Double functorial representation of indexed monoidal structures

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Classical hyperdoctrines

Start with

1. a category $\mathcal{C}$ of "things you want to talk about";
2. associate to each object $X \in \mathcal{C}$ a collection of "facts" that might be true of elements of type X;
3. explain how you do logical operations.

Lawvere's perspective on logic: *quantifiers are adjoints*.

Classical hyperdoctrines

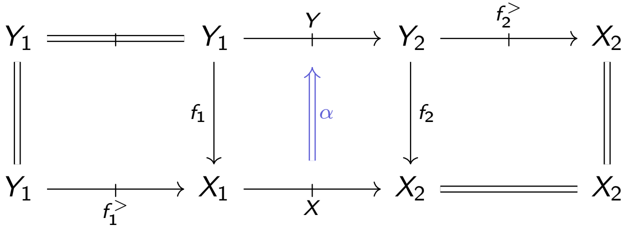
Definition

A regular hyperdoctrine is a functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{Pos}$ such that:

1. Each poset PX is $\wedge$-semilattice;
2. For each morphism $f\colon X \to Y$ in $\mathcal{C}$, the functor $Pf\colon PY \to PX$ has a left adjoint $\exists f$;
3. These adjoints satisfy the Beck-Chevalley condition: for any pullback square $\begin{array}{ccc} A & \xrightarrow{h} & I \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[f]{} & J \end{array}$, the canonical map $\exists h \circ Pk \Rightarrow Pg \circ \exists f$ is invertible;
4. These adjoints satisfy Frobenius reciprocity: for each $f\colon X \to Y$, the canonical map $\exists f(Pf \wedge \mathrm{id}_{PX}) \Rightarrow \mathrm{id}_{PY} \wedge \exists f$ is invertible.

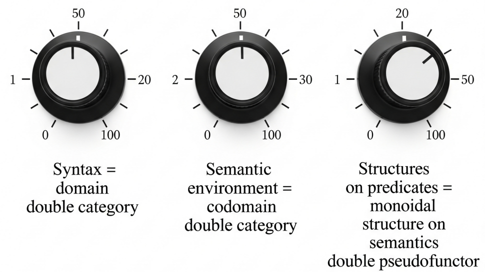
Hyperdoctrines as double pseudofunctors

Theorem: generalised regular hyperdoctrines correspond to those lax symmetric monoidal double pseudofunctors from spans to quintet for which the laxators are companion commuter cells.



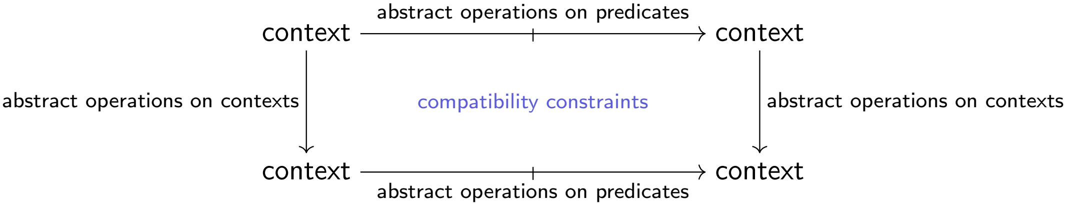
The control panel

Translating classical hyperdoctrines to double categorical language provides a piece for a much grander puzzle.



The big picture: functorial semantics for double categories

You can capture the full syntactic picture in a double category:



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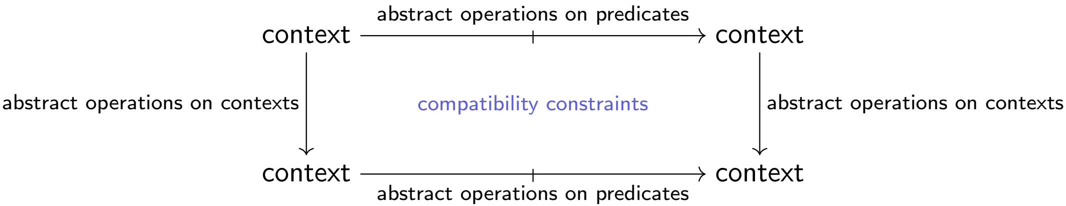

Meaning can be added by a semantics map.

The big picture: functorial semantics for double categories

At a minimum, it should:

- Associate each context to a collection of predicates;
- Turn abstract operations on contexts and predicates into concrete operations on predicates;
- Turn the compatibility constraint into a comparison map of operations.

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This much is achieved by semantics being a double (pseudo)functor.

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This ought to be compatible with the abstract operations on contexts and predicates (Frobenius).

New result: Frobenius corresponds to the monoidal laxators for the lax symmetric monoidal structure on the double pseudofunctor being companion commuter transformations.

Regular double hyperdoctrines

Definition: Let $\mathbb{C}tx$ be a cartesian Beck-Chevalley double category and $\mathbb{D}$ be a symmetric monoidal double category. A *regular double hyperdoctrine over* $\mathbb{C}tx$ *with predicates in* $\mathbb{D}$ *is a lax symmetric monoidal double pseudofunctor* $Q\colon \mathbb{C}tx^{\mathrm{op}} \to \mathbb{D}$ *such that:*

- *For each pair of objects* $A, B \in \mathbb{C}tx$*, the tight component* $\mu_{A,B}\colon QA \otimes QB \to Q(A \times B)$ *of the monoidal laxator of* Q *has a companion in* $\mathbb{D}$*;*
- *For each pair of loose arrows* $X, Y \in \mathbb{C}tx$*, the cell component*

$$\begin{array}{ccc} QX_1 \otimes QY_1 & \xrightarrow{QX \otimes QY} & QX_2 \otimes QY_2 \\ {\scriptstyle \mu_{X_1,Y_1}} \downarrow & \Uparrow {\scriptstyle \mu_{X,Y}} & \downarrow {\scriptstyle \mu_{X_2,Y_2}} \\ Q(X_1 \times Y_1) & \xrightarrow[Q(X \times Y)]{} & Q(X_2 \times Y_2) \end{array}$$

is a companion commuter cell.

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