DOUBLE CATEGORICAL EQUIVALENCES

joint with Lyne Moser and Paula Verdugo

This project started at the program
"TOPOLOGY, REPRESENTATION THEORY AND HIGHER STRUCTURES"
at the Isaac Newton Institute, Isle of Skye

organized by Tara Brendle, Ran Levi, Ehud Meir, Simona Paoli, Ana Ros Camacho and Markus Upmeier.

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Recall...

A 2-category has...

objects $\quad \bullet \quad \bullet \quad \bullet$

morphisms $\quad \bullet \longrightarrow \bullet$

2-cells $\quad$ $$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

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<u>Defn</u> A 2-functor $F : \mathcal{A} \to \mathcal{B}$ is a biequivalence if it's

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horizontal equivalence

$$\begin{tikzcd} FA \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d, phantom, "\Downarrow \cong"] & FA \arrow[d, equal] \\ FA \arrow[rr, equal] & & FA \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[d, equal] & FA \arrow[r] \arrow[d, phantom, "\Downarrow \cong"] & B \arrow[d, equal] \\ B \arrow[rr, equal] & & B \end{tikzcd}$$

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- Looking at $\mathrm{DblCat} = \mathrm{Cat}(\mathrm{Cat})$

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Apply to $\mathcal{A} = \mathrm{DblCat}_h$ (double cats / double functors / hor. nat. tr)

$F : \mathbb{A} \to \mathbb{B}$ weak equiv $\iff \exists\, G : \mathbb{B} \to \mathbb{A}$ + hor. nat. isos

$$FG \cong \mathrm{id}, \quad GF \cong \mathrm{id}.$$

- Recall: $T : \mathcal{A} \longrightarrow \mathcal{A}$ 2-monad $\rightsquigarrow$ there is a model str. on T-algebras with weak equivs = underlying morphism is an equiv. in $\mathcal{A}$ [Lack].

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All these studied by Fiore–Paoli–Pronk [FPP].

Super interesting — but not quite what we're looking for

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HOW? Using homotopy theory.

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UPSHOT: now there is a lot more structure we can use.

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ASSUMPTION 1 Any good notion of equivalence will be part of a model structure.

ASSUMPTION 2 These should be the "canonical trivial fibrations",

i.e. the canonical equivalences should be weak equivalences in a model structure with this class of trivial fibrations.

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$$\varnothing \xrightarrow{i_1} \bullet \qquad\qquad \bullet \quad \bullet \xrightarrow{i_2} \bullet \longrightarrow \bullet \qquad\qquad \bullet \quad \bullet \xrightarrow{i_3} \begin{array}{c}\bullet\\ \downarrow\\ \bullet\end{array}$$

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \xrightarrow{i_4} \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \qquad\qquad \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow\ \Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \xrightarrow{i_5} \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

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<u>Thm</u> [MSV] Every model structure on DblCat with the canonical trivial fibrations is left proper.

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Defn A companion pair in a double category $\mathbb{A}$ is the data of

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\varphi"] & B \arrow[d, equal] \\ B \arrow[r, equal] & B \end{tikzcd} \quad , \quad \begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\psi"] & A \arrow[d, "u"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

such that

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Defn A gregarious equivalence in $\mathbb{A}$ is a companion pair (f, u, φ, ψ) such that:

- f is a horizontal equiv
- u is a vertical equiv.

Defn [Campbell] $F : \mathbb{A} \to \mathbb{B}$ is a gregarious double equiv. if it's:

- surjective on obj. up to gregarious equiv.

  i.e. both
$$\begin{tikzcd} FA \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & B \\ B & \end{tikzcd}$$
  + compatibility

- full on horizontal & vertical mor. up to globular iso

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$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, equal] \arrow[dr, phantom, "\cong"] & FA' \arrow[d, equal] \\ FA \arrow[r, "g"'] & FA' \end{tikzcd} \quad , \quad \begin{tikzcd} FA \arrow[r, equal] \arrow[d, "Fu"'] \arrow[dr, phantom, "\cong"] & FA \arrow[d, "v"] \\ FA' \arrow[r, equal] & FA' \end{tikzcd}$$

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Thm [Campbell, MSV] There is a model str. on DblCat with:

- weak equivalences = gregarious double equivs
- trivial fibrations = canonical trivial fibrations

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Cor [MSV] Any other such model structure has the gregarious double equivs as the weak equivs between fibrant objects.

FIRST GOAL: ✓

NEW GOAL: Further understand / construct ex's of other model structures w/ canonical trivial fibrations.

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In theory, this gives a complete answer to GOAL 2.

In practice, Bousfield localizations can be tricky to understand.

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FACT

$$\left.\begin{array}{l}\cdot \text{ trivial fibrations}\\ \cdot \text{ fibrant objects}\end{array}\right\}\ \text{completely determine the model structure}$$

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Thm [MSV] Any combinatorial model str. on DblCat w/ the canonical trivial fibrations arises from this recipe.

