Arrow algebras
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Section 1

Locales

Locales (point-free spaces)

Locales

A poset $(L, \preccurlyeq)$ is a *frame* or *locale*, if thas finite meets and arbitrary joins with the infinite joins distributing over the finite meets:

$$x \curlywedge \bigcurlyvee_i y_i = \bigcurlyvee_i (x \curlywedge y_i).$$

Let us call a monotone map $f : (L, \preccurlyeq) \to (M, \preccurlyeq)$ which preserves finite meets and arbitrary joins *geometric*. The frames with the geometric morphisms yield a category $\mathsf{Frm}_{\mathrm{geom}}$. The category $\mathsf{Loc}_{\mathrm{geom}}$ of locales is defined as $\mathsf{Frm}^{\mathrm{op}}_{\mathrm{geom}}$.

There is a functor

$$\Omega : \mathsf{Top} \to \mathsf{Loc}_{\mathrm{geom}}$$

which sends a space to its poset of opens. This functor has a right adjoint and this adjunction restricts to an equivalence between *sober spaces* and *locales with enough points*.

Other notions of morphism

Let us make some categorical observations:

- Let us call a monotone map $f : (L, \preccurlyeq) \to (M, \preccurlyeq)$ *cartesian* if it preserves finite meets. This gives rise to other categories $\mathrm{Frm}_{\mathrm{cart}}$ and $\mathrm{Loc}_{\mathrm{cart}}$.

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- If $f, g : (L, \preccurlyeq) \to (M, \preccurlyeq)$ are cartesian morphisms, we can write $f \preccurlyeq g$ if $f(x) \preccurlyeq g(x)$ holds for all $x \in X$. This turns $\mathsf{Frm}_{\mathrm{cart}}$ and $\mathsf{Loc}_{\mathrm{cart}}$ into order-enriched categories.

Other notions of morphism

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- Note that a cartesian map $f : L \to M$ is geometric if and only if it has a right adjoint in $\mathrm{Frm}_{\mathrm{cart}}$; that is, there is a cartesian morphism $g : M \to L$ such that $1_L \preccurlyeq gf$ and $fg \preccurlyeq 1_M$.

Triposes

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Tripos (Hyland, Johnstone, Pitts)

Write PreHey for the category of *preHeyting algebras*. A *tripos* is a pseudofunctor $P : \mathrm{Sets} \to \mathrm{PreHey}^{\mathrm{op}}$ such that:

- for each function $f : Y \to X$, the operation $Pf : PX \to PY$ has both adjoints satisfying the Beck-Chevally condition.
- There is a set Prop and an element $\top \in P(\mathrm{Prop})$ such that for any $A \in P(X)$ there is some map $a : X \to \mathrm{Prop}$ such that $P(a)(\top) \cong A$.

Think: model of higher-order intuitionistic logic with an impredicative and intensional Prop.

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Example: localic tripos

From any locale $(L, \preccurlyeq)$ we obtain a tripos P_L with $P_L(X) := X \to L$ with the pointwise ordering.

Beyond locales?

Tripos-to-topos construction

If P is a tripos, then we can construct a topos out of it by looking at PERs and functional relations between those (in the sense of the tripos).

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There are many interesting of non-localic triposes.

Effective tripos

For any set X, define $P_E(X)$ as the set of functions $X \to \mathrm{Pow}(\mathbb{N})$. If $\varphi, \psi : X \to \mathrm{Pow}(\mathbb{N})$ are two such functions, we will write $\varphi \preccurlyeq \psi$ if there is a partial recursive function f such that for any $x \in X$ and $n \in \varphi(x)$ we have that $f(n)$ is defined and belongs to $\psi(x)$.

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Motivating question

Can we generalise the theory of locales in such a way that other toposes like the effective topos can also be understood as "sheaves over a generalised locale"?

Section 2

Arrow algebras

Arrow structure

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An *arrow structure* is a complete poset $(A, \preccurlyeq)$ together with a binary operation $\rightarrow: A \times A \rightarrow A$ satisfying the following condition:

If $a' \preccurlyeq a$ and $b \preccurlyeq b'$ then $a \rightarrow b \preccurlyeq a' \rightarrow b'$.

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Examples

- Every locale is a complete Heyting algebra with implication given by:

$$x \to y := \bigvee \{z : x \curlywedge z \preccurlyeq y\}.$$

- We also have $(\mathrm{Pow}(\mathbb{N}), \subseteq)$ with

$$X \to Y = \{e : (\forall x \in X)\, e \cdot x \downarrow \text{ and } e \cdot x \in Y\}.$$

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Intuition

We think of the elements of A as truth values or bits of evidence, and we refer to $\preccurlyeq$ as the "evidential ordering" ("subtyping ordering").

