When is $\mathbf{Cat}(\mathcal{Q})$ Cartesian closed?

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joint work with Junche Yu

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Exponentiables, Cartesian closedness

Let $\mathcal{C}$ be a category with finite products.

An object $A \in \mathcal{C}$ is **exponentiable** when

$$- \times A \colon \mathcal{C} \to \mathcal{C}$$

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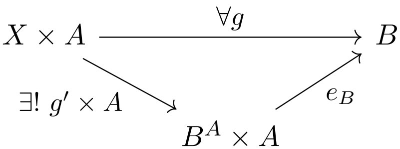
$$\mathcal{C}(X \times A, B) \cong \mathcal{C}(X, B^A)$$



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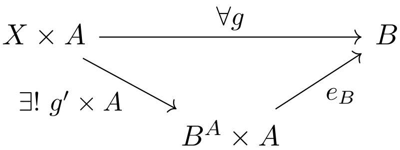

The category $\mathcal{C}$ is **Cartesian closed** when every $A \in \mathcal{C}$ is exponentiable.

Two examples suggesting a question

For any $(A, \leq)$ and $(B, \leq)$ in the category Ord of ordered sets and order-preserving maps, the powerobject $(B^A, \leq)$ exists: it is $B^A = \mathsf{Ord}((A, \leq), (B, \leq))$ with pointwise order.

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Theorem (Clementino and Hofmann, 2006). (A, d) *is exponentiable in* Met *if and only if, for every* $a, b \in A$*, whenever* $d(a, b) = r + s$ *in* $[0, \infty]$ *then, for every* $\varepsilon > 0$*, there exists* $m \in A$ *such that* $d(a, m) < r + \varepsilon$ *and* $d(m, b) < s + \varepsilon$*.*

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More specifically, both Ord and Met are of the form $\mathsf{Cat}(\mathcal{Q})$, the category of $\mathcal{Q}$-enriched categories and functors, each for a suitable quantale $\mathcal{Q}$.

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Which property of $\mathcal{Q}$ makes $\mathsf{Cat}(\mathcal{Q})$ Cartesian closed?

Quantale-enriched categories

A **quantale** $Q = (Q, \bigvee, \circ, 1)$ is a completely ordered monoid in which

$$a \circ (\bigvee_i b_i) = \bigvee_i (a \circ b_i) \quad \text{and} \quad (\bigvee_i a_i) \circ b = \bigvee_i (a_i \circ b)$$

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a set $\mathbb{A}_0$ of "objects",

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For $Q = (\{0, 1\}, \vee, \wedge, 1)$: $\mathbb{A}(x, y) = [\![x \leq y]\!]$ and Cat$(Q) =$ Ord.

For $Q = ([0, \infty], \bigwedge, +, 0)$: $\mathbb{A}(x, y) = d(x, y)$ and Cat$(Q) =$ Met.

But also t-norms and "fuzzy orders", probabilistic metric spaces, monoidal topology, ...

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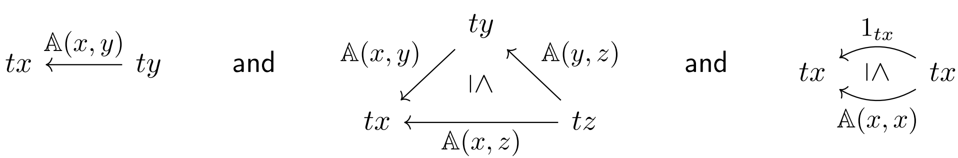

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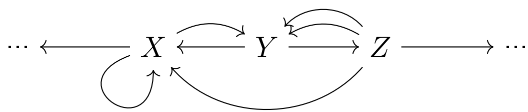

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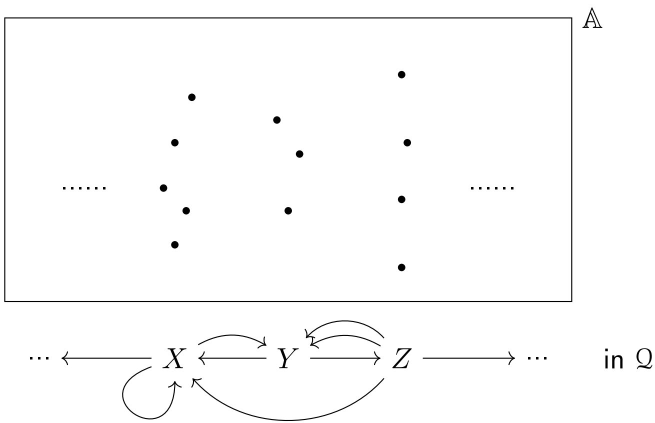

