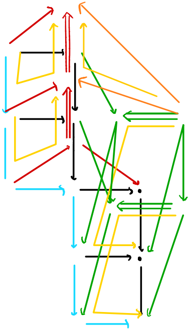
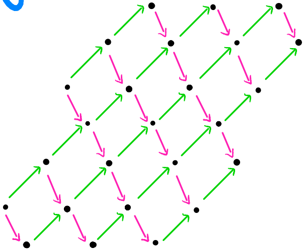
ENHANCEMENTS of QUIVERS with RELATIONS



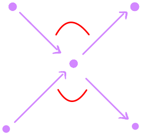


Chiara Sava

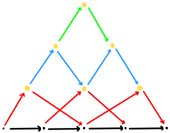
CHARLES UNIVERSITY

INTERNATIONAL CATEGORY THEORY CONFERENCE

CT2025



Brno, 14th July 2025



Representation Theory

Consider the QUIVER (:= oriented graph)

$$A_n \qquad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \xrightarrow{\alpha_3} \bullet \;\cdots\; \xrightarrow{\alpha_{n-1}} \underset{n}{\bullet}$$

We can associate an algebra to the quiver called: PATH ALGEBRA KA_n, K field

$$K\langle \text{paths} \rangle \qquad\qquad pq = \begin{cases} \text{concatenation} \\ 0 \end{cases}$$

In Representation Theory we are interested in studying:

the DERIVED CATEGORY $\mathcal{D}(\mathrm{mod}\, KA_n) := Ch(\mathrm{mod}\, KA_n)[q.isos^{-1}]$

But: here we don't have functorial limits and colimits

Then: we work with the derivator enhancement $\mathbb{D}_K^{A_n}$

Derivator Enhancement

$$A_n \quad \overset{\bullet}{1} \xrightarrow[\alpha_1]{} \overset{\bullet}{2} \xrightarrow[\alpha_2]{} \overset{\bullet}{3} \xrightarrow[\alpha_3]{} \bullet \quad \cdots \quad \xrightarrow[\alpha_{n-1}]{} \overset{\bullet}{n} \qquad \mathcal{D}(\mathrm{mod}\, kA_n) \qquad kA_n$$

Def. The DERIVATOR $\mathbb{D}_k^{A_n}$ is the 2-functor

$$\mathbb{D}_k^{A_n} : \quad \mathrm{Cat}^{op} \longrightarrow \mathrm{CAT}$$

$$J \longmapsto \mathcal{D}(\mathrm{mod}\, k^{A_n \times J})$$

Derivator Enhancement

$$A_n \quad \overset{\cdot}{1} \xrightarrow{\alpha_1} \overset{\cdot}{2} \xrightarrow{\alpha_2} \overset{\cdot}{3} \xrightarrow{\alpha_3} \cdot \;\cdots\; \xrightarrow{\alpha_{n-1}} \overset{\cdot}{n} \qquad \mathcal{D}(\mathrm{mod}\, kA_n/I) \qquad kA_n/I\,,\quad I = \langle \alpha_1 \alpha_2 \rangle$$

Def. The DERIVATOR $\mathbb{D}_k^{A_n/I}$ is the 2-functor

$$\mathbb{D}_k^{A_n} : \quad \mathrm{Cat}^{op} \longrightarrow \mathrm{CAT}$$
$$J \longmapsto \mathcal{D}(\mathrm{mod}\, k^{A_n \times J})$$

?

what if : WE IMPOSE RELATIONS ?

