Locales are dense in Toposes

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Locales and Toposes

- Locales are a notion of space where opens take precedence over points.
- Toposes are a categorification of locales where opens become sheaves.
- Loc is subreflective in Topos.

$$\begin{tikzcd}
\mathsf{Loc} \arrow[r, hook, "\mathsf{Sh}"'] \arrow[r, phantom, "\perp", shift left=2] & \mathsf{Topos} \arrow[l, bend right]
\end{tikzcd}$$

- Loc is a large 2-category which is locally small and locally posetal.
- Topos is a large 2-category which is *not* locally small, but its hom-categories are accessible.

Dense subcategories

Definition

A full subcategory $f : \mathbf{C} \hookrightarrow \mathbf{D}$ is *dense* if any of the following equivalent properties hold:

- for each $d \in \mathbf{D}$, $\operatorname{colim}_{f(c) \to d} f(c) = d$.
- $\operatorname{lan}_f f$ exists, is pointwise, and equals $1_{\mathbf{D}}$.
- $N_f : \mathbf{D} \to \mathrm{Set}^{\mathbf{C}^{\mathrm{op}}} : d \mapsto \mathbf{D}(f-, d)$ is fully faithful.

In this talk we consider exclusively the third condition.

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Remark

It is *not* enough that **C** generates **D** under colimits, it has to do so under *canonical* colimits.

Localic points

Let $\mathcal{E}$ be a topos. For any locale X we have the category of X-points of $\mathcal{E}$.

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This defines a pseudofunctor LPt $\mathcal{E} : \mathsf{Loc}^{\text{op}} \to \mathsf{CAT}$, which is just the nerve of Sh at $\mathcal{E}$.

$$N_{\mathsf{Sh}} = \text{LPt}\ : \mathsf{Topos} \to \mathsf{CAT}^{\mathsf{Loc}^{\text{op}}} : \mathcal{E} \to \mathsf{Topos}(\mathsf{Sh}(-), \mathcal{E}).$$

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Goal

The nerve LPt is fully faithful bicategorically.

$$\mathrm{LPt}_{\ \mathcal{E},\mathcal{F}} : \mathsf{Topos}(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(\mathrm{LPt}\,\mathcal{E}, \mathrm{LPt}\,\mathcal{F}).$$

Outline

- Background on bisites and stacks
- Proof sketch
- Searching for a left adjoint

Sieves in bicategories

Definition

A *sieve* on $x \in \mathcal{K}$ is a fully faithful 1-cell $S \hookrightarrow \text{よ}_x$ in $\mathrm{Cat}^{\mathcal{K}^{\mathrm{op}}}$.

Up to equivalence, this consists of 1-cells with codomain x closed under precomposition *up to isomorphism.*

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Example

Any 1-cell $p : y \to x$ of $\mathcal{K}$ generates a sieve on x via the (bijective on objects, fully faithful) factorisation system.

$$\begin{tikzcd}
\text{よ}_Y \arrow[rr, "p"] \arrow[dr, two heads, "\text{b.o.}"'] & & \text{よ}_X \\
& S_p \arrow[ur, hook, "\text{ff}"'] &
\end{tikzcd}$$

Up to equivalence, a 1-cell belongs to S_p if it factors through p up to isomorphism.

Bisites

Definition

A *topology* J on $\mathcal{K}$ is a class $J(x)$ of *covering sieves* on each $x \in \mathcal{K}$ such that:

- the maximal sieve $\text{よ}_x \hookrightarrow \text{よ}_x$ belongs to $J(x)$,
- for each $S \in J(x)$ and 1-cell $f : y \to x$, the bipullback $R \hookrightarrow \text{よ}_y$ belongs to $J(y)$,

$$\begin{tikzcd}
R \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\cong"] & S \arrow[d, hook] \\
\text{よ}_y \arrow[r, "f"'] & \text{よ}_x
\end{tikzcd}$$

- if all the bipullbacks of $R \hookrightarrow \text{よ}_x$ along the 1-cells $f : y \to x$ of a covering sieve $S \hookrightarrow \text{よ}_x$ are covering, then R itself is covering.

Definition

A *bisite* is a bicategory equipped with a topology.

Loc as a bisite

Definition

Let $J_{\mathcal{O}}(X)$ on be the class of sieves on $X \in \mathsf{Loc}$ which contain at least one open surjection.

In other words, $S \in J_{\mathcal{O}}(X)$ if there is an open surjection $p : Y \to X$ such that $S_p \hookrightarrow S$.

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In other words, $S \in J_{\mathcal{O}}(X)$ if there is an open surjection $p : Y \to X$ such that $S_p \hookrightarrow S$.

This is a topology because open surjections are closed under composition, pullbacks, and identities in Loc.

Remark

Other topologies are available but, for the purposes of this talk, $J_{\mathcal{O}}$ is the most natural.

Stacks over a bisite

Definition

A pseudofunctor $F : \mathcal{K}^{\mathrm{op}} \to \mathsf{CAT}$ is a *stack* on the bisite $(\mathcal{K}, J)$ if it is local with respect to sieve inclusions:

$$F(X) \simeq \mathsf{CAT}^{\mathcal{K}^{\mathrm{op}}}(よ_X, F) \xrightarrow{\quad\sim\quad} \mathsf{CAT}^{\mathcal{K}^{\mathrm{op}}}(S, F)$$

for each $S \hookrightarrow よ_X$ in $J(X)$.