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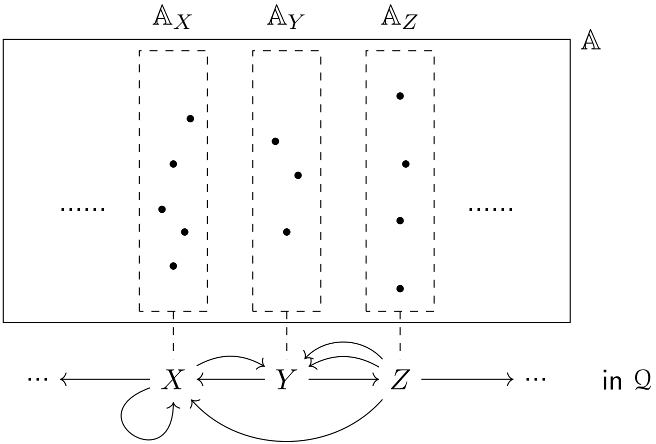

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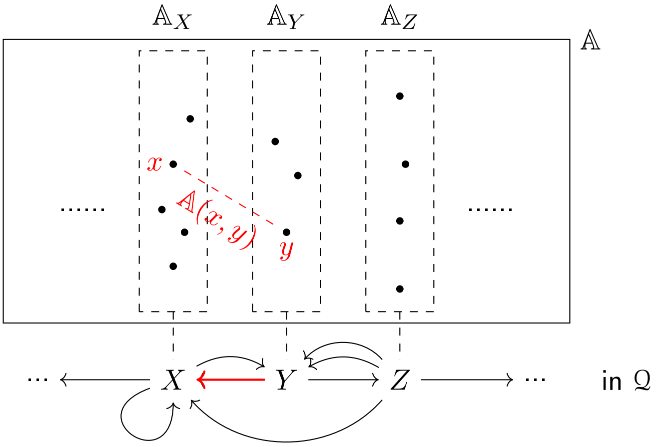

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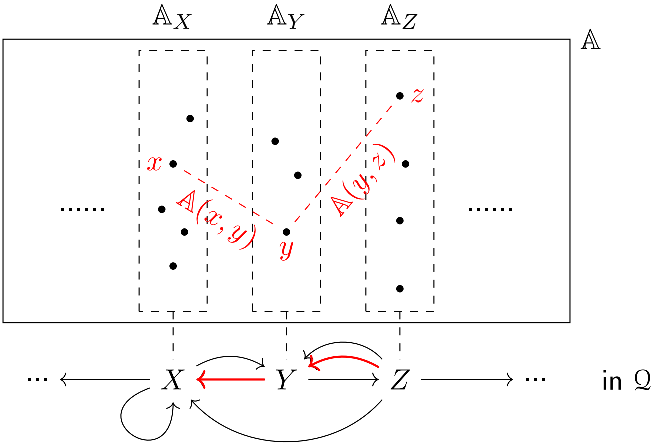

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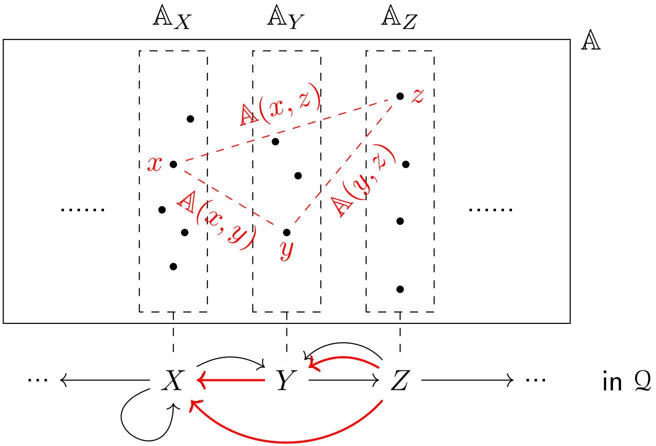

