Extending strong conceptual completeness via virtual ultracategories

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An overview

How to categorify topological spaces?

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lattice of opens
topoi

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The ultraconvergence relation is an extension of the specialization order, that is strong enough to recover the topology.

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A coherent topos (= coherent FO-theory) can be recovered from its ultracategory of points (= models).

We extend this result: a topos with enough points can be recovered from its virtual ultracategory of points.

1) Ultracategories

2) Virtual ultracategories

3) The proof

The usual Stone duality

Classical propositional theories = Boolean Algebras.

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The Stone topology on $\mathrm{Mod}(\mathbb{A})$ arises from the $\mu \in \mathrm{BoolAlg}(2^S, 2)$

$$\begin{tikzcd} & 2^S \arrow[d, "\mu"] \\ \mathbb{A} \arrow[ur, "(v_s)"] \arrow[r, dashed, "\lim_{s:\mu} v_s"'] & 2 \end{tikzcd} \qquad \begin{array}{ccc} \mathrm{Mod}(\mathbb{A})^S & \longrightarrow & \mathrm{Mod}(\mathbb{A}) \\ (v_s)_{s:S} & \longmapsto & \lim_{s:\mu} v_s \end{array}$$

$$\beta(S) \coloneqq \mathrm{BoolAlg}(2^S, 2) = \{\text{ultrafilters on } S\}$$

The $\lim_{s:\mu}(-)$ determine the topology on $\mathrm{Mod}(\mathbb{A})$ by ultraconvergence.

Categorifying to ultracategories

Ultracategories arise by categorifying:

$$\begin{array}{ccc} \text{propositional logic} & 2 & \text{boolean algebras} \\ \rightsquigarrow & \rightsquigarrow & \rightsquigarrow \\ \text{coherent first-order logic} & \mathsf{Set} & \text{small pretopoi} \end{array}$$

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The Stone topology on $\mathsf{Mod}(\mathbb{A})$ comes from $\beta(S) = \mathsf{BoolAlg}(2^S, 2)$
$\rightsquigarrow$ we should look at $\mathsf{Pretop}(\mathsf{Set}^S, \mathsf{Set})$.

Ultraproducts

Łoś's theorem: the ultraproduct functor

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induces an operation $\int_\mu : \mathsf{Mod}(\mathbb{T})^S \longrightarrow \mathsf{Mod}(\mathbb{T})$.

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Actually, ultraproducts are enough!

$$\mathsf{Pretop}(\mathsf{Set}^S, \mathsf{Set}) \simeq \mathsf{Ind}(\mathcal{UF}^{\mathrm{op}}_{\mathsf{Set}^S}) \qquad \text{Joyal 1971}$$

i.e. a pretopos functor $\mathsf{Set}^S \to \mathsf{Set}$ is a filtered colimit of ultraproducts.

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An ultracategory is a category $\mathscr{M}$ together with:

- ultraproduct functors $\int_\mu : \mathscr{M}^\mu \to \mathscr{M}$ $\quad (\mathscr{M}^\mu := \mathrm{colim}_{\mu(S_0)=1} \mathscr{M}^{S_0})$

- functorial reindexing
$f^\# : \int_{s:\mu} A_s \to \int_{t:\nu} A_{f(t)}$, for $f : \nu \to \mu$

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e.g. $\quad$ Set, $\quad$ $\mathrm{Mod}(\mathbb{T})$, $\quad$ Compact Hausdorff spaces ($= \beta$-algebras).

Conceptual completeness

Makkai 1987: $\mathbb{T}$ can be recovered from its ultracategory of models:

$$\mathsf{ev} : \mathbb{T} \xrightarrow{\sim} \mathsf{Ult}(\mathsf{Mod}(\mathbb{T}), \mathsf{Set}) \text{ is an equivalence.}$$

$\rightsquigarrow$ conceptual completeness for coherent logic:

$$\mathsf{Mod} : \mathsf{Pretop}^{\mathrm{op}} \hookrightarrow \mathsf{Ult} \;\text{ is fully faithful.}$$

Whereas the category $\mathsf{Mod}(\mathbb{T}) \in \mathsf{CAT}$ is not enough to recover $\mathbb{T}$,
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a topological answer:

ultraproducts $\sim$ categorified ultraconvergence.

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Lurie's reconstruction theorem:

A coherent topos $\mathscr{E}$ can be recovered from its ultracategory of points

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$\rightsquigarrow$ reconstruction theorem for coherent topoi:

$\mathfrak{pt} : \mathsf{CohTop} \hookrightarrow \mathsf{Ult}^{\mathsf{L}}$ is fully faithful.

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None of these results contain the other!

We will generalize these both results even further.

We can recover a topos with enough points (= theory)
from the virtual ultracategory structure over its points (= models).