Derivator Enhancement

$A_n \quad 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \bullet \; \cdots \; \xrightarrow{\alpha_{n-1}} n \qquad \mathcal{D}(\mathrm{mod}\, kA_n/I) \qquad kA_n/I, \quad I = \langle \alpha_1 \alpha_2 \rangle$

Def. The DERIVATOR $\mathbb{D}_k^{A_n/I}$ is the 2-functor ?

$$\mathbb{D}_k^{A_n} : \mathrm{Cat}^{op} \longrightarrow \mathrm{CAT}$$
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what if: WE IMPOSE RELATIONS?

$$\mathbb{D}_k^{\bullet \to \bullet \to \bullet \to \bullet \; \cdots \; \to \bullet}$$

Derivator Enhancement

$$A_n \quad \underset{1}{\bullet} \xrightarrow{\alpha_1} \underset{2}{\bullet} \xrightarrow{\alpha_2} \underset{3}{\bullet} \xrightarrow{\alpha_3} \bullet \;\cdots\; \xrightarrow{\alpha_{n-1}} \underset{n}{\bullet}$$

$\mathcal{D}(\mathrm{mod}\, kA_n/I)$ $\qquad$ kA_n/I, $\; I = \langle \alpha_1 \alpha_2 \rangle$

Def. The DERIVATOR $\mathbb{D}_k^{A_n/I}$ is the 2-functor ?

$$\mathbb{D}_k^{A_n} : \quad \mathrm{Cat}^{op} \longrightarrow \mathrm{CAT}$$

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what if: WE IMPOSE RELATIONS?

$$\begin{tikzcd}
 & & \bullet \arrow[dr] & & & & \\
\mathbb{D}_k & \bullet \arrow[ur] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet & \cdots \arrow[r] & \bullet
\end{tikzcd}$$

Derivator Enhancement

$$A_n \quad \begin{tikzcd} 1 \arrow[r, "\alpha_1"'] & 2 \arrow[r, "\alpha_2"'] & 3 \arrow[r, "\alpha_3"'] & \bullet & \cdots \arrow[r, "\alpha_{n-1}"'] & n \end{tikzcd}$$

$\mathcal{D}(\mathrm{mod}\, kA_n/I)$ $\qquad kA_n/I$, $I = \langle \alpha_1 \alpha_2 \rangle$

Def. The DERIVATOR $\mathbb{D}_k^{A_n/I}$ is the 2-functor ?

$$\begin{aligned} \mathbb{D}_k^{A_n} : \quad \mathrm{Cat}^{op} &\longrightarrow \mathrm{CAT} \\ J &\longmapsto \mathcal{D}(\mathrm{mod}\, k^{A_n \times J}) \end{aligned}$$

what if : WE IMPOSE RELATIONS ?

SUBDERIVATOR $\quad \mathbb{D}_k^{A_n/I} \subseteq \mathbb{D}_k^{\bullet \to \bullet \to \bullet \to \bullet \cdots \to \bullet}$

Enhancements of quadratic monomial relations

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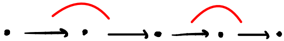
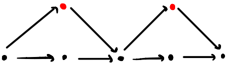
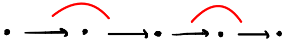
Enhancements of quadratic monomial relations

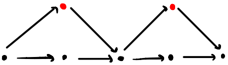


$\mathbb{D}_k$



Enhancements of quadratic monomial relations

- 



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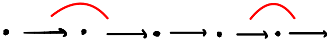
- 

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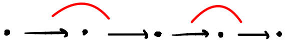
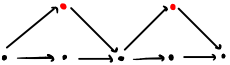
- 

$\mathbb{D}_k$ 

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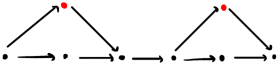
$\mathbb{D}_k$ 

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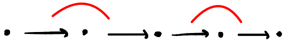
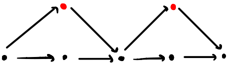
- 

$\mathbb{D}_k$ 

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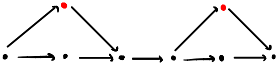
$\mathbb{D}_k$ 

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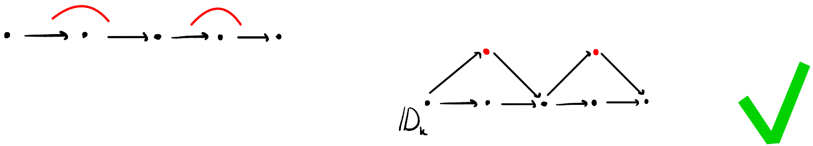

