Cocompleteness of synthetic $(\infty, 1)$-categories

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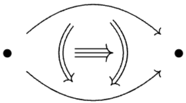
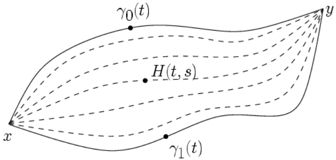
Dedicated to the dear memory of
Thomas Streicher (1958–2025)

∞-category theory

∞-categories:

- **higher morphisms** and **weak composition**





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- Dream: $(\infty, 1)$-category theory should be like 1-category theory, but up-to-homotopy.
- Problems in **analytic**, set-theoretic foundations: heavy encoding and not homotopy-invariant
- Can we do better in **synthetic** foundations?

Synthetic ∞-category theory?!

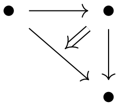
- Various synthetic theories: [RV22], [Cis+25], [MW23], [AM24]; see Fernando's talk
- We'd like to do it in **type theory** to make it amenable to computer formalization.
- Homotopy type theory (HoTT) is a homotopy-theoretic foundation system. But ∞-category is elusive (coherence problem)!
- Directed type theory is a potential way out, but complicated, and applicability is limited: mismatch of type theory with internal logic of Cat.
- **Simplicial HoTT (sHoTT)** strikes a balance: work in ∞-topos of simplicial objects, carve out (complete) Segal objects.
- Does this bring us closer to an "∞-category theory for undergraduates?" [Rie23]
- **Yes!**
  - **Free naturality & coherence theorems**
  - **Reduce to finite-dimensional arguments**
  - **Proofs less model-dependent**
  - **Verification via computer**

Simplicial homotopy type theory

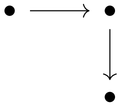
- **Question:** How to extend HoTT to capture ∞-categories?
- **Answer:** (after Riehl–Shulman)**:** HoTT + interval $\mathbb{I} := \Delta^1$
- This encompasses (complete) Segal spaces in the simplicial space model (or any ∞-topos).

$\Delta^1 =$

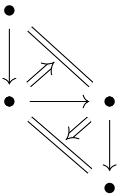
$\Delta^2 =$



$\Lambda^2_1 =$



$\mathbb{E} =$



Synthetic $(\infty, 1)$-category theory

Definition (Synthetic ∞-categories [RS17])

A type A is ...

- **Segal** or a **pre-∞-category** if $A^{\Delta^2} \simeq A^{\Lambda^2_1}$:

$$\{\blacktriangle\} \simeq \{\wedge\}$$

- **Rezk** or a **∞-category** if it is Segal and $A \simeq A^{\mathbb{E}}$:

$$\{\bullet\} \simeq \{\bullet \cong \bullet\}$$

- an **∞-groupoid** if $A^{\mathbb{2}} \simeq A$:

$$\{\bullet\} \simeq \{\bullet \to \bullet\}$$

Definition (hom type)

The **hom type** for $a, b : A$ is: $\hom_A(a, b) := \sum_{f : \mathbb{2} \to A} f(0) = a \times f(1) = b$

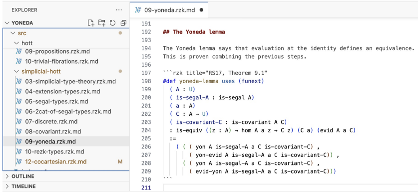
Some synthetic prehistory[2]

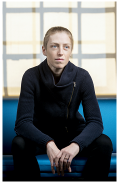
- Basic CT, fibered Yoneda lemma, adjunctions [RS17]
- Limits and colimits [Bar22]
- Cartesian fibrations and generalizations [BW23; Wei24a; Wei24b] (cf. also [RV22])
- Conduché fibrations [Bar24]
- directed univalent universe for ∞-groupoids and left fibrations [GWB24] (using *modal operators*), cf. [Rie18; WL20; Wea24]

[2] There is a plethora of other directed type theories for category theories; e.g. CaTT and talks by Wilfred and Thibaut.