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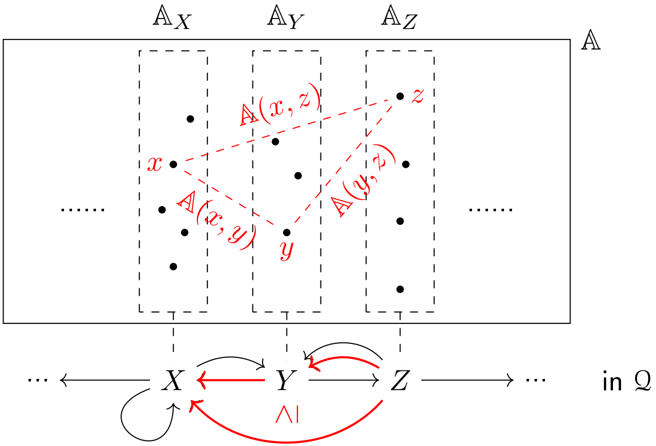

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A quantale is exactly a one-object quantaloid, so we recover all previous examples.

Performing universal constructions on a quantale very often produces a quantaloid. These are used in many new examples: partial (probabilistic) metric spaces, sheaves, ...

Exponentiability in Cat($\mathcal{Q}$)

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objects: $(\mathbb{A} \times \mathbb{B})_X = \mathbb{A}_X \times \mathbb{B}_X$ (for all $X \in \mathcal{Q}_0$),

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Theorem (Clementino, Hofmann and Stubbe, 2009). *A $\mathcal{Q}$-category $\mathbb{A}$ is exponentiable in* Cat($\mathcal{Q}$) *if and only if, for all $x, y \in \mathbb{A}_0$,*

1. *for all* $tx \xleftarrow{\mathbb{A}(x,y)} ty$, $tx \xleftarrow{(f_i)_i} ty$: $\left(\bigvee_i f_i\right) \wedge \mathbb{A}(x,y) = \bigvee_i \left(f_i \wedge \mathbb{A}(x,y)\right)$,
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Exponentiability in Cat($\mathcal{Q}$)

The product $\mathbb{A} \times \mathbb{B}$ of two $\mathcal{Q}$-categories $\mathbb{A}$ and $\mathbb{B}$ exists in Cat($\mathcal{Q}$):

> objects: $(\mathbb{A} \times \mathbb{B})_X = \mathbb{A}_X \times \mathbb{B}_X$ (for all $X \in \mathcal{Q}_0$),
>
> homs: $(\mathbb{A} \times \mathbb{B})((x, y), (x', y')) = \mathbb{A}(x, x') \wedge \mathbb{B}(y, y')$.

Also the terminal $\mathcal{Q}$-category is easy to describe. So Cat($\mathcal{Q}$) has finite products.

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These conditions are always satisfied for ordered sets, and reduce to the "approximate intermediate points" for generalized metric spaces.

(Actually, CHS09 has a characterisation of exponentiable $\mathcal{Q}$-functors—think "Conduché".)

Cartesian closedness of Cat($\mathcal{Q}$)

By definition, Cat($\mathcal{Q}$) is Cartesian closed when **all** $\mathcal{Q}$-categories are exponentiable.

So we can find **necessary conditions** for Cartesian closedness of Cat($\mathcal{Q}$) by requiring exponentiability of **some** $\mathcal{Q}$-categories.

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For any $h\colon X \to Y$ in $\mathcal{Q}$, let $\mathbb{A}_h$ be the $\mathcal{Q}$-category defined as:

objects: $(\mathbb{A}_h)_0 = \{*_1, *_2\}$ with types $t*_1 = Y$ and $t*_2 = X$,

homs: $\mathbb{A}_h(*_1, *_2) = h$, $\mathbb{A}_h(*_2, *_1) = 0_{Y,X}$, $\mathbb{A}_h(*_1, *_1) = 1_Y$ and $\mathbb{A}_h(*_2, *_2) = 1_X$

