A Constructive Small Object Argument

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$(\mathcal{L}, \mathcal{R})$ is a *weak factorisation system* on a category $\mathcal{C}$ if

- Every map $f \in \mathcal{C}$ factors as

$$X \xrightarrow[\in \mathcal{L}]{Lf} Ef \xrightarrow[\in \mathcal{R}]{Rf} Y,$$

- $(\ldots)$

Examples

- (surjections, injections) is a WFS on the category of sets,
- (injections, surjections) is also a WFS on the category of sets,
  Equivalent to the axiom of choice!
- (cofibrations, homotopy equivalences) is a WFS on the category of topological spaces,
- (surjections, injections) is a WFS on the category of groups.

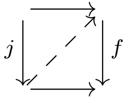
And many more!

We can make the notion of a weak factorisation system more ‘algebraic’, the result is called an *algebraic weak factorisation system*.
It consists of a pair of a comonad and a monad (L, R), its left class is L-**Coalg** and its right class is R-**Alg**.

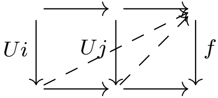
This gives: explicit lifts, explicit factorisations, . . .

Lifting Structures

- Set, $\mathcal{J} \subseteq \mathcal{C}^{\mathbf{2}}$

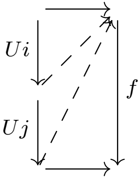


- Category, $U : \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$



(horizontal condition)

- Double category, $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$



(vertical condition)

Small Object Argument

- [Quillen (1967)], small object argument for sets;
- [Garner (2009)], algebraic small object argument for cofibrant generation by a small category;
- [Bourke & Garner (2016)], algebraic small object argument for cofibrant generation by a small double category.

A *constructive* small object argument ...?

- Voevodsky's construction of a model of homotopy type theory in simplicial sets.
- Problem: BCP-obstruction (Bezem, Coquand & Parmann), *constructively unprovable* that this is a model of HoTT.
- Solution 1: definition of a *uniform Kan fibration* in cubical sets [Coquand et al. (2015)].
- Solution 2: definition of an *effective Kan fibration* in simplicial sets [van den Berg & Faber (2022)].
- Important step: a proof that these are the right class in an algebraic weak factorisation system.
- We need a constructive small object argument!

Finitary Small Object Argument

Theorem 1 ([Seip (2024), Theorem 6])

Let $\mathcal{C}$ be a locally small, cocomplete category, and let $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ be a double functor subject to the following conditions

1. *$\mathbb{J}$ is small,*
2. *the object Uj is ω-compact for every object $j \in \mathcal{J}_0$.*

Then the AWFS cofibrantly generated by U exists and is finitary.

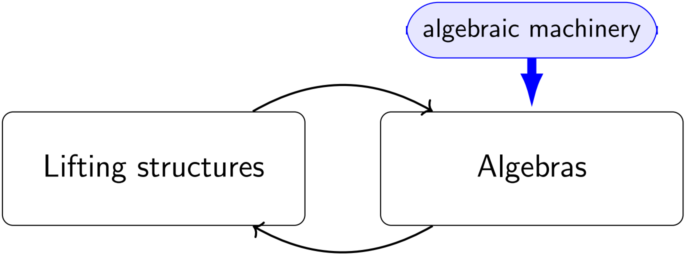
Constructing a Left Adjoint

Proposition ([Bourke (2023), Proposition 13])

$U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ cofibrantly generates an AWFS (L, R) if and only if $V_1 : \mathbb{J}_1^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ has a left adjoint.

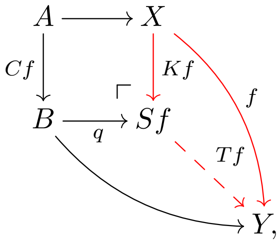
Goal: to give an explicit construction of this adjoint

Proof Strategy



Definition of T

Obtain a ‘one-step factorisation’ $f = Tf \circ Kf : X \to Sf \to Y$



with

$$Cf \cong \int^{j \in \mathcal{J}} \mathcal{C}^2(Uj, f) \cdot Uj.$$

One-step Lifting Structures

A *one-step lifting structure* from f to g consists of a square $(u, v) : f \to g$ equipped with the following lifting operation

$$\begin{array}{ccccc} A_j & \xrightarrow{\sigma_0} & X & \xrightarrow{u} & C \\ {\scriptstyle Uj}\downarrow & \phi_j(\sigma) & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B_j & \xrightarrow[\sigma_1]{} & Y & \xrightarrow[v]{} & D \end{array} \qquad (1)$$

which moreover satisfies the horizontal condition.

