Principal bundles in join restriction categories

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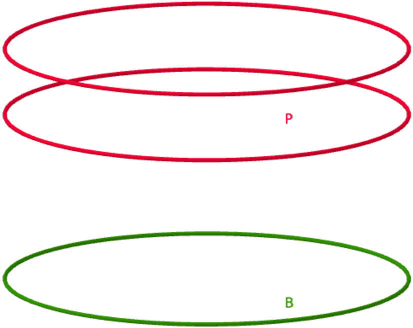
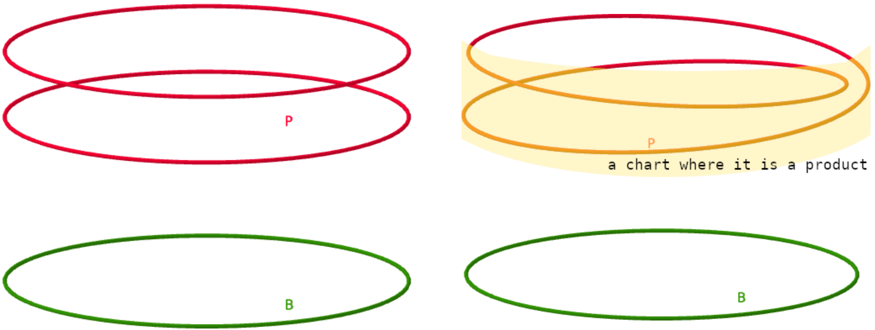
Space times group



Space times group



Three definitions

A principal bundle is...

A map $P \to B$ of topological spaces with a free right G-action on P where P is locally $B \times G$.

Consequence: Changes of charts are the left-multiplication with a group-element $g \in G$.

A map $P \to B$ of manifolds that where P is locally $B \times G$ and the changes of charts are by left multiplication with a group-element $g \in G$.

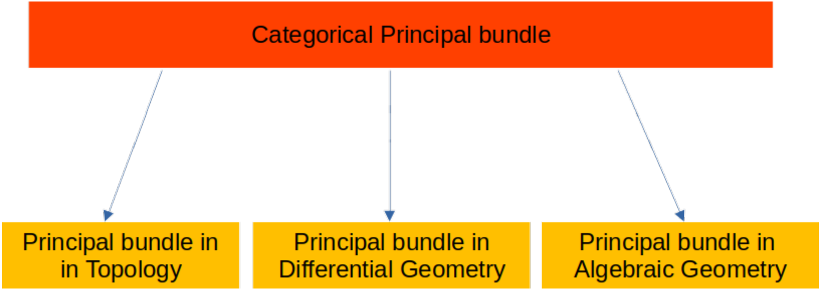
Consequence: There is a right action of G on P.

A map $P \to B$ of schemes with a G-action that is a geometric quotient and locally (iso-)trivial in étale topology.

Consequence: The G-action on P is free.

Common theme: locally space times group

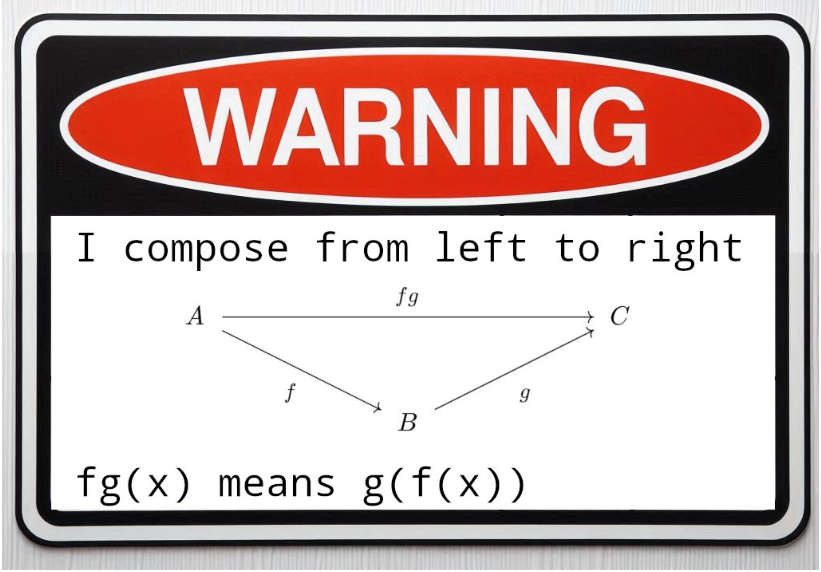
Vision


Categorical Principal bundle
Principal bundle in
in Topology
Principal bundle in
Differential Geometry
Principal bundle in
Algebraic Geometry

