Category Theory 2025

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The Dialectica construction for dependent type theories

based on a joint work in progress with
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Dialectica interpretation (Gödel)

Dialectica Interpretation is based on a theory, called System T, in a many-sorted language $\mathcal{L}$ and such that any formula of T is quantifier free. Whenever A is a formula in the language of arithmetic, then we inductively define a formula A^D in the language $\mathcal{L}$ of the form $\exists x.\forall y.A_D$, where A_D is quantifier free. This interpretation satisfies the following:

Theorem

If HA proves a formula A, then T proves $A_D(t, y)$ where t is a sequence of closed terms.

Dialectica categories (de Paiva)

De Paiva's notion of Dialectica category $\mathrm{Dial}(\mathcal{C})$ associated to a category with finite limits $\mathcal{C}$ is the first attempt of internalising Gödel's Dialectica interpretation.

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An **object** of $\mathrm{Dial}(\mathcal{C})$ is a triple (X, U, α), which we think of as a formula $\exists x \forall u \alpha(x, u)$, where α is a subobject of $X \times U$ in $\mathcal{C}$.

Dialectica categories (de Paiva)

An **arrow** from $\exists x \forall u \alpha(x,u)$ to $\exists y \forall v \beta(y,v)$ is a pair:

$$(F\colon X \longrightarrow Y,\ f\colon X \times V \longrightarrow U)$$

i.e. a pair $(F(x) : Y,\ f(x,v) : U)$ of terms in context satisfying the condition:

$$\alpha(x, f(x,v)) \leq \beta(F(x), v)$$

between the reindexed subobjects, where the squares:

$$\begin{array}{ccc} \alpha(x,f(x,v)) & \longrightarrow & \alpha \\ \downarrow & & \downarrow \\ X \times V & \xrightarrow[\langle \mathrm{pr}_X, f\rangle]{} & X \times U \end{array} \qquad \begin{array}{ccc} \beta(F(x),v) & \longrightarrow & \beta \\ \downarrow & & \downarrow \\ X \times V & \xrightarrow[F \times 1_V]{} & Y \times V \end{array}$$

are pullbacks.

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The notion of morphism of Dial($\mathcal{C}$) is motivated by the definition of the dialectica interpretation for formulas of the form $A \to B$:

$$(A \to B)^D := \exists F \exists f \forall x \forall v(\ A_D(x, f(x, v)) \to B_D(F(x), v)\ ).$$

The action of $(-)^D$ on $A \to B$ is heuristically motivated by the principle of Independence of Premise:

$$\top \vdash (\phi \to \exists x \psi(x)) \to \exists x(\phi \to \psi(x))$$

and Markov principle:

$$\top \vdash (\forall x \phi(x) \to \psi) \to \exists x(\phi(x) \to \psi)$$

by which one can show that:

$$A^D \to B^D \dashv\vdash (A \to B)^D.$$

Dialectica fibrations (Hofstra, Hyland, Biering)

Let $Q\colon \mathcal{E} \to \mathcal{C}$ be a fibration. The **Dialectica fibration** $\mathrm{Dial}(Q)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ associated to Q is defined as follows:

- **Fibres.** The objects of $\mathrm{Dial}(Q)(A)$ are 4-tuples (Γ, X, U, α) where A, X and U are objects of $\mathcal{C}$ and $\alpha \in \mathcal{E}_{\Gamma\times X\times U}$; an arrow:

  $$(\Gamma, X, U, \alpha) \to (\Gamma, Y, V, \beta)$$

  is a triple $(\Gamma \times X \xrightarrow{F} Y,\ \ \Gamma \times X \times V \xrightarrow{f} U,\ \ \phi)$ such that:

  $$\phi : \alpha(\gamma, x, f(\gamma, x, v)) \to \beta(\gamma, F(\gamma, x), v).$$

- **Reindexing.** Whenever g is an arrow $\Delta \to \Gamma$ of $\mathcal{C}$, we define $\mathrm{Dial}(Q)(g)(\Gamma, X, U, \alpha)$ as the predicate:

  $$(\Delta, X, U, \alpha(g(\delta), x, u))$$

  of $\mathrm{Dial}(Q)(\Delta)$.