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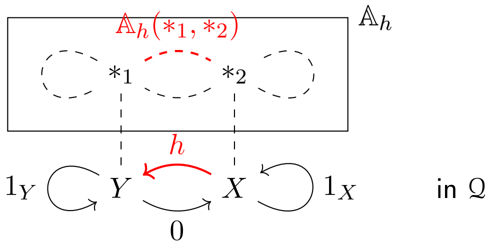
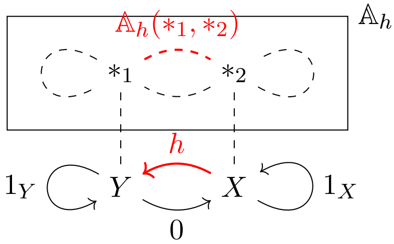


(It is the *collage* (universal cotabulation) of the image of h under the inclusion $\mathcal{Q} \to \mathsf{Dist}(\mathcal{Q})$.)

Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



in $\mathcal{Q}$

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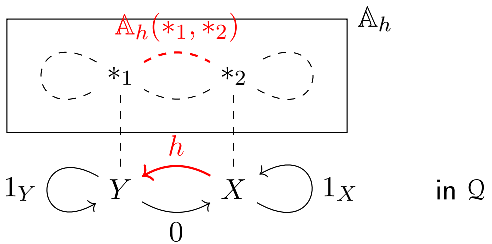
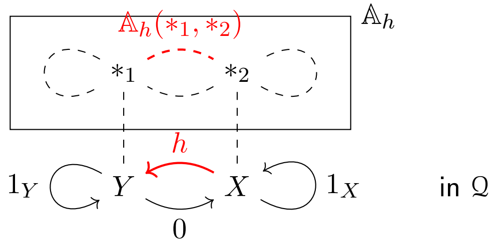


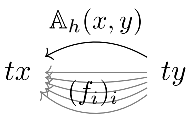
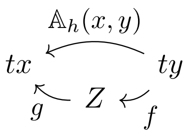
Theorem. $\mathbb{A}_h$ *is exponentiable if and only if, for all* $x, y \in \mathbb{A}_h$,

1. *for all* $tx \xleftarrow[(f_i)_i]{\mathbb{A}_h(x,y)} ty$: $\left(\bigvee_i f_i\right) \wedge \mathbb{A}_h(x,y) = \bigvee_i \left(f_i \wedge \mathbb{A}_h(x,y)\right)$,
2. *for all* $tx \xleftarrow{\mathbb{A}_h(x,y)} ty$, $tx \xleftarrow{g} Z \xleftarrow{f} ty$: $(g \circ f) \wedge \mathbb{A}_h(x,y) = \bigvee_{z \in (\mathbb{A}_h)_Z} \left(g \wedge \mathbb{A}_h(x,z)\right) \circ \left(f \wedge \mathbb{A}_h(z,y)\right)$.

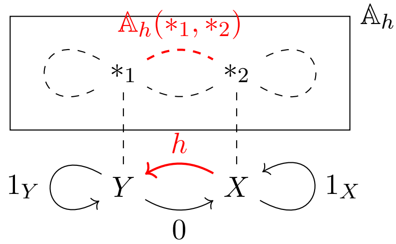
Cartesian closedness of Cat($\mathcal{Q}$)



Theorem. $\mathbb{A}_h$ *is exponentiable ~~if and~~ only if, for $x = *_1$ and $y = *_2$,*

1. *for all*  : $\left(\bigvee_i f_i\right) \wedge \mathbb{A}_h(x,y) = \bigvee_i \left(f_i \wedge \mathbb{A}_h(x,y)\right)$,
2. *for all*  : $(g \circ f) \wedge \mathbb{A}_h(x,y) = \bigvee_{z \in (\mathbb{A}_h)_Z} \left(g \wedge \mathbb{A}_h(x,z)\right) \circ \left(f \wedge \mathbb{A}_h(z,y)\right)$.

Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$

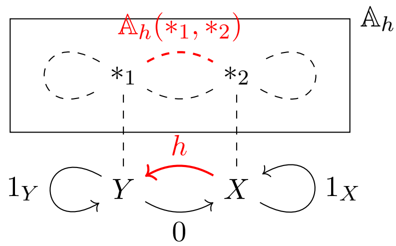


in $\mathcal{Q}$

Theorem. $\mathbb{A}_h$ *is exponentiable ~~if and~~ only if*

1. *for all* $Y \overset{h}{\longleftarrow} X$, $Y \overset{(f_i)_i}{\longleftarrow} X$: $\left(\bigvee_i f_i\right) \wedge h = \bigvee_i \left(f_i \wedge h\right)$,
2. *for all* $Y \overset{h}{\longleftarrow} X$, $Y \overset{g}{\longleftarrow} Z \overset{f}{\longleftarrow} X$: $(g \circ f) \wedge h = \displaystyle\bigvee_{z \in (\mathbb{A}_h)_Z} \Big(g \wedge \mathbb{A}_h(*_1, z)\Big) \circ \Big(f \wedge \mathbb{A}_h(z, *_2)\Big)$.

Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



in $\mathcal{Q}$

Theorem. $\mathbb{A}_h$ *is exponentiable ~~if and~~ only if*

1. $-\wedge h\colon \mathcal{Q}(X,Y) \to \mathcal{Q}(X,Y)$ *preserves suprema,*
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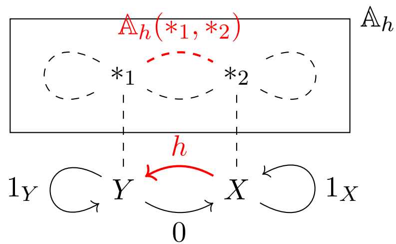

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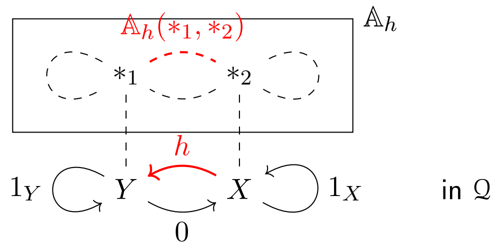
$$= \begin{cases} 0_{X,Y} & \text{if } X \neq Z \neq Y \text{ (so } (\mathbb{A}_h)_Z = \emptyset\text{)}, \\ (g \wedge h) \circ (f \wedge 1_X) & \text{if } X = Z \neq Y \text{ (so } (\mathbb{A}_h)_Z = \{*_2\}\text{)}, \\ (g \wedge 1_Y) \circ (f \wedge h) & \text{if } X \neq Z = Y \text{ (so } (\mathbb{A}_h)_Z = \{*_1\}\text{)}, \\ \Big((g \wedge h) \circ (f \wedge 1_X)\Big) \vee \Big((g \wedge 1_Y) \circ (f \wedge h)\Big) & \text{if } X = Z = Y \text{ (so } (\mathbb{A}_h)_Z = \{*_1, *_2\}\text{)}. \end{cases}$$

Cartesian closedness of $\mathsf{Cat}(\mathcal{Q})$



Theorem. *Every $\mathbb{A}_h$ is exponentiable ~~if and~~ only if*

1. *every* $- \wedge h \colon \mathcal{Q}(X,Y) \to \mathcal{Q}(X,Y)$ *preserves suprema,*
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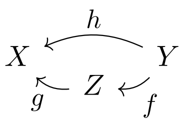

:

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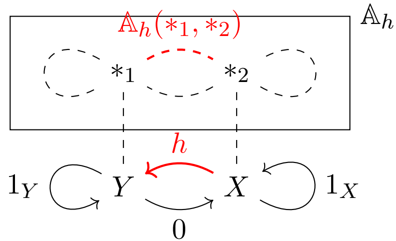
Cartesian closedness of Cat($\mathcal{Q}$)



in $\mathcal{Q}$

Theorem. *Every $\mathbb{A}_h$ is exponentiable ~~if and~~ only if*

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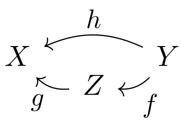
2. *for all*  *:*

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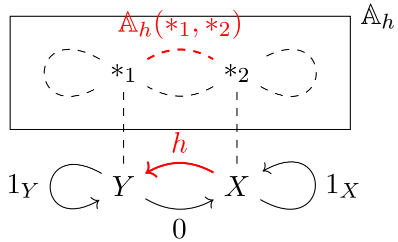