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Remark

For $(\mathsf{Loc}, J_{\mathcal{O}})$ it suffices to check the condition for sieves S_p generated by open surjections.

Lax epimorphisms

Taking iterated commas of any 1-cell $f : \mathbf{C} \to \mathbf{D}$ in a bicategory $\mathcal{K}$ yields a pseudofunctor $\ker f : {\Delta_2}^{\mathrm{op}} \to \mathcal{K}$.

$$\begin{tikzcd}
\mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \arrow[r, shift left=2] \arrow[r, "m" description] \arrow[r, shift right=2] & \mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \arrow[r, shift left=2, "s" description] \arrow[r, shift right=2, "t" description] & \mathbf{C} \arrow[l, "i" description]
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\end{tikzcd}$$

The bicolimit in $\mathcal{K}$ of $\ker f$ weighted by $\Delta_2 \hookrightarrow \mathrm{Cat}$ induces a factorisation:

$$\begin{tikzcd}
\mathbf{C} \arrow[rr, "f"] \arrow[dr] \arrow[rr, phantom, "\cong", shift right=5] & & \mathbf{D}. \\
& \operatorname{colim}^{\Delta_2} \ker f \arrow[ur] &
\end{tikzcd}$$

Definition

If $\operatorname{colim}^{\Delta_2} \ker f \simeq D$ we say that f is a *lax epimorphism* or of *lax descent type* in $\mathcal{K}$.

Example

In Cat, this is the (bijective on objects, fully faithful) factorisation system.

Toposes are stacks

Proposition

For $p : Y \to X$ in Loc, the following are equivalent:

- p is a lax epimorphism in Topos.
- LPt $\mathcal{E}$ is a stack w.r.t. S_p for each topos $\mathcal{E}$.
- LPt $(\mathrm{Set}[\mathbb{O}])$ is a stack w.r.t. S_p.

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In Moerdijk and Vermeulen 2000, it is shown that open surjections are lax epimorphisms in Topos. Hence, every LPt $\mathcal{E}$ is a stack on $(\mathsf{Loc}, J_{\mathcal{O}})$.

Remark

In the non-lax case, this was first noticed in Bunge 1990.

Proof sketch

Fix an open surjection $p : X \to \mathcal{E}$, with $X \in \mathsf{Loc}$. [Joyal and Tierney 1984]

$$\begin{tikzcd}
\mathsf{Topos}(\mathcal{E}, \mathcal{F}) \arrow[r, "\mathrm{LPt}_{\mathcal{E},\mathcal{F}}"] \arrow[d] & \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(\mathrm{LPt}\,\mathcal{E}, \mathrm{LPt}\,\mathcal{F}) \arrow[d] \\
\mathrm{Cocone}(\ker p, \mathcal{F}) \arrow[r] & \mathrm{Cocone}(\ker \mathrm{LPt}\,p, \mathrm{LPt}\,\mathcal{F})
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\end{tikzcd}$$

- the left map is an equivalence since p is a lax epi.
- $\ker \mathrm{LPt}\,p \cong よ \ker p$ and the bottom map is an equivalence by Yoneda.
- $\mathrm{LPt}\,_{\mathcal{E},\mathcal{F}}$ is an equivalence if and only if the right map is.

Proof sketch (cont'd)

The right map

$$\mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(\mathrm{LPt}\,\mathcal{E}, \mathrm{LPt}\,\mathcal{F}) \longrightarrow \mathrm{Cocone}(\ker \mathrm{LPt}\,p, \mathrm{LPt}\,\mathcal{F})$$

can be rewritten as

$$\mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(\mathrm{LPt}\,\mathcal{E}, \mathrm{LPt}\,\mathcal{F}) \xrightarrow{-\circ\tilde{p}} \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(\mathrm{colim}^{\Delta_2} よ \ker p,\ \mathrm{LPt}\,\mathcal{F})$$

where $\tilde{p}$ is as in

$$\begin{tikzcd} よ_X \arrow[rr, "\mathrm{LPt}\ p"] \arrow[dr] & & \mathrm{LPt}\,\mathcal{E}. \\ & \mathrm{colim}^{\Delta_2} よ \ker p \arrow[ur, "\tilde{p}"'] & \end{tikzcd}$$

Proof sketch (cont'd)

Write $[\ker p]$ for $\operatorname{colim}^{\Delta_2} よ \ker p$. We can compute it at each $Y \in \mathsf{Loc}$.

- objects: maps $a : Y \to X$.
- morphisms: lax squares

$$\begin{tikzcd}
Y \arrow[r, "b"] \arrow[d, "a"'] \arrow[rd, phantom, "\Rightarrow"] & X \arrow[d, "p"] \\
X \arrow[r, "p"'] & \mathcal{E}.
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- and $\tilde{p} : [\ker p] \to \mathrm{LPt}\,\mathcal{E}$ is just postcomposition with p.

