurie's reconstruction theorem to all topoi with enough points.

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Virtual ultracategories fit in the framework of generalized multicategories of Cruttwell and Shulman 2010, and ultracategories are recovered as the **representable** ones.

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Proof of the 0-dimensional case

We prove the non-trivial implication,

$$p \text{ is étale} \iff \forall \mu \succcurlyeq p(e), \exists! \nu \succcurlyeq e, p_*(\nu) = \mu$$

and this ν is isomorphic to μ via p .

- p_* is étale
- the hypothesis gives a lift σ
- the topological fact (\star) gives V
- we pullback along δ
not continuous!
- $\delta^* T \subseteq T$ is not open
 and ξ not continuous:
 we restrict to $W \subset V$.

$$W := \{w \in \delta^* V \mid \forall \mu \succcurlyeq w \text{ } (\xi \text{ is defined on } \mu \text{ and } \xi_*(\mu) \succcurlyeq \xi(w))\}$$

(★) a section on a compact of a Hausdorff space can be extended to an open. \square