Characterisation (Trotta, S, de Paiva)

If $\mathcal{C}$ is cartesian closed, then a fibration $Q\colon \mathcal{E} \to \mathcal{C}$ is the Dialectica completion of some doctrine Q'' precisely when Q has the following properties:

1. the fibration Q hs simple Σ and simple Π;
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1. the fibration Q hs simple Σ and simple Π;
2. the fibration Q has enough Σ-*free predicates*: every predicate in context $\gamma : \Gamma$ is of the form $(\Sigma x : X)\alpha(\gamma, x)$ in such a way that every vertical arrow:
$$\alpha(\gamma, x) \to (\Sigma y : Y)\beta(\gamma, y, x)$$
factors uniquely as $\alpha(\gamma, x) \to \beta(\gamma, t(x), x) \to (\Sigma y : Y)\beta(\gamma, y, x)$;
3. the Σ-free objects of Q are stable under Π;
4. the subfibration Q' of the Σ-free predicates of Q has enough Π-*free predicates*.

In this case, the doctrine Q'' such that $\mathrm{Dial}(Q'')$ is Q is made of the *quantifier-free predicates* of Q.

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This is obtained by Hofstra's result $\mathrm{Dial}(Q) \cong (Q^{\Pi})^{\Sigma}$.

Adding dependency between sorts (Trotta, Weinberger, de Paiva)

Fibrations $Q : \mathcal{E} \to \mathcal{C}$ with a family of display maps D in the base category $\mathcal{C}$.

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The **Dialectica fibration** $\mathrm{Dial}(Q, D) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ associated to (Q, D) is defined as follows:

- **Fibres.** The objects of $\mathrm{Dial}(Q)(A)$ are 4-tuples:

$$(\Gamma\ ,\ \Gamma.X \xrightarrow{P_X} \Gamma\ ,\ \Gamma.X.U \xrightarrow{P_U} \Gamma.X\ ,\ \alpha)$$

  where $\alpha \in \mathcal{E}_{\Gamma.X.U}$; an arrow:

$$(\Gamma\ ,\ P_X\ ,\ P_U\ ,\ \alpha) \to (\Gamma\ ,\ \Gamma.Y \xrightarrow{P_Y} \Gamma\ ,\ \Gamma.Y.V \xrightarrow{P_V} \Gamma.Y\ ,\ \beta)$$

  is a triple $(\Gamma.X \xrightarrow{F} \Gamma.Y,\ \ \Gamma.X.V[F] \xrightarrow{f} \Gamma.U,\ \ \phi)$ such that:

$$\phi : \alpha(\gamma, x, f(\gamma, x, v)) \to \beta(\gamma, F(\gamma, x), v).$$

- **Reindexing.** Whenever g is an arrow $\Delta \to \Gamma$ of $\mathcal{C}$, we define $\mathrm{Dial}(Q, D)(g)(\Gamma, X, U, \alpha)$ as the predicate:

$$(\Delta, X[g], U[g.X], \alpha(g(\delta), x, u))$$

  of $\mathrm{Dial}(Q, D)(\Delta)$.

Adding dependency between sorts (Trotta, Weinberger, de Paiva)

Similar results to the non dependent case:

- Dial(Q, D) has dependent Σ and dependent Π.
- Dial(Q, D) is isomorphic to $((Q, D)^{\Pi})^{\Sigma}$.
- Fibrations of the form Dial(Q, D) can be internally characterised in terms of quantifier free elements.

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In both the dependent and the non-dependent case, there would be lots of things to say on the internal logic associated to Dialectica completions (Markov Principle and the Principle of Independence of Premise).

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If $\mathcal{C}$ is cartesian closed than Dial is a pseudomonad in both cases, because there exists a distributive law between the Π-completion and the Σ-completion.

An other important chapter of Dialectica in category theory

- Moss & von Glehn. *Dialectica models of type theory.*
  Dialectica construction base on the gluing construction of fibred dependent type theories.

So far we saw how to Dialectica complete:

- a theory over a proof irrelevant many-sorted signature (without dependency between sorts)
- a theory over a proof relevant many-sorted signature (without dependency between sorts)
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- what about pure dependent type theories?

Why the usual existential (and universal) completion does not work for pure dependent type theories i.e. full comprehension categories

Let $\begin{array}{ccc} \mathcal{E} & \xrightarrow{P} & \mathcal{C}^{\to} \\ & {\scriptstyle Q}\searrow \quad \swarrow{\scriptstyle \mathrm{cod}} & \\ & \mathcal{C} & \end{array}$ be a **comprehension category**.