Outline

- Restriction categories
- Joins

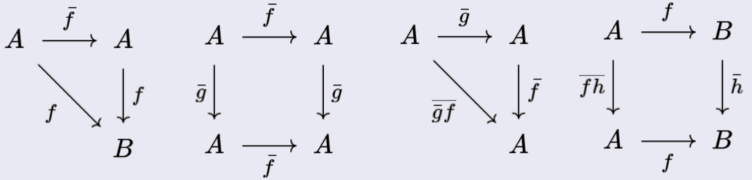
- Fiber bundles
- Principal bundles
- Application to classsical settings


WARNING
I compose from left to right
fg
A
C
f
g
B
fg(x) means g(f(x))

Definition

A **restriction category** has for each map $f : A \to B$ a map $\bar{f} : A \to A$ fulfilling the conditions (R.1) - (R.4).

(R.1) $\bar{f}f = f$ (R.2) $\bar{g}\bar{f} = \bar{f}\bar{g}$ (R.3) $\overline{\bar{g}f} = \bar{g}\bar{f}$ (R.4) $f\bar{h} = \overline{fh}f$



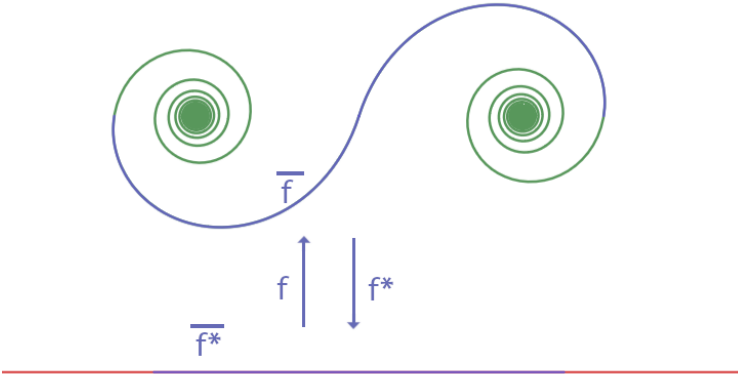
A map f is called **total** if $\bar{f} = 1_A$.

Think partial maps!

Examples

- ParSet:
  - Objects: sets
  - Morphisms: partial maps on subsets
- ParTop:
  - Objects: topological spaces
  - Morphisms: partial continuous maps on open subsets
- ParSmooth:
  - Objects: real finite dimensional vector spaces
  - Morphisms: partial smooth maps on open subsets
- ParMfld:
  - Objects: smooth manifolds
  - Morphisms: partial smooth maps on open subsets

A map f is a **partial isomorphism** if it has a **partial inverse** f^* fulfilling $ff^* = \overline{f}$ and $f^* f = \overline{f^*}$.

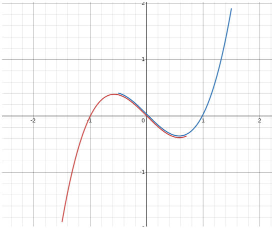


The partial inverse is unique

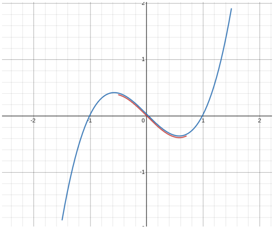
Dominance and Compatibility

Definition

1. Two parallel maps f,g in a restriction category are **compatible**, written $f \smile g$ if $\bar{f}g = \bar{g}f$
2. The map $g : A \to B$ **dominates** the map $f : A \to B$, written as $f \leq g$, if $\bar{f}g = f$. This defines a partial order



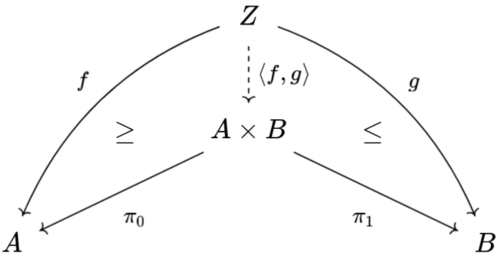
$f \smile g$



$f \leq g$

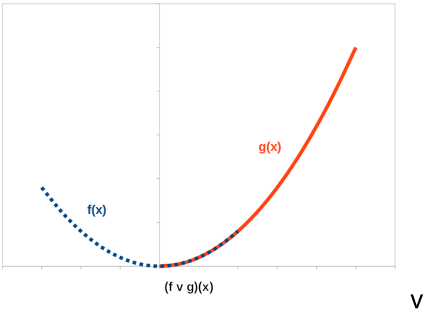
Restriction limits

There is a notion of **restriction limits** in restriction categories where the induced map doesn't need to strictly commute but only commute up to $\leq$.



