Metric Spaces, Entropic Spaces and Convexity

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Lawvere: Thermodynamics (1984)

State space X : $\overline{\mathbb{R}}^{\circ}$-category (entropic space)

Entropy: $\overline{\mathbb{R}}^{\circ}$-functor $X \longrightarrow [-\infty, \infty]$

Baez-Lynch-Moeller: Thermostatics (2021)

State space X : convex space

Entropy : concave map $X \longrightarrow [-\infty, \infty]$

GOAL: Synthesise these to get a category of convex entropic spaces & concave maps.

Inspiration: Fritz-Perrone approach to convexity monad on metric spaces.

Enrich over a commutative quantale $\mathcal{R} = ((\mathrm{ob}\mathcal{R}, \geq), +, 0, \multimap)$

(ie. (co)complete, skeletal, closed symmetric monoidal thin category)

Eg • $\overline{\mathbb{R}}_+ = (([0,\infty], \geq), +, 0)$ $\quad a \multimap b = \max(0, b-a)$

• $\overline{\mathbb{R}} = (([-\infty,+\infty], \geq), \dot{+}, 0)$ $\quad a \multimap b = b \dot{-} a$

• $\overline{\mathbb{R}}^{o} = (([-\infty,+\infty], \leq), \underset{\cdot}{+}, 0)$ $\quad a \multimap b = b \underset{\cdot}{-} a$

$\mathcal{R}$-category X: $\quad X(a,b) \in \mathrm{ob}\,\mathcal{R}$

$\forall a,b,c \in X \quad X(a,b) + X(b,c) \geq X(a,b)$

$0 \geq X(a,a)$

Eg. $\overline{\mathbb{R}}_+$-category is generalization of classical metric space.

• $\mathcal{R}$-category $\underset{\sim}{\mathcal{R}}$ with $\underset{\sim}{\mathcal{R}}(a,b) = a \multimap b$.

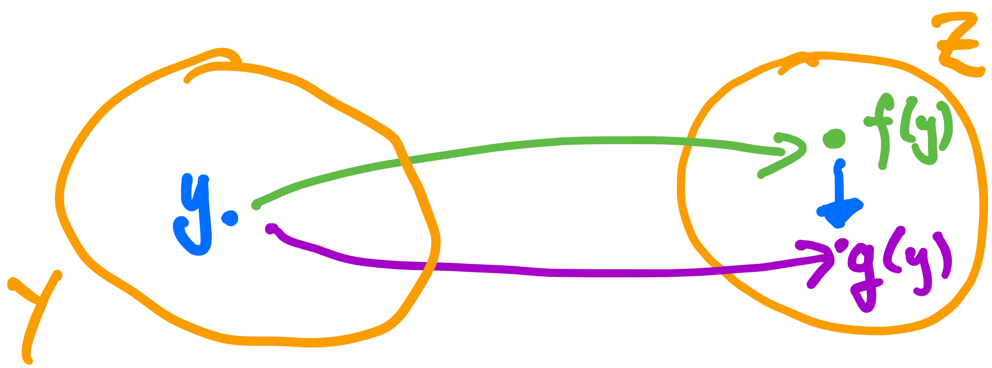
Get a category $\underset{\sim}{R}$-cat

Morphisms: short maps – distance non-increasing maps

closed monoidal:

$$\text{ob}\,[\![Y,Z]\!] = \{\text{short maps } Y \to Z\}$$

$$[\![Y,Z]\!](f,g) = \sup_{y\in Y} Z(f(y), g(y))$$

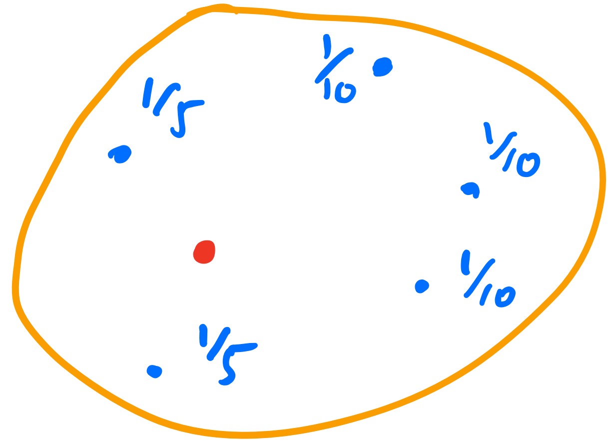


$[\![Y, \underset{\sim}{R}]\!]$ – "scalar-valued functions" or "op-presheaves"

