Mnemetic Lax Idempotent Monads and Compactness

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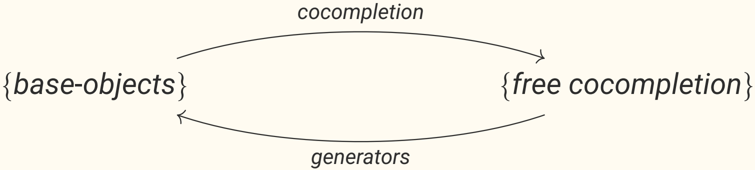
Motivation

We want to understand equivalences of the shape:



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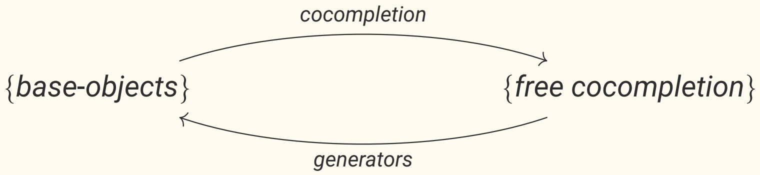
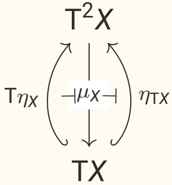


This is for example the shape of Gabriel-Ulmer Duality

Basic Notions of Lax-Idempotent monads

Definition. A pseudo 2-monad $(\mathrm{T} : \mathcal{K} \to \mathcal{K}, \mu, \eta)$ on a 2-category $\mathcal{K}$ is lax-idempotent (or KZ) when we have that for every object $X \in \mathcal{K}$:



[0]Anders Kock. **“Monads for which structures are adjoint to units”.** In: *J. Pure Appl. Algebra* 104.1 (Oct. 1995), pp. 41–59
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Examples of Lax-Idempotent Monads on Categories

Proposition. There is a l.i. monad on Cat (locally small) given by $C \mapsto \text{Ind}(C) \subset \text{Set}^{C^{op}}$ spanned by small filtered colimits of representables.

The unit is the Yoneda embedding $\text{よ} : C \to \text{Ind}(C)$.

On morphisms $F : C \to D$, $\text{Ind}(F)$ is a left kan extension:

$$\begin{array}{ccc} C & \xrightarrow[F]{} & D \\ \downarrow & & \downarrow \\ \text{Ind}(C) & \xrightarrow{lan_{\text{よ}}(\text{よ} \circ F)} & \text{Ind}(D) \end{array}$$

The multiplication $\mu_C : \text{Ind}(\text{Ind}(C)) \to \text{Ind}(C)$ takes $P \in \text{Ind}(\text{Ind}(C))$ to:

$$colim(\int P \to \text{Ind}(C))$$

Examples of Lax-Idempotent Monads on Categories

The same recipe works for other classes of colimits on Cat:

1. Finite Colimits
2. Small Coproducts aka the Families Construction
3. Small Filtered Colimits aka Ind Completion Ind
4. Small Colimits aka Small Presheaf Construction $\mathcal{P}$

And also on Pos:

1. Directed Joins aka Ideal Completion Idl
2. Arbitrary Joins aka Down-Set Monad $\mathcal{D}$

Algebras of Lax-Idempotent Monads

Proposition.[1] The (pseudo-)algebras of a lax-idempotent monad $(\mathsf{T} : \mathcal{K} \to \mathcal{K}, \mu, \eta)$ are pairs $(X, \alpha : \mathsf{T}X \to X)$ such that $\alpha \dashv \eta_X$ and $\alpha\eta_X \cong id$.

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- For $\mathcal{P}$: cocomplete categories

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For l.i. monads being an algebra is a property!

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Forms of compactness

For a Sup-Lattice X, $x \in X$ is completely join prime if for any $S \subseteq X$:

$$x \leqslant \bigvee S \Leftrightarrow \exists s \in S : x \leqslant s$$

When S is downwards closed we can restate this as:

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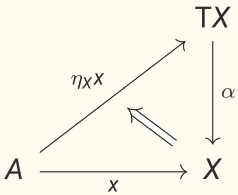
So the "compact" elements with respect to the down-set monad are the ones where the unit η_X behaves like a left adjoint to the algebra α.

General compactness

Definition. A generalized element $x : A \to X$ for an algebra (X, α) is T-compact if we have a natural bijection:

$$\frac{\eta_X x \Rightarrow u}{x \Rightarrow \alpha u}$$

stable under precomposition. Formally this means that:



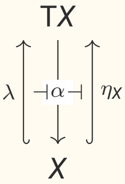
is an absolute left lifting diagram, where the 2-cell is part of the iso $\alpha\eta_X \cong id$. Note, this is best stated via pointwise left lifts!

