Category Theory 2025

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The Dialectica construction for dependent type theories

based on a joint work in progress with Davide Trotta (University of Padua) Jonathan Weinberger (Chapman University) Valeria de Paiva (Topos Institute)

speaker Matteo Spadetto (University of Nantes)

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Dialectica interpretation (Gödel)

Dialectica Interpretation is based on a theory, called System T, in a many-sorted language $\mathcal L$ and such that any formula of T is quantifier free. Whenever A is a formula in the language of arithmetic, then we inductively define a formula A^D in the language $\mathcal L$ of the form $\exists x. \forall y. A_D$, where A_D is quantifier free. This interpretation satisfies the following:

Theorem

If HA proves a formula A, then T proves $A_D(t, y)$ where t is a sequence of closed terms.

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An **object** of Dial(\mathcal{C}) is a triple (X, U, α) , which we think of as a formula $\exists x \forall u \alpha(x, u)$, where α is a subobject of $X \times U$ in \mathcal{C} .

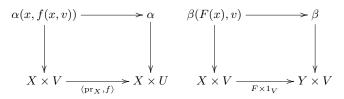
An **arrow** from $\exists x \forall u \alpha(x, u)$ to $\exists y \forall v \beta(y, v)$ is a pair:

$$(F: X \longrightarrow Y, f: X \times V \longrightarrow U)$$

i.e. a pair (F(x):Y, f(x,v):U) of terms in context satisfying the condition:

$$\alpha(x, f(x, v)) \le \beta(F(x), v)$$

between the reindexed subobjects, where the squares:



are pullbacks.

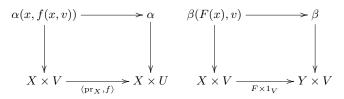
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The notion of morphism of $Dial(\mathcal{C})$ is motivated by the definition of the dialectica interpretation for formulas of the form $A \to B$:

$$(A \to B)^D := \exists F \exists f \forall x \forall v (A_D(x, f(x, v)) \to B_D(F(x), v)).$$

The action of $(-)^D$ on $A \to B$ is heuristically motivated by the principle of Independence of Premise:

$$\top \vdash (\phi \to \exists x \psi(x)) \to \exists x (\phi \to \psi(x))$$

and Markov principle:

$$\top \vdash (\forall x \phi(x) \to \psi) \to \exists x (\phi(x) \to \psi)$$

by which one can show that:

$$A^D \to B^D \dashv \vdash (A \to B)^D$$
.

Dialectica fibrations (Hofstra, Hyland, Biering)

Let $Q: \mathcal{E} \to \mathcal{C}$ be a fibration. The **Dialectica fibration** $Dial(Q): \mathcal{C}^{op} \to \mathbf{Cat}$ associated to Q is defined as follows:

▶ **Fibres.** The objects of $\mathrm{Dial}(Q)(A)$ are 4-tuples (Γ, X, U, α) where A, X and U are objects of $\mathcal C$ and $\alpha \in \mathcal E_{\Gamma \times X \times U}$; an arrow:

$$(\Gamma, X, U, \alpha) \to (\Gamma, Y, V, \beta)$$

is a triple $(\Gamma \times X \xrightarrow{F} Y, \ \Gamma \times X \times V \xrightarrow{f} U, \ \phi)$ such that:

$$\phi:\alpha(\gamma,x,f(\gamma,x,v))\to\beta(\gamma,F(\gamma,x),v).$$

▶ **Reindexing.** Whenever g is an arrow $\Delta \to \Gamma$ of C, we define $\mathrm{Dial}(Q)(g)(\Gamma, X, U, \alpha)$ as the predicate:

$$(\Delta, X, U, \alpha(g(\delta), x, u))$$

of $Dial(Q)(\Delta)$.

Characterisation (Trotta, S, de Paiva)

If $\mathcal C$ is cartesian closed, then a fibration $Q\colon \mathcal E\to \mathcal C$ is the Dialectica completion of some doctrine Q'' precisely when Q has the following properties:

- 1. the fibration Q hs simple Σ and simple Π ;
- 2. the fibration Q has enough Σ -free predicates:

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- 2. the fibration Q has enough Σ -free predicates: every predicate in context $\gamma : \Gamma$ is of the form $(\Sigma x : X)\alpha(\gamma, x)$ in such a way that every vertical arrow:

$$\alpha(\gamma, x) \to (\Sigma y : Y)\beta(\gamma, y, x)$$

factors uniquely as $\alpha(\gamma, x) \to \beta(\gamma, t(x), x) \to (\Sigma y : Y)\beta(\gamma, y, x)$;

- 3. the Σ -free objects of Q are stable under Π ;
- 4. the subfibration Q' of the Σ -free predicates of Q has enough Π -free predicates.

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This is obtained by Hofstra's result $\mathrm{Dial}(Q) \cong (Q^{\Pi})^{\Sigma}$.

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Fibres. The objects of Dial(Q)(A) are 4-tuples:

$$(\Gamma, \Gamma.X \xrightarrow{P_X} \Gamma, \Gamma.X.U \xrightarrow{P_U} \Gamma.X, \alpha)$$

where $\alpha \in \mathcal{E}_{\Gamma,X,U}$; an arrow:

$$(\Gamma, P_X, P_U, \alpha) \rightarrow (\Gamma, \Gamma.Y \xrightarrow{P_Y} \Gamma, \Gamma.Y.V \xrightarrow{P_V} \Gamma.Y, \beta)$$

is a triple $(\Gamma.X \xrightarrow{F} \Gamma.Y, \ \Gamma.X.V[F] \xrightarrow{f} \Gamma.U, \ \phi)$ such that:

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of $Dial(Q, D)(\Delta)$.