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Separators

Let $A = (A, \preccurlyeq, \to)$ be an arrow structure. A *separator* on A is a subset $S \subseteq A$ such that the following are satisfied:

(1) If $a \in S$ and $a \preccurlyeq b$, then $b \in S$.

(2) If $a \to b \in S$ and $a \in S$, then $b \in S$.

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Here k, s and a are defined as follows:

$$\begin{aligned}
\mathsf{k} &:= \bigcurlywedge_{a,b} a \to b \to a \\
\mathsf{s} &:= \bigcurlywedge_{a,b,c} (a \to b \to c) \to (a \to b) \to (a \to c) \\
\mathsf{a} &:= \bigcurlywedge_{a \in A, B \subseteq \mathrm{Im}(\to)} \left(\bigcurlywedge_{b \in B} a \to b\right) \to a \to \bigcurlywedge_{b \in B} b.
\end{aligned}$$

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Examples

1. A frame $(L, \preccurlyeq)$ with $S = \{\top\}$.
2. A frame $(L, \preccurlyeq)$ with S an arbitrary filter.
3. The effective arrow algebra $(\mathrm{Pow}(\mathbb{N}), \subseteq, \rightarrow, \mathrm{Pow}_i(\mathbb{N}))$.

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We will now explain how any arrow algebra gives rise to a tripos (and hence a topos).

Arrow tripos

Proposition

Let $A = (A, \preccurlyeq, \to, S)$ be an arrow algebra. If we preorder A as follows:

$$a \vdash b :\Longleftrightarrow a \to b \in S,$$

then A carries the structure of a preHeyting algebra.

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If $A = (A, \preccurlyeq, \to, S)$ is an arrow algebra and X is a set, then A^X is an arrow algebra as well: implication and the order can be defined pointwise, while

$$\varphi : X \to A \in S^X :\iff \bigcurlywedge_{x \in X} \varphi(x) \in S.$$

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If we put $PX = (A^X, \vdash_{S^X})$, then this defines a tripos: we write $\mathrm{AT}(A)$ for the *arrow tripos* associated to A. This recovers both localic triposes as well as the effective tripos.

Section 3

More examples: pcas

Pcas

Partial combinatory algebra (pca)

$\mathbb{P} = (P, \cdot, \leq, P^{\#})$ is a *partial combinatory algebra (pca)* if:

- $(P, \leq)$ is a poset.
- $\cdot$ is a partial binary operation, such that if $a'b'$ is defined and $a \leq a'$ and $b \leq b'$, then ab is also defined and $ab \leq a'b'$.
- $P^{\#}$ is a *filter*, that is, a subset $P^{\#} \subseteq P$ such that for all $a, b, c \in P$:
  - (i) if $a, b \in P^{\#}$ and ab is defined, then $ab \in P^{\#}$.
  - (ii) if $a \leq b$ and $a \in P^{\#}$, then $b \in P^{\#}$.
  - (iii) there are elements $\mathsf{k}, \mathsf{s} \in P^{\#}$ satisfying:
    - (1) $\mathsf{k}ab\downarrow$ and $\mathsf{k}ab \leq a$;
    - (2) $\mathsf{s}ab\downarrow$;
    - (3) if $ac(bc)\downarrow$, then $\mathsf{s}abc\downarrow$ and $\mathsf{s}abc \leq ac(bc)$.

Remark

The usual notion of a pca is more restrictive. For our purposes, the definition above, which is also the one used in Jetze Zoethout's PhD thesis, is quite convenient.

Tripos from a pca

Examples

1. K_1: the set of natural numbers with Kleene application ($n \cdot m$ is the outcome of the n-th Turing machine on input m, whenever defined) and the discrete order. All elements belong to the filter.
2. Terms in the untyped λ-calculus and $M \leq N$ if $M \twoheadrightarrow_\beta N$. All elements belong to the filter.
3. Write $\mathcal{P} = \mathcal{P}(\mathbb{N})$ and fix a computable bijection $[-] : \mathcal{P}_{\text{fin}}(\mathbb{N}) \times \mathbb{N} \to \mathbb{N}$. Then $X \cdot Y = \{z \in \mathbb{N} \,:\, (\exists \gamma \in \mathcal{P}_{\text{fin}}(Y))\, [\gamma, z] \in X\}$ defines a total binary application on $\mathcal{P}$ and $\mathcal{P}^\# = \{X \in \mathcal{P} \,:\, X \text{ is recursively enumerable}\}$ defines a filter.

