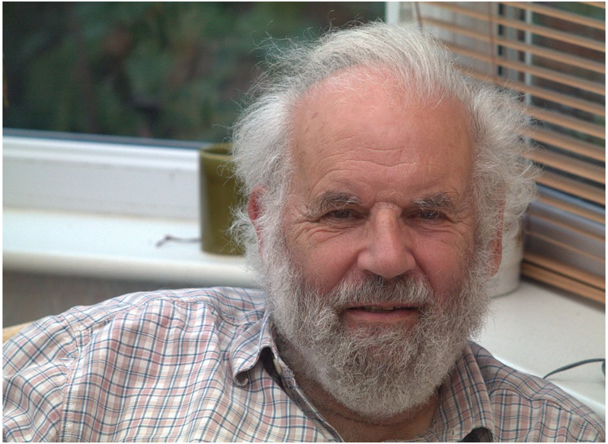
A Kaluzhnin–Krasner embedding theorem for associative algebras?

Tim Van der Linden[1]

Fonds de la Recherche Scientifique–FNRS
Université catholique de Louvain
Vrije Universiteit Brussel

Category Theory 2025 | Brno | 17 July 2025

[1] Joint work with Bo Shan Deval and Xabier García-Martínez.



Ronald Brown
1935–2024

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I am currently working on a talk sketching the influence of his work on categorical algebra.

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- $\mathbb{K}$-Lie algebras, where $\mathbb{K}$ is a field: here $A \wr B = Vect_{\mathbb{K}}(\overline{B}, A) \rtimes B$ and $\overline{B}$ denotes the universal enveloping algebra of B [Petrogradsky, Razmyslov, and Shishkin, 2007];

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Our approach is based on a category-theoretical analysis of the known cases, and includes concrete information about the feasibility of extending the result to other (new) contexts.

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The set-theoretical section s is a group homomorphism iff $\phi_G \circ s = \sigma$, so precisely when we have a morphism of *split* extensions.

Then ϕ_G is uniquely determined by $\phi_A\colon A \to Set(B,A)$.

The same analysis holds for Lie algebras and Hopf algebras.

3. Reduction to split extensions

The Embedding Theorem gives us an embedding of *group extensions*: $A \wr B$ induces

$$0 \longrightarrow Set(B, A) \xrightarrow[\kappa]{} Set(B, A) \rtimes B \underset{\sigma}{\overset{\pi}{\rightleftarrows}} B \longrightarrow 0$$

with $\sigma \coloneqq (0, 1_B) \colon B \to (Set(B, A) \rtimes B) = A \wr B$ the canonical inclusion.

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This alone already implies a strong universality result.

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In a *semi-abelian* category $\mathscr{X}$, for any objects A and B we have

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a universal split extension over B into which each split extension

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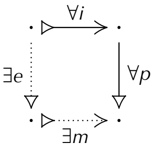

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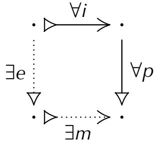

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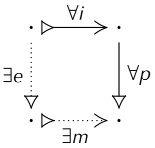

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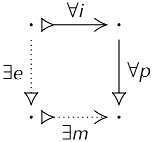

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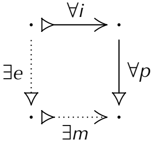
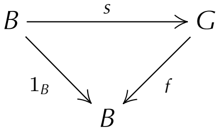


Examples:

- abelian categories: modules over a ring, sheaves of abelian groups;
- pointed varieties of universal algebras with a group operation: groups, rings, Lie algebras, associative algebras, crossed modules;
- loops, Heyting semilattices, cocommutative Hopf algebras, Set_*^{op}.

6. Split extensions and pointed slices

Dominique Bourn calls a split epimorphism $f\colon G \to B$ with a chosen splitting s a **point**, because it is a point in the slice category $(\mathscr{X} \downarrow B)$. We write $Pt_B(\mathscr{X}) \coloneqq (1_B \downarrow (\mathscr{X} \downarrow B))$.



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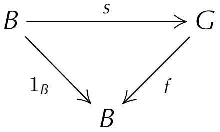

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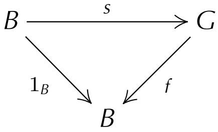

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Protomodularity is equivalent to the condition that these functors reflect isomorphisms.