Joins

If partial maps f and g coincide on the intersection of their domains, i.e. $f \smile g$, we can put them together to a big function $f \vee g$.



Joins

Definition

A **join restriction category** X has for each $A, B \in X$ and each compatible set $S \subset Hom(A, B)$ a map

$$\bigvee_{s \in S} s : A \to B$$

and it fulfills the following properties

1. $\overline{\bigvee_{s \in S} s} = \bigvee_{s \in S} \bar{s}$
2. $g(\bigvee_{s \in S} s) = \bigvee_{s \in S} gs$
3. it is a join (supremum) with respect to the partial ordering $\leq$, i.e. $f_i \leq \bigvee f_i$ and $f_i \leq g \; \forall i \Rightarrow \bigvee f_i \leq g$

This allows to glue morphisms together.

The ingredients

Restrictions: $\bar{f} = 1|_{\mathrm{dom}(f)}$
Inverses: $f f^* = \bar{f}$, $f^* f = \bar{f^*}$
Dominance: $g \leq f \Leftrightarrow g = f|_{\mathrm{dom}(f)}$
Compatibility: $f \smile g \Leftrightarrow f|_{\mathrm{dom}(g)} = g|_{\mathrm{dom}(f)}$
Limits: universal property up to $\leq$
Joins: For $f \smile g$: $f \vee g \geq f$ and $f \vee g \geq g$

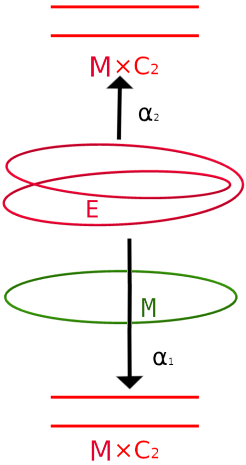
Let $\mathbb{X}$ be a join-restriction category.

Definition

A **fiber bundle** over $M \in \mathrm{Ob}(\mathbb{X})$ with typical fiber $F \in \mathrm{Ob}(\mathbb{X})$ is an object $E \in \mathrm{Ob}(\mathbb{X})$ with a total map $q : E \to M$ and a family of partial isomorphisms $(\alpha_i : E \to M \times F)_{i \in I}$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\alpha_i} & M \times F \\ {\scriptstyle \bar{\alpha}_i}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ E & \xrightarrow[q]{} & M \end{array}$$

commutes and $\bigvee_{i \in I} \bar{\alpha}_i = 1_E$ and $\bar{\alpha}_i^* = e_i \times 1$ for a map $e_i = \bar{e}_i : M \mapsto M$.



Group objects

Definition

In a category with products, a **group object** is an object $G \in \mathrm{Ob}(\mathbb{X})$ together with (total) morphisms

$$u : 1 \to G \qquad m : G \times G \to G \qquad i : G \to G$$

fulfilling associativity, unit and inverse properties.

Lie groups are group objects in the category of smooth manifolds. Topological groups are group objects in the category of topological spaces.

Definition

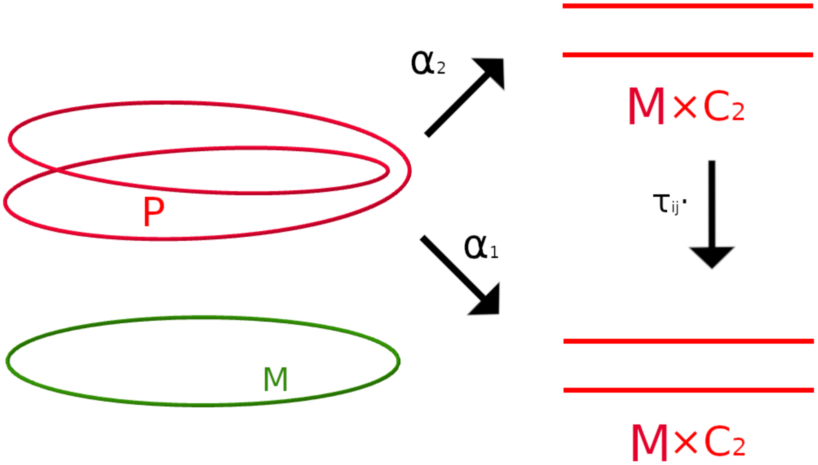
Let (G, u, m, i) be a group object and M any other object in a join restriction category. Then a G-atlas on M consists of partial maps $\tau_{ij} : M \to G$ such that

$$(\tau_{ij}, \tau_{jk})m \leq \tau_{ik} \qquad \tau_{ii} \leq !u \qquad \tau_{ji} = \tau_{ij} i$$