Universal Property

This induces a presheaf

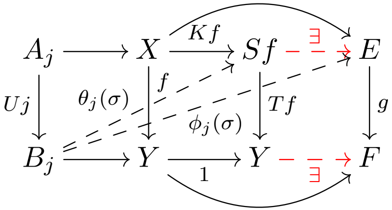
$$J\text{-1-Step}\colon (\mathcal{C}^{\mathbf{2}})^{op} \times \mathcal{C}^{\mathbf{2}} \to \mathrm{Set}$$

Theorem 2

We have an isomorphism $\phi_{f,g} \colon \mathcal{C}^{\mathbf{2}}(Tf, g) \cong J\text{-1-Step}(f, g)$ natural in each variable.

Corollary 1

The one-step lifting structure $(Kf, 1) : f \to Tf$ *is initial in the category of one-step lifting structures for* f *against* $\mathcal{J}$.

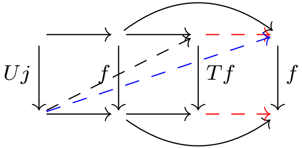


Generation by a Category

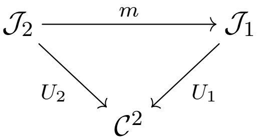
Proposition 1

We have an isomorphism T-$\mathbf{Alg} \cong \mathcal{J}^{\pitchfork}$ over $\mathcal{C}^{\mathbf{2}}$.

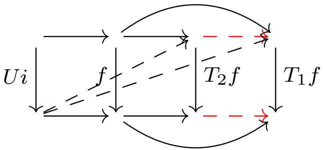
Use the universal property of T:

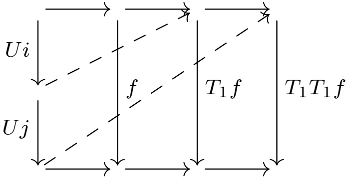


Generation by a Double Category



$\mathcal{J}_1^{\pitchfork} \cong T_1$-**Alg** and $\mathcal{J}_2^{\pitchfork} \cong T_2$-**Alg** by Proposition 1. Using the universal property of T, we have two induced natural transformation $\gamma : T_2 \Rightarrow T_1$ and $\lambda : T_2 \Rightarrow T_1T_1$.





Special Algebras

Call a T_1-algebra $\beta : Tf \to f$ special if the following diagram commutes

$$\begin{array}{ccccc} T_2 f & & \xrightarrow{\gamma_f} & & T_1 f \\ \downarrow{\scriptstyle \lambda_f} & & & & \downarrow{\scriptstyle \beta} \\ T_1 T_1 f & \xrightarrow[T_1 \beta]{} & T_1 f & \xrightarrow[\beta]{} & f. \end{array}$$

This gives a full subcategory $T_1\text{-}\mathbf{Alg}_s$ of $T_1\text{-}\mathbf{Alg}$.

Generation by a Double Category

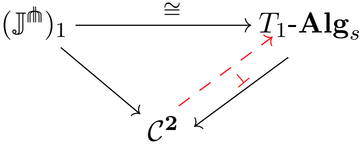
Proposition 2

We have an isomorphism $T_1\text{-}\mathbf{Alg}_s \cong (\mathbb{J}^{\pitchfork})_1$ *over* $\mathcal{C}^{\mathbf{2}}$.

We already know that $T_1\text{-}\mathbf{Alg} \cong \mathcal{J}_1^{\pitchfork}$. Furthermore, the lifting structures satisfying the vertical condition are precisely the ones whose induced T_1-algebra is special.

vertical condition $\leftrightsquigarrow$ special algebra

Constructing the Left Adjoint



Proposition 3

If T_1 preserves colimits of n-chains for some limit ordinal n, then the free special algebra exists.

The proof uses *algebraic chains*.

Theorem 3

Let $\mathcal{C}$ be a locally small, cocomplete category, and suppose we have a small double category $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ where Uj is λ-presentable for each $j \in \mathcal{J}_0$. Then the AWFS cofibrantly generated by $\mathbb{J}$ exists.

For the constructive case (Theorem 1), take $\lambda = \omega$.

Presheaf Categories

Corollary 2

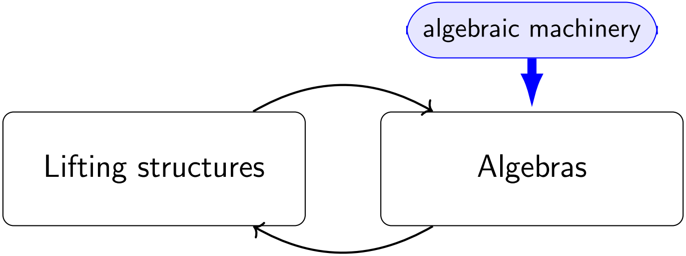
Let $\mathbb{C}$ be a small category and let $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\widehat{\mathbb{C}})$ be a double functor subject to the following conditions

1. *$\mathbb{J}$ is small,*
2. *Uj is finitely generated for every object $j \in \mathcal{J}_0$.*

Then the AWFS cofibrantly generated by U exists and is finitary.

Example: effective Kan fibrations.

Summary



vertical condition $\leftrightsquigarrow$ special algebra

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