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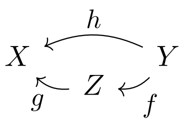
2. *for all* 

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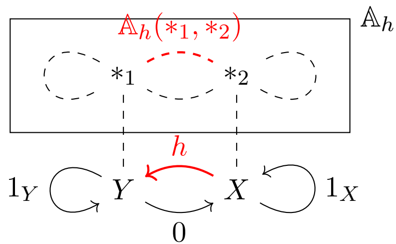

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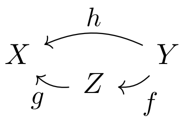

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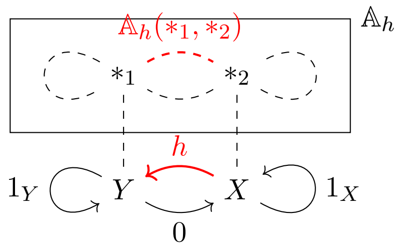

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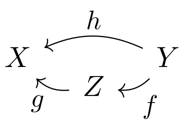

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Proof: "case analysis" for the exponentiability of any $\mathcal{Q}$-category. □

More examples

For a quantale $Q = (Q, \circ, 1)$, Cat(Q) is Cartesian closed if and only if the underlying suplattice of Q is a locale and

$$\text{for all } a, b, c \in Q\text{:} \quad (a \circ b) \wedge c = \Big((a \wedge c) \circ (b \wedge 1)\Big) \vee \Big((1 \wedge a) \circ (b \wedge c)\Big),$$

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$$\text{for all } a, b, c \in Q\colon \quad (a \circ b) \wedge c = ((a \wedge c) \circ b) \vee (a \circ (b \wedge c)).$$

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   $$[0,1] \times [0,1] \to [0,1] \colon (x, y) \mapsto x \circ y = \begin{cases} 0 & \text{if } x, y \leq \frac{1}{2} \\ x \wedge y & \text{otherwise} \end{cases}$$

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4. the only *continuous* t-norm satisfying the above condition is the Gödel t-norm $[0,1] \times [0,1] \to [0,1]\colon (x, y) \mapsto x \circ y = x \wedge y$ (see [Lai and Zhang, 2016]),

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$$[0,1] \times [0,1] \to [0,1]\colon (x, y) \mapsto x \circ y = \begin{cases} 0 & \text{if } x, y \leq \frac{1}{2} \\ x \wedge y & \text{otherwise} \end{cases}$$
4. the only *continuous* t-norm satisfying the above condition is the Gödel t-norm $[0,1] \times [0,1] \to [0,1]\colon (x, y) \mapsto x \circ y = x \wedge y$ (see [Lai and Zhang, 2016]),
5. the latter condition was observed in [Lai and Luo, 2025], but only for so-called complete subquantales of continuous t-norms (and with a different proof).

More examples

Some quantaloidal examples:

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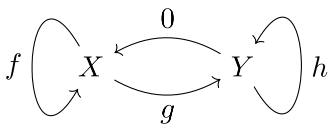

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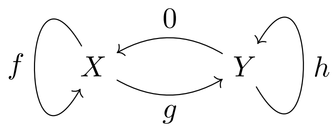

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3. The free quantaloid $\mathcal{PC}$ on a (small) category $\mathcal{C}$ is given by:

$(\mathcal{PC})_0 = \mathcal{C}_0$,
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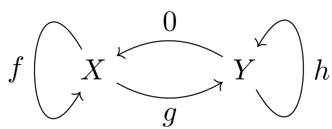

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   $\mathcal{PC}$ is always locally localic, and satisfies the conditions in the Theorem if and only if

   when two morphisms compose in $\mathcal{C}$, then at least one of them is an identity.

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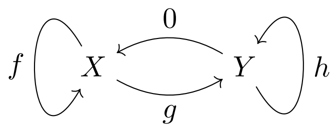

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   In particular, the only free quantale $\mathcal{P}M$ (on a monoid $(M, \circ, 1)$) satisfying this condition is when $M = \{*\}$ (and so $\mathcal{P}M = (\{0,1\}, \vee, \wedge, 1)$).

   (This corrects a mistake in an example in [Clementino, Hofmann and Stubbe, 2009].)

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