If $\mathbb{P} = (P, \cdot, \leq, P^\#)$ is a pca, then $(DP, \subseteq, \to, S)$ is an arrow algebra, where:

- DP is the collection of downsets in P,
- $X \to Y := \{z \in P \,:\, (\forall x \in X)\, zx\downarrow \text{ and } zx \in Y\}$,
- $S = \{X \in DP \,:\, (\exists x \in X)\, x \in P^\#\}$.

Section 4

Nuclei and morphisms

Nuclei

Nucleus

Let $A = (A, \preccurlyeq, \to, S)$ be an arrow algebra. A mapping $j : A \to A$ will be called a *nucleus* if the following three properties are satisfied:

(1) $a \preccurlyeq b$ implies $ja \preccurlyeq jb$ for all $a, b \in A$.

(2) $\curlywedge_{a \in A} a \to ja \in S$.

(3) $\curlywedge_{a,b \in A} (a \to jb) \to (ja \to jb) \in S$.

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Examples

Let $A = (A, \preccurlyeq, \to, S)$ be an arrow algebra and $a \in A$. Then the following define nuclei:

- $jx = (x \to a) \to a$
- $jx = a \to x$
- $jx = x \vee a$, where $\vee$ is the join in the logical ordering.

Subalgebras from nuclei

Proposition

Let $(A, \preccurlyeq, \to, S)$ be an arrow algebra and $j : A \to A$ be a nucleus on it. Then $A_j = (A, \preccurlyeq, \to_j, S_j)$ with

$$\begin{aligned} a \to_j b \quad &:\equiv \quad a \to jb \\ a \in S_j \quad &:\Leftrightarrow \quad ja \in S \end{aligned}$$

is also an arrow algebra.

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Theorem

The arrow tripos associated to A_j is a subtripos of the one associated to A. Indeed, any subtripos of $\mathrm{AT}(A)$ is of this form.

Cartesian morphism of arrow algebras

Cartesian morphisms of arrow algebras (Tarantino)

Let $\mathcal{A} = (A, \preccurlyeq, \to, S_A)$ and $\mathcal{B} = (B, \preccurlyeq, \to, S_B)$ be arrow algebras. Then a *cartesian morphism* $f : \mathcal{A} \to \mathcal{B}$ is a function $f : A \to B$ satisfying:

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3. for any subset $X \subseteq A \times A$,
if $\curlywedge_{(a,a') \in X} a \to a' \in S_A$ then $\curlywedge_{(a,a') \in X} f(a) \to f(a') \in S_B$.

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This leads to a category $\mathrm{ArrAlg}_{\mathrm{cart}}$ of arrow algebras and cartesian morphisms between those.

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This leads to a category $\mathrm{ArrAlg}_{\mathrm{cart}}$ of arrow algebras and cartesian morphisms between those. We will regard this category as pre-order enriched with $f \preccurlyeq g : \mathcal{A} \to \mathcal{B}$ if $\curlywedge_{a \in A} fa \to ga \in S_B$.

Correspondence to morphisms of triposes

Geometric morphisms of arrow algebras (Tarantino)

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As shown by Tarantino, these morphisms have the following properties:

- Morphisms of arrow algebras between locales coincide with locale morphisms.
- Morphisms between arrow algebras deriving from pcas correspond to computationally dense morphisms of pcas.
- Morphisms of arrow algebras correspond to geometric morphisms between the associated triposes.
- Morphisms between arrow algebras can be factored as a surjection followed by an embedding, where these surjections and embeddings induce surjections and embeddings on the level of triposes. The embeddings of arrow algebras are induced by (unique) nuclei.

Section 5

Comparison to work of Miquel

Implicative algebras

Our work on arrow algebras is heavily inspired by the work of Alexandre Miquel on *implicative algebras*.

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2. Every arrow algebra is equivalent to an implicative algebra.

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2. Every arrow algebra is equivalent to an implicative algebra.
3. However, there are many naturaly occurring examples of arrow algebras which are not implicative algebras. For instance, if $\mathbb{P}$ is a pca, then $D\mathbb{P}$ is an implicative algebra iff the application in $\mathbb{P}$ is total.
4. Also, we have a notion of morphism of arrow algebras and a neat factorisation of these morphisms into surjections and inclusions.

THANK YOU!

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