Locales are dense in Toposes

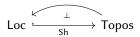
Errol Yuksel

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CT2025 Brno

Locales and Toposes

- Locales are a notion of space where opens take precedence over points.
- Toposes are a categorification of locales where opens become sheaves.
- Loc is subreflective in Topos.



- Loc is a large 2-category which is locally small and locally posetal.
- Topos is a large 2-category which is *not* locally small, but its hom-categories are accessible.

Dense subcategories

Definition

A full subcategory $f: \mathbf{C} \hookrightarrow \mathbf{D}$ is *dense* if any of the following equivalent properties hold:

- for each $d \in \mathbf{D}$, $\operatorname{colim}_{f(c) \to d} f(c) = d$.
- lan_f f exists, is pointwise, and equals 1_D.
- $N_f: \mathbf{D} \to \mathsf{Set}^{\mathbf{C}^{\mathsf{op}}}: d \mapsto \mathbf{D}(f-,d)$ is fully faithful.

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Remark

It is *not* enough that **C** generates **D** under colimits, it has to do so under *canonical* colimits.

Localic points

Let $\mathcal E$ be a topos. For any locale X we have the category of X-points of $\mathcal E.$

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This defines a pseudofunctor LPt $\mathcal{E}:\mathsf{Loc}^\mathsf{op}\to\mathsf{CAT},$ which is just the nerve of Sh at $\mathcal{E}.$

$$N_{\mathsf{Sh}} = \mathsf{LPt} \, : \mathsf{Topos} o \mathsf{CAT}^{\mathsf{Loc}^\mathsf{op}} : \mathcal{E} o \mathsf{Topos}(\mathsf{Sh}(-), \mathcal{E}).$$

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Goal

The nerve LPt is fully faithful bicategorically.

$$\operatorname{LPt}_{\,\mathcal{E},\mathcal{F}}:\operatorname{\mathsf{Topos}}(\mathcal{E},\mathcal{F})\stackrel{\sim}{\longrightarrow}\operatorname{\mathsf{CAT}^{\mathsf{Loc}^{\mathrm{op}}}}(\operatorname{\mathsf{LPt}}\,\mathcal{E},\operatorname{\mathsf{LPt}}\,\mathcal{F}).$$

Outline

- Background on bisites and stacks
- Proof sketch
- Searching for a left adjoint

Sieves in bicategories

Definition

A sieve on $x \in \mathcal{K}$ is a fully faithful 1-cell $S \hookrightarrow \sharp_x$ in $\mathsf{Cat}^{\mathcal{K}^{\mathsf{op}}}$.

Up to equivalence, this consists of 1-cells with codomain x closed under precomposition up to isomorphism.

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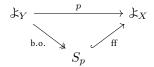
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Example

Any 1-cell $p:y\to x$ of $\mathcal K$ generates a sieve on x via the (bijective on objects, fully faithful) factorisation system.



Up to equivalence, a 1-cell belongs to S_p if it factors through p up to isomorphism.

Bisites

Definition

A topology J on ${\mathcal K}$ is a class J(x) of covering sieves on each $x\in {\mathcal K}$ such that:

- for each $S \in J(x)$ and 1-cell $f: y \to x$, the bipullback $R \hookrightarrow \sharp_y$ belongs to J(y),

$$R \longrightarrow S$$

$$\downarrow \qquad \cong \qquad \downarrow$$

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• if all the bipullbacks of $R \hookrightarrow \mathcal{L}_x$ along the 1-cells $f: y \to x$ of a covering sieve $S \hookrightarrow \mathcal{L}_x$ are covering, then R itself is covering.

Definition

A bisite is a bicategory equipped with a topology.

Loc as a bisite

Definition

Let $J_{\mathcal{O}}(X)$ on be the class of sieves on $X \in \mathsf{Loc}$ which contain at least one open surjection.

In other words, $S \in J_{\mathcal{O}}(X)$ if there is an open surjection $p:Y \to X$ such that $S_p \hookrightarrow S$.

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In other words, $S \in J_{\mathcal{O}}(X)$ if there is an open surjection $p: Y \to X$ such that $S_p \hookrightarrow S$.