Definition

A **principal G-bundle** over M consists of a fiber bundle $(q : P \to M, (\alpha_i)_{i \in I})$ with fiber G and a G-atlas τ_{ij} on M such that

$$u_{ij} = \alpha_i^* \alpha_j = (\pi_0, (\pi_0 \tau_{ji}, \pi_1)m) : M \times G \to M \times G$$


P
M
α2
α1
M×C2
τij·
M×C2

Theorem (Cockett, S.)

Let $q : P \to M$ be a principal G-bundle. Then there exists a total map

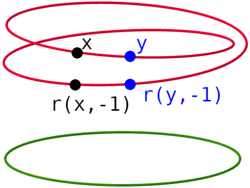
$$r : P \times G \to P$$

such that

$$\begin{CD} P \times G \times G @>{r \times 1}>> P \times G \\ @V{1 \times m}VV @VV{a}V \\ P \times G @>>{a}> P \end{CD} \qquad \begin{array}{ccc} P & \xrightarrow{1 \times u} & P \times G \\ {\scriptstyle 1}\downarrow & \swarrow{\scriptstyle r} & \\ P & & \end{array}$$

$$\begin{CD} P \times G @>{r}>> P \\ @V{\pi_0}VV @VV{p}V \\ P @>>{q}> M \end{CD} \qquad \begin{CD} P \times G @>{r}>> P \\ @V{\alpha_i \times 1}VV @VV{\alpha_i}V \\ M \times G \times G @>>{1 \times m}> M \times G \end{CD}$$

commute.



Vertical bundle

Given a principal G-bundle $P \xrightarrow{q} M$ in a tangent join restriction category, the **vertical bundle** $T_0(P)$ and the **tangent space at the unit** T_uG are the pullbacks

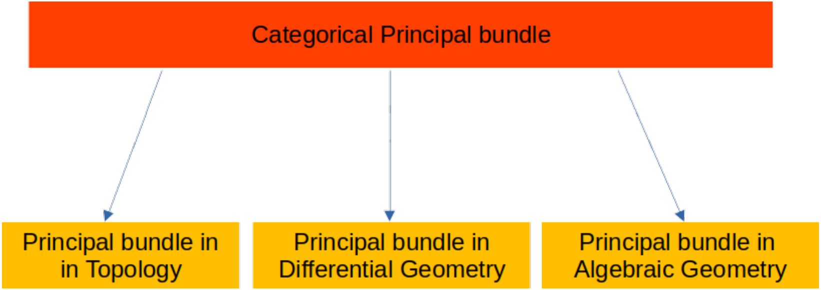
$$
\begin{array}{ccc}
T_0(P) & \xrightarrow{Tq^*(0)} & T(P) \\
{\scriptstyle pq}\downarrow & \lrcorner & \downarrow{\scriptstyle T(q)} \\
M & \xrightarrow[0]{} & T(M)
\end{array}
\qquad
\begin{array}{ccc}
T_u(G) & \xrightarrow{p_u^*} & T(G) \\
{\scriptstyle !}\downarrow & \lrcorner & \downarrow{\scriptstyle p} \\
1 & \xrightarrow[u]{} & G
\end{array}
.
$$

Theorem (Cockett, S.)

If $T_u(G)$ exists, the vertical bundle T_0G exists and

$$T_0(P) \cong P \times T_u(G).$$

Same as classically?



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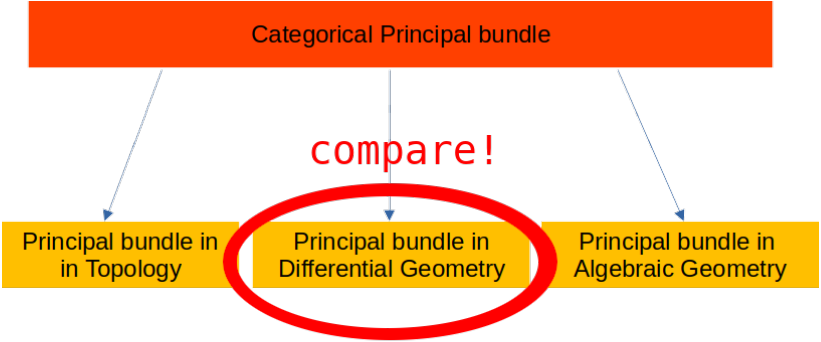


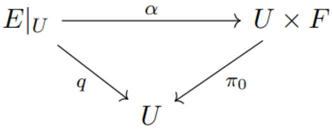
Same as classically?