Convex space A

$\forall n \in \mathbb{N}$, $\alpha_1, \ldots \alpha_n \geq 0$, $\sum^{!} \alpha_i = 1$, $x_1, \ldots, x_n \in A$

$\exists$ $\sum \alpha_i x_i \in A$
(barycentre)
+ axioms



Eg
- Any $\mathbb{R}$-vector space or affine space
- $[0, \infty]$ where $\alpha \infty + (1-\alpha) a = \infty$ $(\alpha > 0)$
- $[-\infty, +\infty]$ where $\alpha(+\infty) + (1-\alpha)(-\infty) = +\infty$
  (or can define the other way)

Algebras for convexity monad $C : \mathbf{Set} \to \mathbf{Set}$

$C(S) = \{ \sum \alpha_i \ulcorner x_i \urcorner \mid \sum^{!} \alpha_i = 1 \}$ formal

Convex quantale:

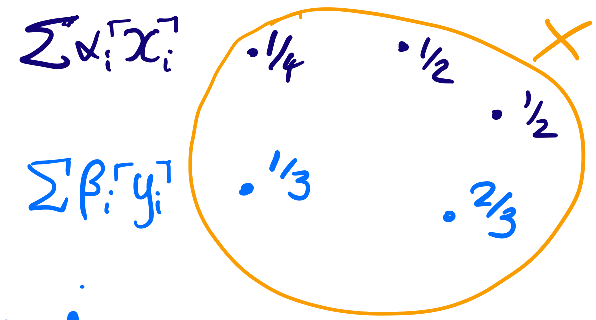
quantale $\mathcal{R}$ with $\mathrm{ob}\,\mathcal{R}$ a convex set and

- $a_i \geq b_i \implies \sum \alpha_i a_i \geq \sum \alpha_i b_i$
- $\sum \alpha_i (a_i \multimap b_i) \geq (\sum \alpha_i a_i) \multimap (\sum \alpha_i b_i)$

Eg
- $\overline{\mathbb{R}}_+ = ([0,\infty], \geq)$
- $\overline{\mathbb{R}} = ([-\infty,+\infty], \geq)$ $+\infty$ dominates
- $\overline{\mathbb{R}}^\circ = ([-\infty,+\infty], \leq)$ $-\infty$ dominates

Monad $C : \mathrm{Set} \to \mathrm{Set}$, want $\mathbb{R}\text{-cat} \to \mathbb{R}\text{-cat}$

$$C^{DD}(X)\left(\sum \alpha_i \ulcorner x_i \urcorner, \sum \beta_j \ulcorner y_j \urcorner\right) = ??$$



Consider $\sum \alpha_i \ulcorner x_i \urcorner$ as a finitely supported probability measure.

Get a set map

$$C(X) \times \mathrm{ob}[\![X, \mathbb{R}]\!] \to \mathrm{ob}\,\mathbb{R}\,; \qquad \left(\sum \alpha_i \ulcorner x_i \urcorner, f\right) \mapsto \underbrace{\sum \alpha_i f(x_i)}_{\mathbb{R}\text{ convex}}$$

Gives

$$C(X) \to \mathrm{ob}[\![[\![X, \mathbb{R}]\!], \mathbb{R}]\!]$$ ← short maps

Pull back the metric

$$C^{DD}(X)\left(\sum \alpha_i \ulcorner x_i \urcorner, \sum \beta_j \ulcorner y_j \urcorner\right) = \sup_{f \in [\![X, \mathbb{R}]\!]} \sum \alpha_i f(x_i) \multimap \sum \beta_j f(y_j)$$

This has an optimal transport interpretation in terms of costs.

We get a category of algebras and algebra maps for $C^{\mathcal{D}}$.