This is a topology because open surjections are closed under composition, pullbacks, and identities in Loc.

Remark

Other topologies are available but, for the purposes of this talk, $J_{\mathcal{O}}$ is the most natural.

Stacks over a bisite

Definition

A pseudofunctor $F:\mathcal{K}^{\mathrm{op}}\to\mathsf{CAT}$ is a stack on the bisite (\mathcal{K},J) if it is local with respect to sieve inclusions:

$$F(X) \simeq \operatorname{CAT}^{\mathcal{K}^{\operatorname{op}}}(\sharp_X, F) \xrightarrow{\hspace{1cm}} \operatorname{CAT}^{\mathcal{K}^{\operatorname{op}}}(S, F)$$

for each $S \hookrightarrow \sharp_X$ in J(X).

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Remark

For (Loc, J_O) it suffices to check the condition for sieves S_p generated by open surjections.

Lax epimorphims

Taking iterated commas of any 1-cell $f: \mathbf{C} \to \mathbf{D}$ in a bicategory \mathcal{K} yields a pseudofunctor $\ker f: \Delta_2^{\mathrm{op}} \to \mathcal{K}$.

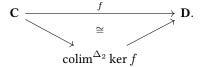
$$\mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \xrightarrow{\longrightarrow} \mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \xleftarrow{\longrightarrow} i \xrightarrow{s} \mathbf{C}$$

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$$\mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \xrightarrow{\longrightarrow m} \mathbf{C} \Rightarrow_{\mathbf{D}} \mathbf{C} \xleftarrow{\longleftarrow i} \xrightarrow{s} \xrightarrow{\longleftarrow} \mathbf{C}$$

The bicolimit in K of ker f weighted by $\Delta_2 \hookrightarrow \mathsf{Cat}$ induces a factorisation:



Definition

If $\operatorname{colim}^{\Delta_2} \ker f \simeq D$ we say that f is a lax epimorphism or of lax descent type in \mathcal{K} .

Example

In Cat, this is the (bijective on objects, fully faithful) factorisation system.

Toposes are stacks

Proposition

For $p:Y\to X$ in Loc, the following are equivalent:

- p is a lax epimorphism in Topos.
- LPt \mathcal{E} is a stack w.r.t. S_p for each topos \mathcal{E} .
- LPt $(Set[\mathbb{O}])$ is a stack w.r.t. S_p .

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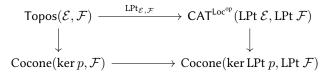
In Moerdijk and Vermeulen 2000, it is shown that open surjections are lax epimorphisms in Topos. Hence, every LPt $\mathcal E$ is a stack on (Loc, $J_{\mathcal O}$).

Remark

In the non-lax case, this was first noticed in Bunge 1990.

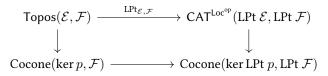
Proof sketch

Fix an open surjection $p:X\to\mathcal{E},$ with $X\in\mathsf{Loc}.$ [Joyal and Tierney 1984]



Proof sketch

Fix an open surjection $p: X \to \mathcal{E}$, with $X \in \text{Loc.}$ [Joyal and Tierney 1984]



- the left map is an equivalence since p is a lax epi.
- ker LPt $p \cong \mbox{$\sharp$}$ ker p and the bottom map is an equivalence by Yoneda.
- LPt \mathcal{E},\mathcal{F} is an equivalence if and only if the right map is.