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$\mathrm{DblCat}_{h,eqp}$	every hor. mor. has a companion & a conjoint	equivalence invariance of formal category theory [Ver]

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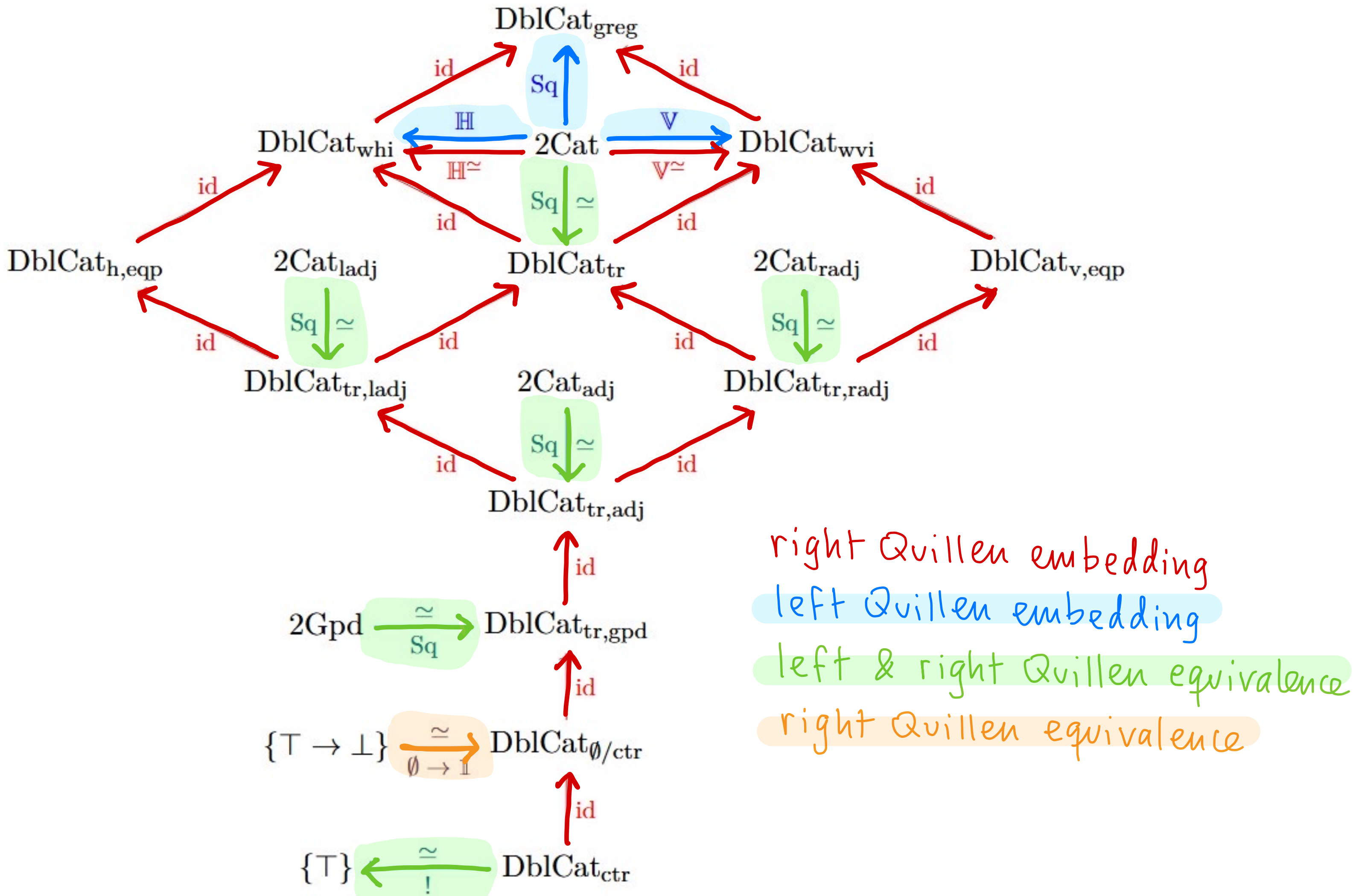
DblCat_{greg}	all	canonical, "initial" [Campbell]
DblCat_{tr}	every hor. & ver. mor. has a companion	homotopy theory of 2-categories
$\mathrm{DblCat}_{h,eqp}$	every hor. mor. has a companion & a conjoint	equivalence invariance of formal category theory [Ver]
DblCat_{whi}	every hor equiv. has a companion	right Quillen nerve [Moser] [MSV2] $N: \mathrm{DblCat}_{whi} \longrightarrow \mathrm{Dbl}(\infty,1)\mathrm{Cat}$

EXAMPLES:

	fibrant objects	properties
$\mathrm{DblCat}_{\mathrm{greg}}$	all	canonical, "initial" [Campbell]
$\mathrm{DblCat}_{\mathrm{tr}}$	every hor. & ver. mor. has a companion	homotopy theory of 2-categories
$\mathrm{DblCat}_{h,\mathrm{eqp}}$	every hor. mor. has a companion & a conjoint	equivalence invariance of formal category theory [Ver]
$\mathrm{DblCat}_{\mathrm{whi}}$	every hor equiv. has a companion	right Quillen nerve $N: \mathrm{DblCat}_{\mathrm{whi}} \longrightarrow \mathrm{Dbl}(\infty,1)\mathrm{Cat}$ [Moser] [MSV2]
$\mathrm{DblCat}_{\mathrm{tr,gpd}}$	double groupoids + every hor. & ver. mor. has a companion	homotopy theory of 2-groupoids


DblCat_greg
DblCat_whi
2Cat
DblCat_wvi
DblCat_h,eqp
2Cat_ladj
DblCat_tr
2Cat_radj
DblCat_v,eqp
DblCat_tr,ladj
2Cat_adj
DblCat_tr,radj
DblCat_tr,adj
2Gpd
DblCat_tr,gpd
{⊤ → ⊥}
DblCat_∅/ctr
∅ → 𝟙
{⊤}
DblCat_ctr
id
Sq
ℍ
𝕍
ℍ^≃
𝕍^≃
≃
!
right Quillen embedding
left Quillen embedding
left & right Quillen equivalence
right Quillen equivalence

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