Formalizing ∞-categories in Rzk

- Kudasov has developed the Rzk proof assistant, implementing sHoTT: `https://rzk-lang.github.io/`
- Using Rzk we initiated the first ever formalizations of ∞-category theory.
- In spring 2023, with Kudasov and Riehl we formalized the (discrete fibered) Yoneda lemma of ∞-category theory: `https://emilyriehl.github.io/yoneda/`
- alongside many other results
- Many proofs in this ∞-dimensional setting *easier* than in dimension 1!
- Formalization helped find a mistake in original paper
- More students & researchers joined us developing a library for ∞-category theory: `https://rzk-lang.github.io/sHoTT/` **Join us!**





Additions: modalities and axioms

To capture more categorical structure *modal operators* [Shu18; Gra+20]:

- **Opposite** op: $\langle \mathrm{op} \mid A \rangle$ has its n-simplices reversed
- **Discretization/core** $\flat$: $\langle \flat \mid A \rangle \to A$ is the maximal subgroupoid of A
- **Codiscretization** $\sharp$: $A \to \langle \sharp \mid A \rangle$ is localization at $\partial\Delta^n \to \Delta^n$ (for closed types)
- **Twisted arrows** tw: $\langle \mathrm{tw} \mid A \rangle$ has as n-simplices:

$$\begin{array}{ccccccccc} a_n & \longleftarrow & \dots & \longleftarrow & a_2 & \longleftarrow & a_1 & \longleftarrow & a_0 \\ \downarrow & & & & & & & & \\ a_{n+1} & \longrightarrow & \dots & \longrightarrow & a_{2n-2} & \longrightarrow & a_{2n-1} & \longrightarrow & a_{2n} \end{array}$$

We furthermore need some coherence conditions and axioms about modal and simplicial interaction; in particular **Birkhoff/Blechschmidt duality** or **synthetic quasi-coherence** [Koc77; Ble23; CCH24; PS25; SY25; Wil25; Mye25]:

Axiom (Blechschmidt duality/synthetic quasi-coherence [Ble23; CCH24])

Let A be a finitely presented $\mathbb{I}$-algebra, i.e., $A \simeq \mathbb{I}[x_1, \dots, x_n]/(r_1 = s_1, \dots, r_n = s_n)$, then the evaluation map is an equivalence:

$$\lambda a, f. f(a) : A \simeq (\hom_{\mathbb{I}}(A, \mathbb{I}) \to \mathbb{I})$$

The universe of spaces

Theorem

1. *There is a synthetic ∞-category $\mathcal{S}$ whose terms are ∞-groupoids.*
2. *$\mathcal{S}$ classifies (amazing) left fibrations:* $$\begin{array}{ccc} E & \longrightarrow & \mathcal{S}_* \\ {\scriptstyle \xi}\downarrow & & \downarrow{\scriptstyle \pi} \\ B & \xrightarrow{\chi_\xi} & \mathcal{S} \end{array}$$
3. *$\mathcal{S}$ is closed under Σ, identity types, and finite (co)limits.*
4. *$\mathcal{S}$ is **directed univalent**:* $$\mathrm{arrtofun} : (\Delta^1 \to \mathcal{S}) \simeq \left(\sum_{A,B:\mathcal{S}} (A \to B) \right)$$

NB: Bootstrapping is done in *cubical* outer layer: use tiny interval [Lic+18; WL20; Wea24; Ril24] and simplicial types as subtopos [KV20; Sat19; SW21; Wil25].

Towards synthetic higher algebra

We can internally define presheaf categories $\mathrm{PSh}(C) :\equiv \langle \mathrm{op} | C \rangle \to \mathcal{S}$.

Definition (∞-monoids)

The category Mon_∞ of ∞-*monoids* is the full subcategory[a] of $\mathrm{PSh}(\Delta)$ defined by the predicate

$$\varphi(X :_\flat \mathrm{PSh}(\Delta)) :\equiv \prod_{n:\mathrm{Nat}} \mathrm{isEquiv}(\langle X(\iota_k)_{k<n} \rangle : X(\Delta^n) \to X(\Delta^1)^n)$$

[a] need the codiscrete modality $\sharp$

This encodes the structure of a homotopy-coherent monoid. Multiplication is given through

$$\mu_X : X(\Delta^1) \simeq X(\Delta^1)^2 \to X(\Delta^1).$$

Definition (∞-groups)

The category Grp_∞ of ∞-*groups* is the full subcategory of Mon_∞ defined by the predicate

$$\varphi(X :_\flat \mathrm{Mon}_\infty) :\equiv \mathrm{isEquiv}(\lambda x, y. \langle x, \mu_X(x,y) \rangle : X(\Delta^1)^2 \to X(\Delta^1)^2)$$

One can show that both these categories have the right type of morphisms.