General compactness on Categories

- For $\mathfrak{X}$ cocomplete, $x : \mathbf{1} \to \mathfrak{X}$ is $\mathcal{P}$-compact iff it is atomic in the sense that
$$\hom(x, -) : \mathfrak{X} \to \mathsf{Set}$$
preserves small colimits.
- For $\mathfrak{X}$ Ind-cocomplete, $x : \mathbf{1} \to \mathfrak{X}$ is Ind-compact when it is finitely presentable, i.e.
$$\hom(x, -) : \mathfrak{X} \to \mathsf{Set}$$
preserves all filtered colimits.

Continuous Algebras

A continuous algebra for a l.i. monad T is an algebra (X, α) with a further left adjoint:



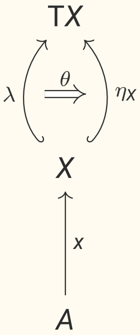
Morphisms are the algebra morphisms commuting with the left adjoint. Continuous algebras form a non-full sub-2-category T-*Cont* of T-*Alg*.

Examples.

1. For $\mathcal{D}$: completely distributive lattices
2. For Idl: continuous domains
3. For Ind: continuous categories

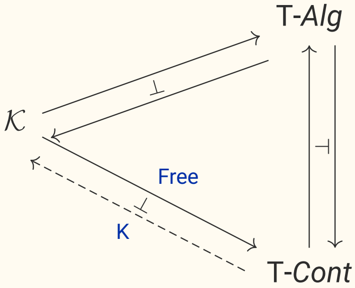
Continuous Algebras

Lemma. For a continuous algebra (X, α), a generalized element $x : A \to X$ is compact iff we have that the following induced 2-cell $\theta x : \lambda x \Rightarrow \eta_X x$ is invertible.



Theorem

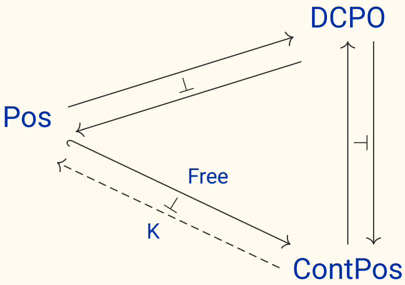
Theorem. (Frey, S.) If the 2-category $\mathcal{K}$ has inverters of adjoint cylinders, then we have the following adjunction with commuting left adjoints



where K takes any continuous algebra to the inverter of the induced cell $\theta : \lambda \Rightarrow \eta_A$.

Example

In the case of Idl, we get:



Moreover Free : T-*Cont* $\to \mathcal{K}$ is full and faithful (local equivalence).

Example

Proposition. For $\mathsf{T} = \mathsf{OP}, \mathsf{Idl}, \mathcal{D}$, the induced functor $\mathsf{Free} : \mathsf{T}\text{-}\mathit{Cont} \to \mathcal{K}$ is full and faithful.

Question. Is it always the case?

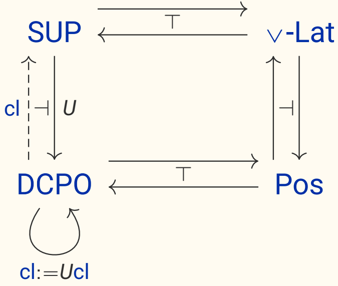
Partial Answer. No: the atomic objects of $\mathcal{P}(X)$ only recover the Cauchy completion of X.

Wish? Maybe the adjunction is always idempotent?

Counter Example

Counter-Example. The adjunction between Free $\dashv$ K : T-*Cont* $\to \mathcal{K}$ need not be idempotent.

Consider the square of adjunctions:



Unlabeled = free-forgetful → all monadic

Prop. cl $\dashv$ U exists and is lax idempotent.

Counter Example

Prop. (Frey) The lax-idempotent monad $\mathsf{cl} : \mathsf{DCPO} \to \mathsf{DCPO}$ induces a non-idempotent adjunction:

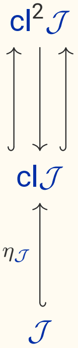
$$\mathsf{Free} \dashv \mathsf{K} : \mathsf{cl}\text{-}Cont \to \mathsf{DCPO}$$

Consider the DCPO $\mathcal{J}^2$ with underlying set $\mathbb{N} \times \mathbb{N}^\infty$ such that:

$$\begin{aligned}(m, n) \leqslant (m', n') \text{ iff } & m = m' \text{ and } n \leqslant n' (\leqslant \infty) \\ \text{or } & n' = \infty \text{ and } n \leqslant m'\end{aligned}$$

[2] Peter T. Johnstone. **"Scott is not always sober".** In: *Continuous Lattices*. Berlin, Germany, Oct. 2006, pp. 282–283. DOI: 10.1007/BFb0089911.

