The commuting tensor product of multicategories

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CT2025
Masaryk University, Brno

14 July 2025

GENERAL SECRETARIAT FOR RESEARCH AND TECHNOLOGY
H.F.R.I. Hellenic Foundation for Research & Innovation

Context and motivation

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	$\mathbb{B}$Mod		
objects	rings		
vertical 1-cells	homomorphisms		
horizontal 1-cells	bimodules $M \colon R \mathrel{\mapstochar\rightarrow} S$		

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commuting tensor	$R \otimes S$		
horizontal 1-cells	bimodules $M\colon R \to\!\!\!\mapsto S$		
commuting tensor	$R{\otimes}R' \overset{M\otimes M'}{\to\!\!\!\mapsto} S{\otimes}S'$		

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Fix a fibrant double category $(\mathbb{C}, \boxtimes, I)$

Fix an oplax monoidal fibrant double category $(\mathbb{C}, \boxtimes, I)$

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$$\begin{array}{ccccc}
A_1 \boxtimes A_2 & & \xrightarrow{(y_1 \circ x_1) \boxtimes (y_2 \circ x_2)} & & C_1 \boxtimes C_2 \\
\| & & \Downarrow \xi & & \| \\
A_1 \boxtimes A_2 & \xrightarrow{x_1 \boxtimes x_2} & B_1 \boxtimes B_2 & \xrightarrow{y_1 \boxtimes y_2} & C_1 \boxtimes C_2
\end{array}$$

Fix an oplax monoidal closed and fibrant double category $(\mathbb{C}, \boxtimes, I)$

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Theorem: The double category $\mathbb{B}\mathrm{im}(\mathbb{C})$ is oplax monoidal closed.

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- Monads with *multimaps* form a representable closed multicategory.

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- Define $m \otimes n$ on bimodules of monads, using $\boxtimes$ on free ones and extending via reflexive coequalizers.