Cofinal functors & Quillen’s Theorem A

Definition (Cofinal functors)

A functor $f :_\flat C \to D$ is *right cofinal* if for every $X :_\flat D \to \mathcal{S}$ we have:

$$\varinjlim_{D} X \simeq \varinjlim_{C} X \circ f$$

Proposition (Characterization of right cofinality)

A functor is right cofinal iff it is left orthogonal to all right fibrations.

Theorem (Quillen’s Theorem A)

A functor $f :_\flat C \to D$ *is right cofinal if and only if* $L_{\Box}(C \times_D d/D) \simeq \mathbf{1}$ *for each* $d :_\flat D$.

Sifted colimits

Definition

A crisp ∞-category C is *sifted* if $\varinjlim_{C} : \mathcal{S}^C \to \mathcal{S}$ preserves finite products.

With Quillen's Theorem A we get:

Proposition

A crisp ∞-category C is sifted if and only if for all $n : \mathbb{N}$ the map $C^! : C \to C^n$ is right cofinal.

Theorem

If C has finite coproducts and sifted colimits then C is cocomplete.

Filtered colimits

Definition

An ∞-category is **finite** if it is generated by $\mathbf{0}$, $\mathbf{1}$, or ▯ under pushouts.

Definition

A crisp ∞-category C is *filtered* if $\varinjlim : \mathcal{S}^C \to \mathcal{S}$ preserves finite limits.

Definition

A crisp ∞-category C is *weakly filtered* if $C^! : \mathcal{C}^X \to \mathcal{C}$ is right cofinal for all finite ∞-categories X.

We can adapt [SW25] to prove:

Theorem

If C has finite and filtered colimits then it is cocomplete.

Spectra

Stable homotopy theory studies the limit behavior of spaces upon repeatedly suspending them. Spaces get replaced by *spectra* which correspond to symmetric monoidal ∞-groupoids and are central to higher algebra:

Definition (The category of spectra)

The ∞-category of *spectra* is defined as the limit (in the ambient universe): $\mathrm{Sp} :\equiv \varprojlim(\mathcal{S}_* \overset{\Omega}{\leftarrow} \mathcal{S}_* \overset{\Omega}{\leftarrow} \ldots)$.

Proposition

Sp *is closed under finite limits and filtered colimits.*

Following [Cno25], using the cofinality of $\mathbb{N}$ we can prove:

Proposition

$\Omega : \mathrm{Sp} \to \mathrm{Sp}$ *is an equivalence.*

Proposition

Sp *is finitely cocomplete, and pushouts coincide with pullbacks.*

Corollary

Sp *is cocomplete.*

Ordinary homology theories and smash product

For a commutative ring R, consider the Eilenberg–Mac Lane spectrum functor $1 \mapsto \mathrm{H}R : \mathcal{S} \to \mathrm{Sp}$.

Theorem

The family of functors $H_i : \mathcal{S} \to \mathrm{Ab}$ defined by $H_i\, X :\equiv \pi_i\, \mathrm{H}(X; R)$ satisfies the Eilenberg–Steenrod axioms.

Via directed univalence we can define the smash product $- \wedge - : \mathcal{S}_* \times \mathcal{S}_* \to \mathcal{S}_*$, immediately recovering the results proven in Book HoTT such as associativity [Lju24]. Using directed univalence again, we can lift the following to a functor on spectra:

Definition

The **smash product** of spectra $X, Y : \mathrm{Sp}$ is given by

$$X \otimes Y :\equiv \varinjlim_{i,j:\mathbb{N}} \Omega^{i+j} \Sigma^{\infty} (X_i \wedge Y_j).$$

Outlook

- synthetic ∞-category Cat and (co)cartesian straightening-unstraightening
- more on Conduché fibrations and the $(\infty, 2)$-topos perspective
- synthetic ∞-monads, ∞-operads, (symmetric) monoidal ∞-categories, ...
- $(\mathrm{Sp}, \otimes, \mathbb{S})$ as an s.m.c. (or the unit in presentable stable ∞-categories)
- internal higher topos theory
- metatheory of (higher?) type theories internally in type theory
- computational version and metatheory of modal sHoTT
- synthetic higher and differential geometry [Sch13; SS12; Shu18; Wel18; CCH24; MR23]
- more formalization
- ...

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