Extensive Morphisms

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Joint work with Michael Hoefnagel

Extensive Categories

Definition (Extensive category)

A category $\mathbb{C}$ with finite coproducts is *extensive* whenever the functor

$$+ : (\mathbb{C} \downarrow X_1) \times (\mathbb{C} \downarrow X_2) \to (\mathbb{C} \downarrow (X_1 + X_2))$$

is an equivalence for all objects X_1 and X_2 in $\mathbb{C}$.

- "Extensive categories are categories in which sums exist and are well-behaved".

Carboni, A., Lack, S. and Walters, R.F.C. (1993). Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84:145–158.

- Examples: Set, Top, Pos, Grphs, GSet, Diff, CRing^{op}.

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We study two variants of extensivity:

1. Extensive morphisms:[1] restrict extensivity to morphisms in a category;
2. Near-sums: exchange $+$ for another bifunctor $\oplus$.

[1] Hoefnagel, M., Theart, E. (2025). On extensivity of morphisms. *Theory and Applications of Categories* (accepted to appear).

Motivation: Often in categories which are not extensive, there are large classes of morphisms which exhibit extensive behaviour.

A characterisation of extensive categories

Proposition

A category $\mathbb{C}$ with binary coproducts is extensive if and only if the following conditions hold:

1. each morphism in $\mathbb{C}$ admits a pullback along any coproduct inclusion;
2. given a commutative diagram

$$\begin{array}{ccccc} A_1 & \longrightarrow & A & \longleftarrow & A_2 \\ \downarrow & & \downarrow & & \downarrow \\ X_1 & \longrightarrow & X & \longleftarrow & X_2 \end{array}$$

   where the bottom row is a coproduct diagram, the top row is a coproduct diagram if and only if the two squares are pullbacks.

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Extensive morphisms

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Let $\mathbb{C}$ be a category with finite coproducts. A morphism $f : A \to X$ is *extensive* whenever the following conditions hold:

1. f admits a pullback along every coproduct inclusion;
2. given a commutative diagram

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   where the bottom row is a coproduct diagram, the top row is a coproduct diagram if and only if the two squares are pullbacks.

- A category is extensive if and only if each morphism is;
- The composite of two extensive morphisms is extensive.

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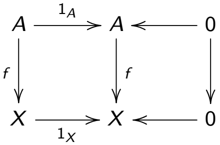
- This definition shares similarities to that of *crisp morphisms* in:

Manes, E. (1992). *Predicate Transformer Semantics*. Cambridge Tracts in Theoretical Computer Science, Cambridge University Press.

Extensive morphisms in pointed categories

- Note that extensivity trivialises pointed categories since initials must be strict.
- Any extensive morphism f in a pointed category must have trivial kernel by the following diagram:



Extensive morphisms in pointed categories

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$$\begin{array}{ccccc} A & \xrightarrow{1_A} & A & \longleftarrow & 0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow \\ X & \xrightarrow[1_X]{} & X & \longleftarrow & 0 \end{array}$$

Examples

- A morphism f in Set_* is extensive if and only if $\ker f \simeq 0$.
- If each split monomorphism $\mathbb{C}$ is a coproduct inclusion, then a morphism f in $\mathrm{Pt}_{\mathbb{C}}(A)$ is extensive if and only if $\ker f \simeq 0$.
- In the category of finitely generated abelian groups, a group is cyclic if and only if each monomorphism into it is extensive.

Definition

A morphism $f : X \to Y$ is *coextensive* in $\mathbb{C}$ when it is extensive in $\mathbb{C}^{\mathrm{op}}$.

Coextensivity within certain classes of morphisms

- For $\mathbb{C}$ finitely complete: Identity morphisms in $\mathbb{C}$ are coextensive iff product projections are extremal epimorphisms.

In any variety of algebras this is equivalent to the variety admitting at least one constant, since $0 \times 1 = 0 \to 1$ must be surjective.

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- For $\mathbb{C}$ Barr-exact: Regular epimorphisms are coextensive if and only if $\mathbb{C}$ has the *Fraser-Horn Property* (if θ is an equivalence relation on $X_1 \times X_2$, then $\theta = \theta_1 \times \theta_2$ for θ_i equivalence relations on X_i).

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- For $\mathbb{C}$ (finitely) complete: Product projections being coextensive implies that $\mathbb{C}$ satisfies the (finite) *strict refinement property*. (Converse holds when product projections are extremal epimorphisms).

A category $\mathbb{C}$ satisfies the *strict refinement property* if, for any two product diagrams $(X \xrightarrow{a_i} A_i)_{i \in I}$ and $(X \xrightarrow{b_j} B_j)_{j \in J}$, one can form a commutative square as below for each i and j

$$\begin{array}{ccc} C_{ij} & \xleftarrow{\beta_{i,j}} & B_j \\ {\scriptstyle \alpha_{i,j}} \uparrow & & \uparrow {\scriptstyle b_j} \\ A_i & \xleftarrow[a_i]{} & X \end{array}$$

such that $(A_i \xrightarrow{\alpha_{i,s}} C_{i,s})_{s \in J}$ and $(B_j \xrightarrow{\beta_{t,j}} C_{t,j})_{t \in I}$ are product diagrams.

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- For $\mathbb{C}$ (finitely) complete: Product projections being coextensive implies that $\mathbb{C}$ satisfies the (finite) *strict refinement property*. (Converse holds when product projections are extremal epimorphisms).

- For $\mathbb{C}$ Barr-exact: Split monomorphisms are coextensive iff $\mathbb{C}$ is coextensive.

Conclusion

- Extensivity can be viewed both as a property of sums, i.e. of the functors

$$+ : (\mathbb{C} \downarrow X_1) \times (\mathbb{C} \downarrow X_2) \to (\mathbb{C} \downarrow (X_1 + X_2)),$$

  and as a property of morphisms in a category. Extensive morphisms embodies this second point of view.
- Through coextensive morphisms, various algebraic properties enjoy the benefits of categorical generalisation.

References

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