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Let Γ be an object of $\mathcal{C}$. According to the dependent Σ completion, a new predicate over Γ would be a triple $(\Gamma\ ,\ \Gamma.A_1 \to \Gamma\ ,\ A_2)$ hence a triple (Γ, A_1, A_2).

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But now:

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- or we need to let P act over these new predicates

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In the latter case, we obtain a new display map $\Gamma.A_1.A_2 \to \Gamma.A_1 \to \Gamma$, hence we are forced to add predicates of the form (Γ, A_1, A_2, A_3) ... and so on.

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This in particular verifies that: whenever Γ is a context, A is a type in context Γ, and B is a type in context $\Gamma.A$, there is a choice of:

a predicate $(\Sigma A)B$ in context Γ and an isomorphism $\Gamma.A.B \to \Gamma.(\Sigma A)B$

such that:

$$\begin{array}{ccc} \Gamma.A.B & \longrightarrow & \Gamma.(\Sigma A)B \\ \downarrow{\scriptstyle P_B} & & \downarrow{\scriptstyle P_{(\Sigma A)B}} \\ \Gamma.A & \xrightarrow{P_A} & \Gamma \end{array}$$

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This property characterises extensional Σ-types.

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This construnction defines a pseudomonad.

An analogous property characterising Π-types?

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If a dependent type theory has *function variable contexts* (Bossi/Valentini, Garner):

$$\frac{\lfloor \gamma : \Gamma \rfloor \; A(\gamma) : \textsc{Type} \qquad \lfloor \gamma, x : A(\gamma) \rfloor \; B(\gamma, x) : \textsc{Type}}{\lfloor \gamma : \Gamma, v(\_) : B(\gamma, \_) \rfloor}$$

meaning that, under the hypotheses:

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we can build two different contexts:

$$\lfloor \gamma, x, y \rfloor \qquad \lfloor \gamma, v(\_) \rfloor$$

then such a characterisation for Π-types is available.

Comprehension categories of function variable contexts

Let $(\mathcal{C}, Q : \mathcal{E} \to \mathcal{C}, P : \mathcal{E} \to \mathcal{C}^{\to})$ be a comprehension category such that, whenever:

$$\Gamma \text{ is a context}, \quad A_1 \text{ is a type in context } \Gamma, \quad A_2 \text{ is a type in context } \Gamma.A_1, \\ \dots \\ \text{and } A_n \text{ is a type in context } \Gamma.A_1. \dots .A_{n-1}$$

the re-indexing functor:

$$\mathcal{C}/\Gamma \to \mathcal{C}/\Gamma.A_1^n$$
$$f \mapsto f^{n\bullet}$$

has a chosen right $I_{\Gamma.A_1^n}$-relative coadjoint:

$$\mathcal{D}/\Gamma.A_1^n \to \mathcal{C}/\Gamma$$
$$P_B \mapsto R_{\Gamma|B} : \Gamma|B \to \Gamma$$

where $I_{\Gamma.A_1^n}$ is the full forgetful functor $\mathcal{D}/\Gamma.A_1^n \hookrightarrow \mathcal{C}/\Gamma.A_1^n$. Then we say that $(\mathcal{C}, Q, P, R)$ is a **comprehension category of function variable contexts** (fvccc).

Characterisation of models of Π-types

A fvccc $(\mathcal{C}, Q, P, R)$ is a model of extensional Π-types if and only if:

whenever Γ is a context, A is a type in context Γ, and B is a type in context $\Gamma.A$

there is a choice of:

a type $(\Pi A)B$ in context Γ and an isomorphism of contexts $\Gamma|B \to \Gamma.(\Pi A)B$

such that:

$$\begin{array}{ccc} \Gamma|B & \longrightarrow & \Gamma.(\Pi A)B \\ & {\scriptstyle R_{\Gamma|B}} \searrow \quad \swarrow {\scriptstyle P_{(\Pi A)B}} & \\ & \Gamma & \end{array}$$

commutes.

Σ and Π completing a fvccc

Given an fvccc $(\mathcal{C}, Q, P, R)$:

- The fvccc $(\mathcal{C}, Q, P, R)^{\Sigma}$ has for predicates the *finite compositions* of display maps of the fvccc $(\mathcal{C}, Q, P, R)$.
- The fvccc $(\mathcal{C}, Q, P, R)^{\Pi}$ has for predicates the *function variable contexts* of the fvccc $(\mathcal{C}, Q, P, R)$.

These operations *preserve the comprehension category structure*, hence there is hope for defining a notion of Dialectica construction for pure dependent type theories.

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