- Algebras are convex $\mathbb{R}$-categories: convex spaces which are $\mathbb{R}$-categories in a compatible way.

[Eg. $\sum \alpha_i X(x_i, y_i) \geq X(\sum \alpha_i x_i, \sum \alpha_i y_i)$

This is sometimes sufficient.]

- Algebra maps are convex linear maps

$f: X \to Y$ s.t. $f(\sum \alpha_i x_i) = \sum \alpha_i f(x_i)$

These are the wrong maps!
We want concave/convex maps.

We can fix this.

C^{DD} actually gives an $\mathbb{R}$-cat-enriched monad

$$C^{DD} : \mathbb{R}\text{-CAT} \longrightarrow \mathbb{R}\text{-CAT}$$

The "underlying category" of any $\mathbb{R}$-category is a poset

$$x \geqslant_X y \iff 0 \geq X(x, y)$$

This gives a 2-monad

$$C^{DD} : \mathbb{R}\text{-CAT} \longrightarrow \mathbb{R}\text{-CAT}$$

objects: $\mathbb{R}$-categories
morphisms: short maps
2-morphisms: $\geqslant$

- Strict algebras: convex $\mathbb{R}$-categories
- Lax & colax algebra maps: convex / concave maps

$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd} \qquad \begin{tikzcd} C^{DD}(X) \arrow[r] \arrow[d, "C^{DD}(f)"'] \arrow[dr, phantom, "\Swarrow"] & X \arrow[d, "f"] \\ C^{DD}(Y) \arrow[r] & Y \end{tikzcd} \qquad \sum \alpha_i f(x_i) \geq f\left(\sum \alpha_i x_i\right)$$

convex

Lawvere: Thermodynamics.

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Entropy: $\bar{\mathbb{R}}^{\circ}$-functor $X \to [-\infty, \infty]$

Baez-Lynch-Moeller: Thermostatics.

State space X : convex space

Entropy : concave map $X \to [-\infty, \infty]$

GOAL: Synthesise these to get a category of convex entropic spaces & concave maps.

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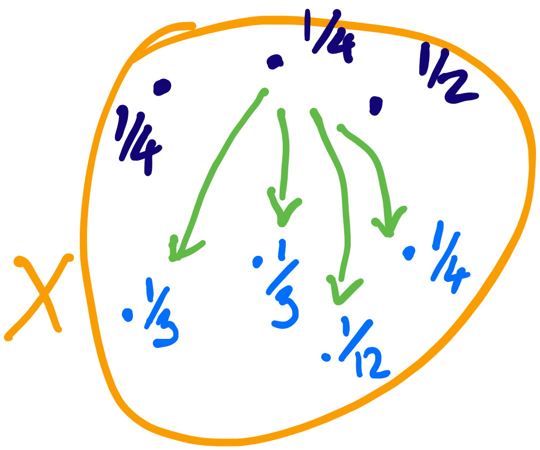
There's another way to define a distance between convex linear combinations.

Optimal transport plan of goods: minimize cost.

$$C^{01}(X)\left(\textstyle\sum' \alpha_i \ulcorner x_i \urcorner, \sum' \beta_j \ulcorner y_j \urcorner\right) = \inf \left\{ \sum \gamma_{ij} X(x_i, y_j) \;\middle|\; \sum_j \gamma_{ij} = \alpha_i,\ \sum_i \gamma_{ij} = \beta_j \right\}$$



γ_{ij} is amount of goods $x_i \rightsquigarrow y_j$

$\gamma_{ij} X(x_i, y_j)$ = cost of transportation

(aka Wassostein or Kantorovich-Rubinstein metric)

Not clear to me how to interpret this in category theory terms.

"Weak duality" holds:

$$C^{01}(X)\left(\textstyle\sum' \alpha_i \ulcorner x_i \urcorner, \sum' \beta_j \ulcorner y_j \urcorner\right) \geqslant C^{DD}(X)\left(\textstyle\sum' \alpha_i \ulcorner x_i \urcorner, \sum' \beta_j \ulcorner y_j \urcorner\right)$$