$\rightsquigarrow$ this is a reconstruction theorem for topoi with enough points:

$$\mathfrak{pt} : \mathrm{GTop}_{wep} \hookrightarrow \mathrm{vUlt} \quad \text{is fully faithful.}$$

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$\mathscr{E}$ = sheaves over a compact Hausdorff space

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$$\begin{array}{ccl} \mathrm{Sh}(X) & \xrightarrow{\sim} & \mathrm{Ult}^{\mathrm{L}}(\mathfrak{pt}(X), \mathrm{Set}) \\ E & \longmapsto & \begin{cases} x \mapsto E_x \\ + \text{ coherent maps } \sigma : E_a \to \int_{x:\mu} E_x \ , \ \text{for } a = \lim \mu. \end{cases} \end{array}$$

A sheaf can be reconstructed from its stalks + some data (the σ's).

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We generalize from X compact Hausdorff to T topological space.

We still have the construction,

$$E \in \mathrm{Sh}(T) \quad \longmapsto \quad \begin{cases} x \mapsto E_x \\ + \text{ coherent maps } \sigma_{a,\mu} : E_a \to \int_{x:\mu} E_x \ , \ \text{for } a \preccurlyeq \mu. \end{cases}$$

Can we reconstruct the sheaf E from this data?

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For the reverse construction, the étale space is given by:

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To show it is étale, we use the following characterization.

Lemma (S.) : For $p : E \to T$ continuous,

$$p \text{ is étale} \quad \text{iff} \quad \forall\, \mu \succcurlyeq p(e),\ \exists!\ \nu \succcurlyeq e,\ p_*(\nu) = \mu$$

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p is étale iff $\forall\, \mu \succcurlyeq p(e),\ \exists!\ \nu \succcurlyeq e,\ p_*(\nu) = \mu$
and this ν is isomorphic to μ via p.

Categorifying ultraconvergence

$a \preccurlyeq (b_s)_{s:\mu}$ means $\forall U \text{ open },\ U \ni a \to (\forall s : \mu,\ U \ni b_s)$

For a, $(b_s)_{s:\mu}$ points of $\mathscr{E}$ topos, we denote by $\alpha : a \multimap (b_s)_{s:\mu}$ for

$$\begin{tikzcd} & \mathrm{Set}^S \arrow[d, "\int_\mu"] \\ \mathscr{E} \arrow[ur, "(b_s^*)"] \arrow[r, "a^*"'] \arrow[r, phantom, shift left=4, "\Uparrow\alpha"] & \mathrm{Set} \end{tikzcd} \qquad \alpha \in \mathrm{Nat}_{E:\mathscr{E}}(E_a, \textstyle\int_{s:\mu} E_{b_s})$$

it is a proof-relevant version of "$a \preccurlyeq (b_s)_{s:\mu}$".

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For a, $(b_s)_{s:\mu}$ points of $\mathscr{E}$ topos, we denote by $\alpha : a \multimap (b_s)_{s:\mu}$ for

$$\begin{tikzcd} & \mathrm{Set}^S \arrow[d, "\int_\mu"] \\ \mathscr{E} \arrow[ur, "(b_s^*)"] \arrow[r, phantom, "\Uparrow\alpha"] \arrow[dr, "a^*"'] & \mathrm{Set} \end{tikzcd} \qquad \alpha \in \mathrm{Nat}_{E:\mathscr{E}}(E_a, \textstyle\int_{s:\mu} E_{b_s})$$

it is a proof-relevant version of "$a \preccurlyeq (b_s)_{s:\mu}$".

$a \multimap (b_s)_{s:\mu}$ is an ultraarrow wih codomain an ultrafamily of objects,

these ultraarrows form a generalized multicategorical structure on the points of $\mathscr{E}$ that we denote $\mathfrak{pt}(\mathscr{E}) \in \mathrm{vUlt}$.

$\rightsquigarrow$ the notion of virtual ultracategory.

(generalized multicategories are introduced in Cruttwell and Shulman 2010)

Virtual ultracategories

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A virtual ultracategory is a generalized multicategory with arrows having the shape $a \prec (b_s)_{s:\mu}$. Explicitly:

- a class of objects X,
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$$\mathfrak{pt} : \mathrm{GTop} \longrightarrow \mathrm{vUlt}$$

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Ultracategories are recovered as the representable virtual ultracategories.

Statement of the theorem

Theorem (S.): Let $\mathscr{E}$ be a topos with enough points, the functor

$$\mathrm{ev} : \mathscr{E} \xrightarrow{\sim} \mathrm{vUlt}(\mathfrak{pt}(\mathscr{E}), \mathrm{Set}) \text{ is an equivalence.}$$

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This theorem generalizes, and gives a new proof, to Lurie's result.

1) Ultracategories

2) Virtual ultracategories

3) The proof

Strategy

We already proved the case $\mathscr{E} = \mathsf{Sh}(\text{topological space})$; to extend from the topological case to the general case, we use representation of topoi by topological groupoids.

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Butz and Moerdijk 1998: any topos with enough points $\mathscr{E}$ can be represented by a topological groupoid $(T_\bullet)$,

i.e. there is a is a universal descent cocone:

$$\begin{tikzcd} \mathrm{Sh}(T_1 \times_{T_0} T_1) \arrow[r, "m"] & \mathrm{Sh}(T_1) \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathrm{Sh}(T_0) \arrow[l, "u"] \arrow[r, dashed, bend left, "\pi"] & \mathscr{E} \end{tikzcd}$$

and so, $\mathscr{E} = \mathrm{Sh}_{\mathrm{eq}}(T_\bullet)$ the topos of equivarient sheaves over $(T_\bullet)$.