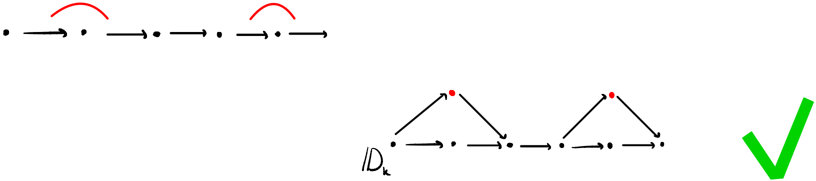

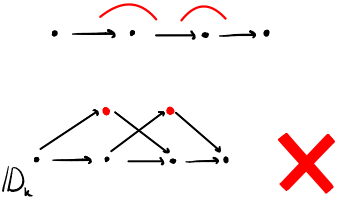

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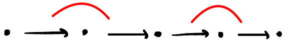
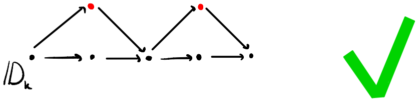
- 



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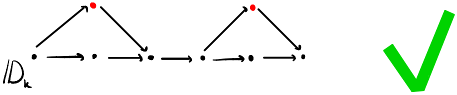

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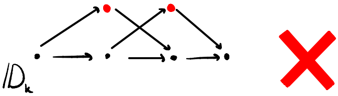
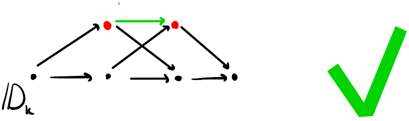
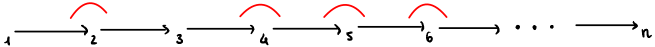
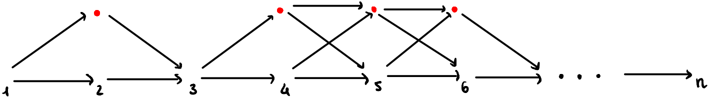




Enhancements of quadratic monomial relations

Ex. A_n/I



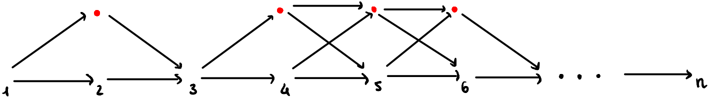


$$\mathbb{D}_k^{A_n/I} = \mathbb{D}_k$$

Enhancements of quadratic monomial relations

Ex. A_n/I

$$\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & 3 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 6 \arrow[r] & \cdots \arrow[r] & n \end{tikzcd}$$



$\mathbb{D}_k^{A_n/I} = \mathbb{D}_k$

Prop. [S.] If I is generated by compositions of two consequent arrows, $\mathbb{D}_k^{A_n/I}$ is the derivator enhancement of $\mathcal{D}(\mathrm{mod}\, kA_n/I)$.

Enhancements of quadratic monomial relations

Ex. A_n/I

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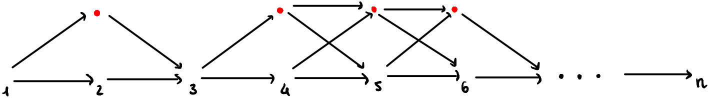

$\mathbb{D}_k^{A_n/I} = \mathbb{D}_k$

Prop. [S.] If I is generated by compositions of two consequent arrows, $\mathbb{D}_k^{A_n/I}$ is the derivator enhancement of $\mathcal{D}(\mathrm{mod}\, kA_n/I)$.

Thm. [S.] (particular case) If I is generated by compositions of two consequent arrows,

$$\mathbb{D}_k^{A_n/I} \cong \mathbb{D}_k^{A_n} \quad \forall n \geq 3.$$

Thank you!