1. The Kaluzhnin–Krasner *Universal Embedding Theorem*

For (non-abelian) groups A and B, the *Universal Embedding Theorem* says that the *(unrestricted) wreath product* $A \wr B$ acts as a universal receptacle for any group G viewed as an extension from A to B.

Theorem [Krasner et Kaloujnine, 1951]

For any group extension $0 \to A \to G \to B \to 0$, $\quad (A \lhd G,\ B \cong G/A)$
the group G can be embedded into the wreath product $A \wr B$
via a group homomorphism $\phi_G \colon G \to A \wr B$. □

- By definition, $A \wr B$ **is the group** $Set(B, A) \rtimes B$, where
  - $Set(B, A) \coloneqq \{h \colon B \to A$ **function**$\}$ with pointwise multiplication $(hh')(b) = h(b)h'(b)$, for $h, h' \colon B \to A$ and $b \in B$;
  - the action of B on $Set(B, A)$ is $(h^{b'})(b) = h(bb')$, for $b, b' \in B$.
- Fix a set-theoretical section $s \colon B \to G$ of $f \colon G \to B$. $\quad (f \circ s = 1_B)$
- $\phi_G \colon G \to A \wr B \colon g \mapsto (h_g, f(g))$ where $h_g \colon B \to A \colon b \mapsto s(b) \cdot g \cdot s(b \cdot f(g))^{-1}$.

It is not hard to check that this is indeed a group monomorphism.
Note that ϕ_G depends on the choice of s.

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  $\mathscr{X}(B, A) \cong \mathscr{X}(\top \times B, A) \cong \mathscr{X}(\top, A^B) \cong \mathscr{X}(\bot, A^B) = \{¡_{A^B}\colon \bot \to A^B\}$
- However, recall how $\mathit{Set}(B, A)$ occurs when constructing $A \wr B$ in Gp!

How to "correct" the concept so that it applies to *Gp*, *Lie*$_{\mathbb{K}}$, etc.?

Would not work:

- closedness for a different product such as $\otimes$; (too abelian)

7. Constructing function spaces: cartesian closedness

A category with finite products is *cartesian closed* when any two objects A, B have an *exponential* A^B. Formally, "times B" is left adjoint to "to the power B": $(-) \times B \dashv (-)^B$

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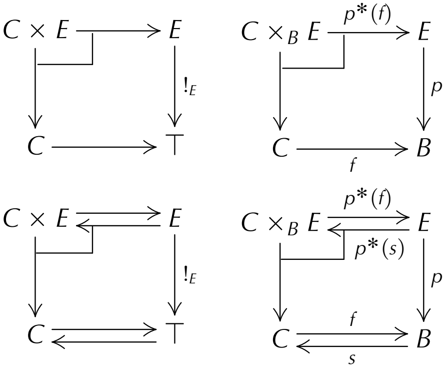
How to "correct" the concept so that it applies to *Gp*, $\mathit{Lie}_{\mathbb{K}}$, etc.?

Would not work:

- closedness for a different product such as $\otimes$; (too abelian)
- *local* or *algebraic* cartesian closedness. (no zeroes; too weak)

	plain	local	
plain			$p^* : (\mathscr{X} \downarrow B) \to (\mathscr{X} \downarrow E)$
algebraic			$p^* : Pt_B(\mathscr{X}) \to Pt_E(\mathscr{X})$



8. Flavours of cartesian closedness

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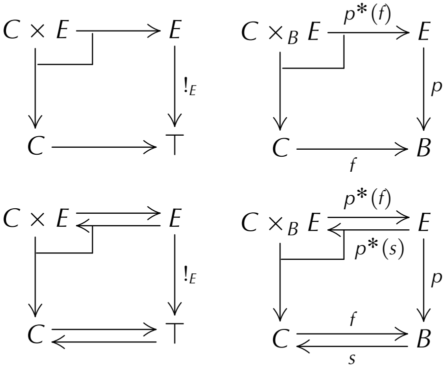

Which pullback functors are left adjoint?

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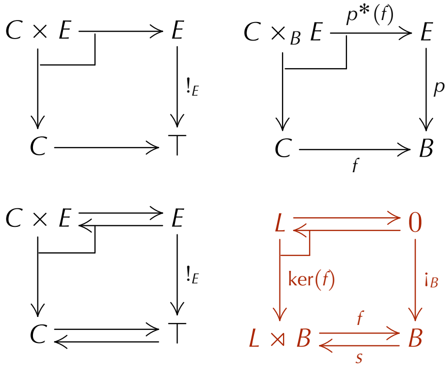

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In a semi-abelian category $\mathscr{X}$, the condition (LACC) reduces to the special case $E = 0$.