Similar results to the non dependent case:

- ▶ Dial(Q, D) has dependent Σ and dependent Π .
- ▶ Dial(Q, D) is isomorphic to $((Q, D)^{\Pi})^{\Sigma}$.
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If $\mathcal C$ is cartesian closed than Dial is a pseudomonad in both cases, because there exists a distributive law between the Π -completion and the Σ -completion.

An other important chapter of Dialectica in category theory

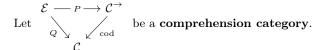
Moss & von Glehn. Dialectica models of type theory. Dialectica construction base on the gluing construction of fibred dependent type theories.

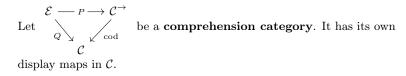
So far we saw how to Dialectica complete:

- ▶ a theory over a proof irrelevant many-sorted signature (without dependency between sorts)
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- ▶ what about pure dependent type theories?





Let
$$C$$
 be a **comprehension category**. It has its own display maps in C .

Let Γ be an object of \mathcal{C} . According to the dependent Σ completion, a new predicate over Γ would be a triple $(\Gamma, \Gamma, A_1 \to \Gamma, A_2)$ hence a triple (Γ, A_1, A_2) .

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In the latter case, we obtain a new display map $\Gamma.A_1.A_2 \to \Gamma.A_1 \to \Gamma$, hence we are forced to add predicates of the form (Γ, A_1, A_2, A_3) ... and so on.

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This in particular verifies that: whenever Γ is a context, A is a type in context Γ , and B is a type in context Γ . A, there is a choice of:

a predicate $(\Sigma A)B$ in context Γ and an isomorphism $\Gamma.A.B \to \Gamma.(\Sigma A)B$ such that:

$$\begin{array}{c|c} \Gamma.A.B & \longrightarrow & \Gamma.(\Sigma A)B \\ \downarrow & & \downarrow \\ \downarrow & & \downarrow \\ \Gamma.A & \stackrel{P_A}{\longrightarrow} & \Gamma \end{array}$$

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This construction defines a pseudomonad.

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If a dependent type theory has function variable contexts (Bossi/Valentini, Garner):

$$\frac{\lfloor \gamma : \Gamma \rfloor \ A(\gamma) : \text{Type} \qquad \lfloor \gamma, x : A(\gamma) \rfloor \ B(\gamma, x) : \text{Type}}{\lfloor \gamma : \Gamma, \upsilon(_) : B(\gamma, _) \rfloor}$$

meaning that, under the hypotheses:

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we can build two different contexts:

$$[\gamma, x, y]$$
 $[\gamma, v(_)]$

then such a characterisation for Π -types is available.

Comprehension categories of function variable contexts

Let $(C, Q : E \to C, P : E \to C^{\to})$ be a comprehension category such that, whenever:

 Γ is a context, A_1 is a type in context Γ , A_2 is a type in context $\Gamma.A_1$,

and A_n is a type in context $\Gamma.A_1....A_{n-1}$

the re-indexing functor:

$$\mathcal{C}/\Gamma \to \mathcal{C}/\Gamma.A_1^n$$
$$f \mapsto f^{n \bullet}$$

has a chosen right I_{Γ,A_1^n} -relative coadjoint:

$$\mathcal{D}/\Gamma.A_1^n \to \mathcal{C}/\Gamma$$

 $P_B \mapsto R_{\Gamma|B} : \Gamma|B \to \Gamma$

where I_{Γ,A_1^n} is the full forgetful functor $\mathcal{D}/\Gamma,A_1^n \hookrightarrow \mathcal{C}/\Gamma,A_1^n$. Then we say that (\mathcal{C},Q,P,R) is a comprehension category of function variable contexts (fvccc).

Characterisation of models of Π -types

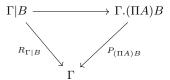
A fvccc (C, Q, P, R) is a model of extensional Π -types if and only if:

whenever Γ is a context, A is a type in context Γ , and B is a type in context $\Gamma.A$

there is a choice of:

a type
$$(\Pi A)B$$
 in context Γ and an isomorphism of contexts $\Gamma|B \to \Gamma.(\Pi A)B$

such that:



commutes.

Σ and Π completing a fvccc

Given an fvccc (C, Q, P, R):

- ▶ The fvccc $(C, Q, P, R)^{\Sigma}$ has for predicates the *finite compositions* of display maps of the fvccc (C, Q, P, R).
- ▶ The fvccc $(C, Q, P, R)^{\Pi}$ has for predicates the function variable contexts of the fvccc (C, Q, P, R).

These operations *preserve the comprehension category structure*, hence there is hope for defining a notion of Dialectica construction for pure dependent type theories.

References



de Paiva. The Dialectica categories.

Hyland. Proof theory in the abstract.

Biering. Dialectica interpretations: a categorical analysis.

Hofstra. The Dialectica monad and its cousins.

Moss and Glehn. Dialectica models of type theory.

Spadetto, Trotta, de Paiva. The Gödel fibration.

Trotta, Weinberger, de Paiva. Skolem, Gödel, and Hilbert fibrations.

Bossi, Valentini. An intuitionistic theory of types with assumptions of high-arity variables

Garner. On the strength of dependent products in the type theory of Martin Löf