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It remains to show that

$$\mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(\mathrm{LPt}\,\mathcal{E}, \mathrm{LPt}\,\mathcal{F}) \xrightarrow{-\circ\tilde{p}} \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}([\ker p],\ \mathrm{LPt}\,\mathcal{F})$$

is an equivalence, i.e., that $\mathrm{LPt}\,\mathcal{F}$ is local with respect to $\tilde{p} : [\ker p] \to \mathrm{LPt}\,\mathcal{E}$.

Proof sketch (fin)

It is easy to check that $\tilde{p}$ is

- pointwise fully faithful,
- $J_{\mathcal{O}}$-dense: in any bipullback

$$\begin{tikzcd}
S \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "\cong"] & {[\ker p]} \arrow[d, hook, "\tilde{p}"] \\
\text{よ}_Y \arrow[r, "q"'] & \mathrm{LPt}\,\mathcal{E}
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  the sieve $S \hookrightarrow \text{よ}_Y$ is covering.

As in 1-category theory, we can show that such a map belongs to the saturation of sieves inclusions.

So, the stack $\mathrm{LPt}\,\mathcal{F}$ is local w.r.t. $\tilde{p}$ and we are done: the nerve of Sh is 2-fully faithful.

$$\mathrm{LPt}_{\mathcal{E},\mathcal{F}} : \mathsf{Topos}(\mathcal{E},\mathcal{F}) \xrightarrow{\sim} \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(\mathrm{LPt}\,\mathcal{E}, \mathrm{LPt}\,\mathcal{F}).$$

Some details

Let $(\mathcal{K}, J)$ be a bisite. Assume that $f : A \hookrightarrow B$ in $\mathrm{Pstk}(\mathcal{K})$ is fully faithful and J-dense.

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$$D_J : \mathcal{K}_J \to \mathrm{Pst}(\mathcal{K})^{\to} : (x \in \mathcal{K}, S \in J(x)) \mapsto S \hookrightarrow \text{よ}_x .$$

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Consider also the following weight.

$$W_f : {\mathcal{K}_J}^{\mathrm{op}} \to \mathsf{Cat} : (x, S) \mapsto \mathrm{Pstk}(\mathcal{K})^{\to}(S \hookrightarrow よ_x,\ A \overset{f}{\hookrightarrow} B)_{\mathsf{cart}}$$

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Then one can show that

$$\mathrm{colim}^{W_f} D_J \simeq f$$

in $\mathrm{Pstk}(\mathcal{K})^{\to}$. If C is a stack on $(\mathcal{K}, J)$, then

$$\mathrm{Pstk}(\mathcal{K})(f : A \to B, C) = \lim^{W_f} \mathrm{Pstk}(\mathcal{K})(S \hookrightarrow よ_x, C)$$

is a bilimit of equivalences, hence an equivalence itself.

Can we do better?

We have seen that Topos embeds in localic prestacks.

The embedding LPt is a nerve and so preserves all bilimits. Could it have a left biadjoint?

$$\begin{tikzcd}
\mathsf{Topos} \arrow[r, hook, bend right, "\mathrm{LPt}"'] \arrow[r, phantom, "\perp"] & \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}} \arrow[l, dashed, bend right]
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Disclaimer

The following slides are work in progress.

Small prestacks

As expected, there is the issue of size: $\mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}$ is too large.

But one can still try: in Di Liberti 2022 the author provides a *relative* left biadjoint on *small* prestacks.

$$\begin{tikzcd}
 & \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}_{\mathrm{small}} \arrow[dl, dashed, "\Theta"'] \arrow[d, hook] \\
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The existence of Θ follows from the theory of biKan extension, and is realised by the formula

$$\Theta(F) \simeq \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}(F, \mathrm{LPt}\,\mathrm{Set}[\mathbb{O}]).$$

Small stacks

Maybe $\mathrm{Lpt} : \mathsf{Topos} \hookrightarrow \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}$ factors through $\mathsf{CAT}_{\mathrm{small}}^{\mathsf{Loc}^{\mathrm{op}}}$, and makes Θ is a genuine biadjoint.

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There is still hope: Lpt lands in small stacks.

$$\begin{tikzcd}
\mathsf{Topos} \arrow[r, hook] \arrow[rrr, hook, bend left, "\mathrm{LPt}"] & \mathsf{Stk}(\mathsf{Loc})_{\mathrm{small}} \arrow[r, hook] & \mathsf{Stk}(\mathsf{Loc}) \arrow[r, hook] & \mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}
\end{tikzcd}$$

Why? We saw that $\tilde{p} : [\ker p] \to \mathrm{LPt}\,\mathcal{E}$ made $\mathrm{LPt}\,\mathcal{E}$ the stackification of $[\ker p]$. So $\mathrm{LPt}\,\mathcal{E}$ is a small bicolimit of representables in $\mathsf{Stk}(\mathsf{Loc})$.

It remains to define Θ on $\mathrm{Stk}(\mathsf{Loc})_{\mathrm{small}}$. We cannot simply restrict as small stacks are not small prestacks.

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\end{tikzcd}$$

This is work in progress.

Thank you!

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