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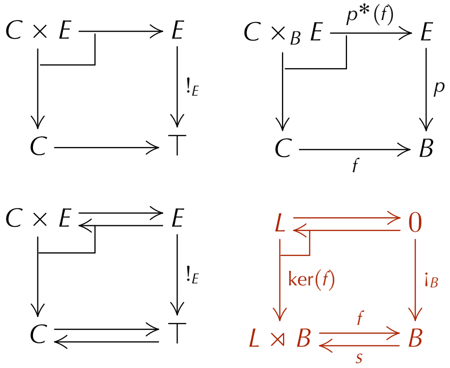

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Which pullback functors are left adjoint?

In a semi-abelian category $\mathscr{X}$, the condition (LACC) reduces to the special case $E = 0$.
If $\mathscr{X}$ is a variety, then this amounts to cocontinuity of the kernel functor $¡_B^*$.

9. Local algebraic cartesian closedness

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This means that for any object A of $\mathscr{X}$, we have

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Examples: $KR(A) \cong Set(B, A)$ in Gp, and $KR(A) \cong Vect_{\mathbb{K}}(\overline{B}, A)$ in $Lie_{\mathbb{K}}$.

4. Reduction to split extensions—and a **necessary** condition

Any split extension $S = (0 \to A \to G \rightleftarrows B \to 0)$ from A to B *naturally* embeds into the wreath product split extension as in

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\;k\;} & G & \underset{s}{\overset{f}{\rightleftarrows}} & B & \longrightarrow & 0 \\
 & & {\scriptstyle \phi_A}\downarrow & & {\scriptstyle \phi_G}\downarrow & & \| & & \\
0 & \longrightarrow & Set(B,A) & \xrightarrow[\kappa]{} & A \wr B & \underset{\sigma}{\overset{\pi}{\rightleftarrows}} & B & \longrightarrow & 0.
\end{array}$$

This becomes, **unavoidably**:

Theorem [Deval, García-Martínez and VdL, 2024]

In a *semi-abelian* category $\mathscr{X}$, for any objects A and B we have

$$R(A) = \big(0 \to KR(A) \to A \wr B \rightleftarrows B \to 0\big),$$

a universal split extension over B into which each split extension

$$S = \big(0 \to A \to G \rightleftarrows B \to 0\big)$$

embeds, iff $\mathscr{X}$ is ***locally algebraically cartesian closed***,
in which case $K \dashv R$ for each chosen object B
and the embedding is given by the S-component $\eta_S \colon S \to RK(S)$ of the adjunction unit. □

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Theorem [García-Martínez and VdL, 2019]

Over an infinite field $\mathbb{K}$ with $\mathit{char}(\mathbb{K}) \neq 2$, the only non-abelian (LACC) variety of non-associative $\mathbb{K}$-algebras is $\mathit{Lie}_{\mathbb{K}}$. □

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There is no Kaluzhnin–Krasner Theorem for associative algebras!

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  … Compactly generated Hausdorff groups are (LACC) through.
  So, finding the optimal setting for KK here seems an interesting problem.

Thank you!

References

- L. Bartholdi, O. Siegenthaler, and T. Trimble, *Wreath products of cocommutative Hopf algebras*, arXiv:1407.3835, 2014
- D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory Appl. Categ., 1998
- B. S. Deval, X. García-Martínez, and TVdL, *A universal Kaluzhnin–Krasner embedding theorem*, Proc. Amer. Math. Soc., 2024
- X. García-Martínez and TVdL, *A characterisation of Lie algebras via algebraic exponentiation*, Adv. Math., 2019
- J. R. A. Gray, *Algebraic exponentiation in general categories*, Appl. Categ. Structures, 2012
- G. Janelidze, L. Márki and W. Tholen, *Semi-abelian categories*, J. Pure Appl. Algebra, 2002
- M. Krasner et L. Kaloujnine, *Produit complet des groupes de permutations et problème de groupes. II*, Acta Sci. Math. (Szeged), 1951
- V. M. Petrogradsky, Yu. P. Razmyslov, and E. O. Shishkin, *Wreath products and Kaluzhnin–Krasner embedding for Lie algebras*, Proc. Amer. Math. Soc., 2007