Prop. $\mathcal{J}$ is cl-compact in cl$\mathcal{J}$, thus clKcl$\mathcal{J} \neq$ cl$\mathcal{J}$, giving us an iteration:



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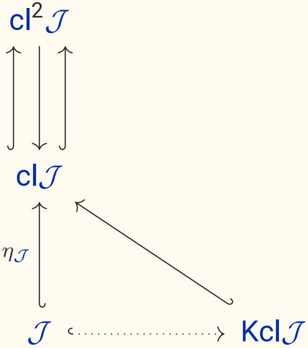

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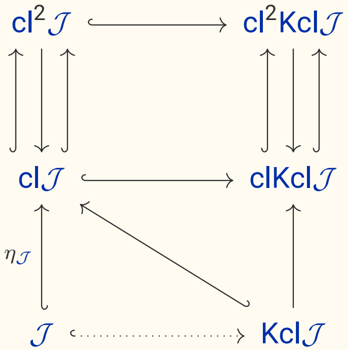

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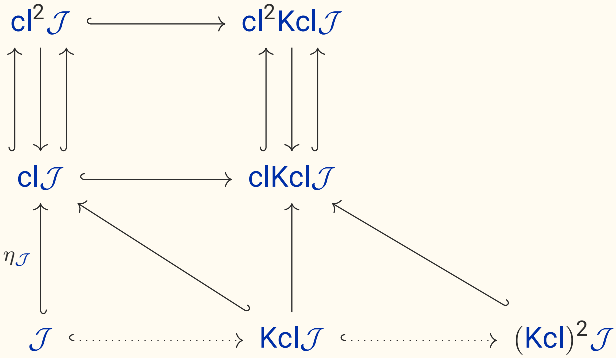

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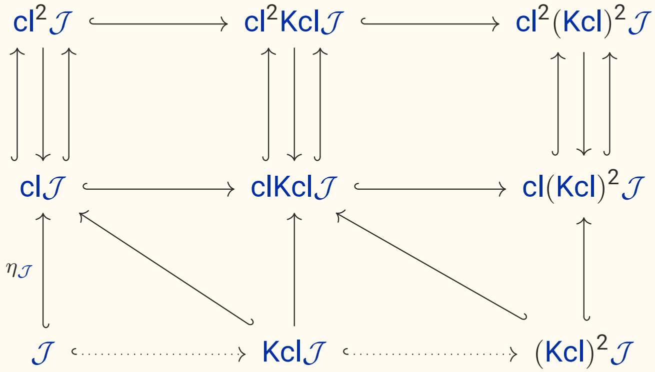

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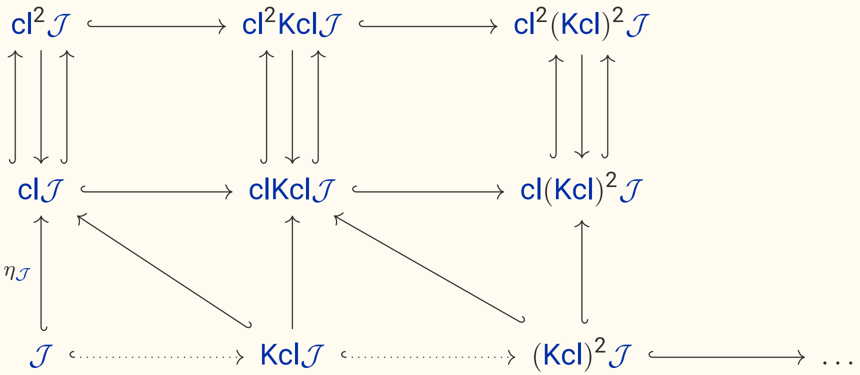

Mnemetic Monads

Definition. T is mnemetic iff the unit of the monad is the inverter of the 2-cell depicted:

$$
\begin{array}{c}
\mathsf{T}^2X \\
\mathsf{T}\eta_X \uparrow \overset{\theta}{\Longrightarrow} \uparrow \eta_{\mathsf{T}X} \\
\mathsf{T}X \\
\eta_X \uparrow \\
X
\end{array}
$$

Idea. Thus Mnemetic Monads are those where the the unit of the monad already contains all the generators.

Mnemetic Monads

Prop. Given a 2-category $\mathcal{K}$ with inverters of adjoint cylinders with a l.i. monad $\mathsf{T} : \mathcal{K} \to \mathcal{K}$, we have that Free $\dashv$ K is:

1. a coreflection iff T is mnemetic
2. idempotent iff T is pre-mnemetic
3. non-idempotent otherwise

Conclusions

What we saw:

- An abstract criterion for compactness
- A way of using it to extract theorems about free algebras

Ongoing work:

- Understand the free representable multicategory l.i. monad on multicategories through this lens
- Way-Below arrows and Continuous Algebras
- Absolute left lifts work best when viewed as pointwise left lifts: 2-cat $\to$ equipments