The higher algebra of monoidal bicategories

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July 16th, 2025

CT 2025
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Point of departure

Problem

- Weak (n, k)-categories are complicated objects.
- In particular, so are objects in higher categorified algebra.
- There are different ways to formalize and manage them.

Classical approach in low dimensions

- Define fully weak structures “by hand”.
- Establish a set of coherence theorems to replace fully weak structures by semi-strict ones.

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Low dimensional coherence theorems

Examples

1. Monoidal categories $\simeq$ strict monoidal categories (Mac Lane).
2. Bicategories $\simeq$ 2-categories (Street, Lack).
3. Monoidal bicategories $\simeq$ Gray monoids (Gordon–Power–Street).

Application 1: A classic theorem

Theorem (Joyal–Street)

Let $\mathcal{C}$ be a monoidal category.

1. *$\mathcal{C}$ braided $\Rightarrow$ $Mon(\mathcal{C})$ monoidal.*
2. *$\mathcal{C}$ symmetric $\Rightarrow$ $Mon(\mathcal{C})$ symmetric.*

The forgetful functor

$$Mon(\mathcal{C}) \to \mathcal{C}$$

is as monoidal as $\mathrm{Mon}(\mathcal{C})$ is.

Application 2: A 2-dimensional generalization

Main Theorem 1

Let $\mathcal{C}$ be a monoidal bicategory.

1. $\mathcal{C}$ braided $\Rightarrow$ PsMon($\mathcal{C}$) monoidal.
2. $\mathcal{C}$ sylleptic $\Rightarrow$ PsMon($\mathcal{C}$) braided.
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Let $\mathcal{C}$ be a Gray monoid.

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Point of departure II

"Orthogonal" idea in low dimensions

- Embed low dimensional category theory in ∞-category theory.
- Use the ∞-categorical machinery.

Story of the talk

1. Some tools of higher algebra.
2. The higher algebra of bicategories.
3. An ∞-categorical proof of Main Theorem 1.

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Higher algebra

Core notions

1. ∞-operads = symmetric ∞-multicategories.
2. Symmetric monoidal ∞-categories = "representable" ∞-operads.
3. For O, $\mathcal{V} \in \mathrm{Op}_\infty$ get $\mathrm{Alg}_{\mathcal{O}}(\mathcal{V}) \in \mathrm{Cat}_\infty$.
4. For $\mathcal{O} \in \mathrm{Op}_\infty$, $\mathcal{C} \in \mathrm{SMonCat}_\infty$ get $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \in \mathrm{SMonCat}_\infty$.
5. For $\mathcal{V} \in \mathrm{Op}_\infty$, get

$$\mathrm{Alg}_{\mathcal{V}}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathcal{V} \otimes_{\mathrm{BV}} \mathcal{O}}(\mathcal{C})$$

where $\mathcal{V} \otimes_{\mathrm{BV}} \mathcal{O}$ is the Boardman-Vogt tensor product.

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Examples

1. Any $\mathcal{C} \in \mathrm{Cat}_\infty^\Pi$ gives $\mathcal{C}^\times \in \mathrm{SMonCat}_\infty$. E.g. $\mathrm{Cat}_\infty^\times$ itself.
2. May's little cubes ∞-operads:

$$E_0 \to E_1 \to \cdots \to E_n \to \cdots \to E_\infty.$$

2.1 $\mathrm{MonCat}_\infty \simeq \mathrm{Alg}_{E_1}(\mathrm{Cat}_\infty^\times)$.

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Theorem (Dunn's Additivity Theorem)

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Monoidal bicategories

Opening observation

Although BiCat $\not\subseteq \mathrm{Cat}_\infty$, we can use BiCat $\in \mathrm{Cat}_\infty$!

And we can reduce the higher algebra of bicategories to the higher algebra of 2-categories as well:

Theorem (Lack)

$Ho_\infty(2\text{-}Cat) \simeq Ho_\infty(BiCat)$.

What are the monoids in $\mathrm{Ho}_\infty(2\text{-Cat})^\times$?

Definition

A Gray monoid is a strict monoid in Gray $= (2\text{-Cat}, \otimes_{\mathrm{Gr}})$.

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The homotopy theory of Gray monoids

Theorem (Lack)

Gray is a nice monoidal model category.

Corollary

The adjunction

$$GrMon \underset{U}{\overset{F}{\rightleftarrows}} 2\text{-}Cat$$

allows to transfer the canonical model structure on 2-Cat to a model structure on GrMon such that a morphism f in GrMon is a (trivial) fibration if and only if $U(f)$ is so in 2-Cat.
Furthermore,

$$Ho_\infty(GrMon) \simeq Alg_{E_1}(Ho_\infty(2\text{-}Cat)^\times).$$

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Main Theorem 2

For all $\mathcal{C} \in$ 2-Cat there is a bijection between equivalence classes of

1. braided monoidal structures and E_2-algebra structures,
2. sylleptic monoidal structures and E_3-algebra structures,
3. symmetric monoidal structures and E_∞-algebra structures

on $\mathcal{C}$.

The same applies to the corresponding structures on 2-functors.

Related to work by Gurski and Gurski–Osorno.

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A proof of Main Theorem 1

Proposition A

The functor

$$\mathrm{PsMon}(-)\colon \mathrm{GrMon} \to 2\text{-}\mathrm{Cat}$$

preserves trivial fibrations and finite products. It hence induces a cartesian monoidal functor

$$\mathrm{Alg}_{E_1}(-)\colon \mathrm{Alg}_{E_1}(\mathrm{Ho}_\infty(2\text{-}\mathrm{Cat})^\times) \to \mathrm{Ho}_\infty(2\text{-}\mathrm{Cat})^\times$$

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Proposition B

Let $\mathcal{C} \in \mathrm{Alg}_{E_n}(\mathsf{Ho}_\infty(\text{2-Cat})^\times)$. Let $0 \leq m \leq n$. Then

$$\mathsf{Alg}_{E_m}(\mathcal{C}) := \mathsf{Alg}^{(m)}_{E_1}(\mathcal{C})$$

is an E_{n-m}-algebra. The forgetful 2-functor

$$\mathsf{Alg}_{E_m}(\mathcal{C}) \to \mathcal{C}$$

is E_{n-m}-monoidal.

Proof of Proposition B

1. If $\mathcal{C}^\times$ is cartesian monoidal, then so is $\mathrm{Alg}_{E_n}(\mathcal{C}^\times)$. In particular, so is $\mathrm{Alg}_{E_1}(\mathrm{Ho}_\infty(2\text{-Cat})^\times)$.
2. The functor

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   is symmetric monoidal.
3. All symmetric monoidal functors $F\colon \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ lift to a symmetric monoidal functor $F\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^\otimes) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{D}^\otimes)$ for any $\mathcal{O} \in \mathrm{Op}_\infty$.
4. This proves the case for $n = 1$ by the Additivity Theorem. Then simply iterate.

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Proposition A + Proposition B. □

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Conjecture

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Theorem

Assume the conjecture holds. Let $\mathcal{C}$ be a Gray monoid.

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   - 2.1 *$\Rightarrow$ $\mathrm{BrMon}(\mathcal{C})$ symmetric monoidal,*
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Summary

- The higher algebra of BiCat considered as an ∞-category captures the strong monoidal aspects of the bicategorical algebra studied by Joyal–Street, Day–Street, and many others (Main Theorem 2).
- This simplifies proofs and constructions in bicategorical algebra (e.g. Main Theorem 1).

□