A problem!

$$
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\end{tikzcd}
\qquad \mathbf{GTop}
$$

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\begin{aligned}
\mathscr{E} &\simeq \mathrm{Cocone}_{\mathrm{desc}}(\mathrm{Sh}(T_\bullet); \mathrm{Set}[\mathbb{O}]) \\
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Problem: $\mathfrak{pt} : \mathrm{GTop} \to \mathrm{vUlt}$ does not preserve colimits (right adjoint!)

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Proposition: A functor of v-ultracats $F : X \to Y$ surjective on objects and such that any $F(x) \multimap (b_s)_{s:\mu}$ can be lift to some $x \multimap (y_s)_{s:\mu}$ is effective descent.

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2) We consider the groupoid $(T_{\bullet}^{\text{amp}})$ of small models with an ample indexing, so that $\mathfrak{pt}(\pi)$ satisfies the property of 1.

Definition: An indexing $\alpha : \kappa \rightharpoonup\!\!\!\!\rightharpoonup M$ is ample if $|\kappa \setminus \mathrm{dom}(\alpha)| = |\kappa|$.

$$\begin{tikzcd} \kappa \arrow[d, two heads, "\alpha"'] \arrow[dr, dashed, two heads, "(\beta_s)"] & \\ M_p \arrow[r] & \int_{s:\mu} M_{q_s} \end{tikzcd}$$

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3) A theorem of Wrigley 2023 ensures that the ample condition is not too strong, i.e., that $(T_\bullet^{\mathsf{amp}})$ represents $\mathscr{E}$.

The duality

We have shown:

$\mathrm{ev} : \mathscr{E} \xrightarrow{\sim} \mathrm{vUlt}(\mathrm{pt}(\mathscr{E})_{\restriction X}, \mathrm{Set})$ is an equivalence for $X \subseteq \mathrm{pt}(\mathscr{E})$ small separating.

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We can generalize:

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Proof: note that $\mathfrak{pt}(\mathscr{E}) \simeq \mathrm{colim}_A\, \mathfrak{pt}(\mathscr{E})_{\restriction A}$, where A range over small separating sets of points of $\mathscr{E}$, to reduce to the above case. $\square$

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We get a pseudoidempotent 2-adjunction

$$\begin{tikzcd} \mathrm{GTop} \arrow[r, phantom, "\perp"] \arrow[r, bend right, "\mathfrak{pt} := \mathrm{GTop}(\mathrm{Set}, -)"'] & \mathrm{vUlt}_{bounded} \arrow[l, bend right, "\mathrm{vUlt}(-, \mathrm{Set}) =: \mathfrak{sh}"'] \end{tikzcd}$$

inducing a fully faithful embedding $\mathfrak{pt} : \mathrm{GTop}_{\mathrm{wep}} \hookrightarrow \mathrm{vUlt}$.

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This adjunction induces a fully faithful embedding $\mathfrak{pt} : \mathsf{GTop}_{\mathsf{wep}} \hookrightarrow \mathsf{vUlt}$ extending Lurie's reconstruction theorem to all topoi with enough points.

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Virtual ultracategories fit in the framework of generalized multicategories of Cruttwell and Shulman 2010, and ultracategories are recovered as the representable ones.

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Proof of the 0-dimensional case

We prove the non-trivial implication,

$$p \text{ is étale} \quad \Leftarrow \quad \begin{array}{l} \forall\ \mu \succcurlyeq p(e),\ \exists!\ \nu \succcurlyeq e,\ p_*(\nu) = \mu \\ \text{and this } \nu \text{ is isomorphic to } \mu \text{ via } p. \end{array}$$

- p_* is étale
- the hypothesis gives a lift σ
- the topological fact $(\star)$ gives V
- we pullback along δ
  not continuous!
- $\delta^* T \subseteq T$ is not open
  and ξ not continuous:
  we restrict to $W \subseteq V$.

$$\begin{tikzcd}
 & & \{\nu \mid \nu \cong p_*(\nu)\} \arrow[d, "p_*"', "\text{ét}"] & \\
\{\mu \mid \mu \succcurlyeq p(e)\} \arrow[urr, dashed, bend left, "\sigma"] \arrow[rr, hook] \arrow[dr, hook] & & \beta(T) & E \arrow[ul, "\delta"'] \arrow[d] \\
 & V \arrow[ur, hook, "\text{open}"] \arrow[uur, dashed] & & T \arrow[ul, "\delta"'] \\
 & & \delta^* V \arrow[ul, "\delta"] \arrow[ur, hook] \arrow[uur, dashed, "\xi"] &
\end{tikzcd}$$

$$W \coloneqq \{w \in \delta^* V \mid \forall \mu \succcurlyeq w\ (\xi \text{ is defined on } \mu \text{ and } \xi_*(\mu) \succcurlyeq \xi(w))\}$$

$(\star)$ a section on a compact of a Hausdorff space can be extended to an open. $\square$