Definition

A **classical fiber bundle** (E, q, M, F) consists of manifolds E, M, F and a smooth mapping $q : E \to M$; furthermore each $x \in M$ has an open neighbourhood U such that

$$E|_U := q^{-1}(U) \cong U \times F$$

via a fiber respecting diffeomorphism.



Just the same as a principal bundle in partial manifolds?

Same as classically?

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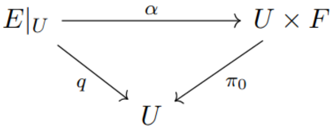
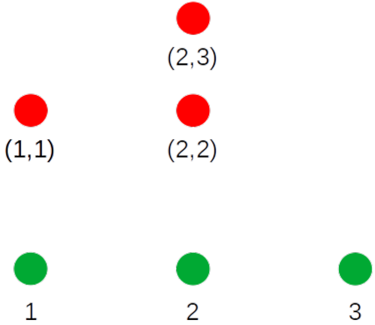


Just the same as a principal bundle in partial manifolds? No!

Not the same as classically

$M = \{1,2,3\}$, $E = \{(1,1),(2,1),(2,2)\}$ with $q = \pi_0$ form a principal $\{1\}$-bundle of smooth manifolds.



The difference

$$
\begin{array}{ccc}
q^{-1}(U) & \xrightarrow{\alpha_U} & U \times F \\
& {\scriptstyle q} \searrow \quad \swarrow {\scriptstyle \pi_0} & \\
& U &
\end{array}
\qquad\qquad
\begin{array}{ccc}
V & \xrightarrow{\alpha_i} & U \times F \\
& {\scriptstyle q} \searrow \quad \swarrow {\scriptstyle \pi_0} & \\
& U &
\end{array}
$$

The difference

$$\begin{array}{ccc} q^{-1}(U) & \xrightarrow{\alpha_U} & U \times F \\ & {\scriptstyle q} \searrow \quad \swarrow {\scriptstyle \pi_0} & \\ & U & \end{array} \qquad\qquad \begin{array}{ccc} V & \xrightarrow{\alpha_i} & U \times F \\ & {\scriptstyle q} \searrow \quad \swarrow {\scriptstyle \pi_0} & \\ & U & \end{array}$$

Remember that $\bar{\alpha}_i^* = e_i \times 1_F$

Definition

A fiber bundle is **totally fibered** if $\overline{qe_i} = \bar{\alpha}_i$ for all $i \in I$.

Application to classical settings

- **Differential Geometry:** Let G be a Lie-Group. A *classical principal G-bundle* is exactly the same as a totally fibered principal G-bundle in the join-restriction category $\mathrm{ParMfld}$ with an epic projection map $q : P \to M$.
- **Topology:** For a topological group G, the category of *classical principal G-bundles* over M is isomorphic to the subcategory of categorical principal bundles where
  - the base space of every object is M,
  - every bundle $P \xrightarrow{q} M$ is totally fibered,
  - the projection map $q : P \to M$ is surjective, and
  - every morphism $(f, g) : (P, M, q) \to (P', M, q')$ has the identity as its second component: $g = 1_M$
- **Algebraic Geometry:** How can one get étale-partial maps in $\mathrm{CAlg}_R^{\mathrm{op}}$ as a join-restriction category? Working on it with Geoff Vooys, but it is hard.

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Turing Categories

A **Turing category** is

- a Cartesian restriction category
- with a Turing object, i.e. an object T
  - with a family of (partial) maps $\tau_{X,Y} : T \times X \to Y$
  - such that for every morphism $f : Z \times X \to Y$ there is a total morphism $h_f : Z \to T$ making

$$\begin{array}{ccc} T \times X & \xrightarrow{\tau_{X,Y}} & Y \\ {\scriptstyle h\times X}\uparrow & \nearrow_{f} & \\ Z \times X & & \end{array}$$

commute.

In a Turing category with Turing object T, **what does it mean for $T \xrightarrow{p} M$ to be a principal bundle?**

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