Proof sketch (cont'd)

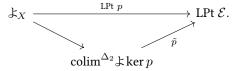
The right map

$$\mathsf{CAT}^{\mathsf{Loc}^\mathsf{op}}(\mathsf{LPt}\ \mathcal{E}, \mathsf{LPt}\ \mathcal{F}) \longrightarrow \mathsf{Cocone}(\ker \mathsf{LPt}\ p, \mathsf{LPt}\ \mathcal{F})$$

can be rewritten as

$$\mathsf{CAT}^{\mathsf{Loc}^{\mathsf{op}}}(\mathsf{LPt}\;\mathcal{E},\mathsf{LPt}\;\mathcal{F}) \xrightarrow{\phantom{\mathsf{Cop}}\phantom{\mathsf{Cop}}\phantom{\mathsf{CAT}}\phantom{\mathsf{Cop}}\phantom{\mathsf{Cop}}\phantom{\mathsf{Cop}}} \mathsf{CAT}^{\mathsf{Loc}^{\mathsf{op}}}(\mathsf{colim}^{\Delta_2} \sharp \ker p,\;\mathsf{LPt}\;\mathcal{F})$$

where \tilde{p} is as in



Proof sketch (cont'd)

Write $[\ker p]$ for $\operatorname{colim}^{\Delta_2} \sharp \ker p$. We can compute it at each $Y \in \operatorname{Loc}$.

- objects: maps $a: Y \to X$.
- morphisms: lax squares

$$\begin{array}{ccc} Y & \stackrel{b}{\longrightarrow} & X \\ a \downarrow & \Rightarrow & \downarrow p \\ X & \stackrel{p}{\longrightarrow} & \mathcal{E}. \end{array}$$

• and $\tilde{p}: [\ker p] \to \mathsf{LPt}\; \mathcal{E}$ is just postcomposition with p.

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It remains to show that

$$CAT^{Loc^{op}}(LPt \mathcal{E}, LPt \mathcal{F}) \xrightarrow{-\circ \tilde{p}} CAT^{Loc^{op}}([\ker p], LPt \mathcal{F})$$

is an equivalence, i.e., that LPt $\mathcal F$ is local with respect to $\tilde p:[\ker p]\to \mathsf{LPt}\ \mathcal E.$

Proof sketch (fin)

It is easy to check that \tilde{p} is

- pointwise fully faithful,
- $J_{\mathcal{O}}$ -dense: in any bipullback

$$\begin{array}{ccc} S & & & [\ker p] \\ \downarrow & & \downarrow \tilde{p} \\ & & \downarrow_Y & & \\ & & & q \end{array} \text{LPt } \mathcal{E}$$

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Proof sketch (fin)

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$$S \longrightarrow [\ker p]$$

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the sieve $S \hookrightarrow \sharp_Y$ is covering.

As in 1-category theory, we can show that such a map belongs to the saturation of sieves inclusions.

So, the stack LPt $\mathcal F$ is local w.r.t. $\tilde p$ and we are done: the nerve of Sh is 2-fully faithful.

$$\operatorname{LPt}_{\mathcal{E},\mathcal{F}}:\operatorname{Topos}(\mathcal{E},\mathcal{F})\stackrel{\sim}{\longrightarrow}\operatorname{CAT}^{\operatorname{Loc}^{\operatorname{op}}}(\operatorname{LPt}\mathcal{E},\operatorname{LPt}\mathcal{F}).$$

Let (\mathcal{K},J) be a bisite. Assume that $f:A\hookrightarrow B$ in $\mathrm{Pstk}(\mathcal{K})$ is fully faithful and J-dense.

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Consider also the following weight.

$$W_f: \mathcal{K}_J^{\mathrm{op}} \to \mathsf{Cat}: (x, S) \mapsto \mathsf{Pstk}(\mathcal{K})^{\to}(S \hookrightarrow \sharp_x, \ A \stackrel{f}{\hookrightarrow} B)_{\mathsf{cart}}$$

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Then one can show that

$$\operatorname{colim}^{W_f} D_J \simeq f$$

in $\operatorname{Pstk}(\mathcal{K})^{\rightarrow}$. If C is a stack on (\mathcal{K}, J) , then

$$\operatorname{Pstk}(\mathcal{K})(f:A\to B,C) = \lim^{W_f} \operatorname{Pstk}(\mathcal{K})(S \hookrightarrow \mathfrak{k}_x,C)$$

is a bilimit of equivalences, hence an equivalence itself.

Can we do better?

We have seen that Topos embeds in localic prestacks.

The embedding LPt is a nerve and so preserves all bilimits. Could it have a left biadjoint?

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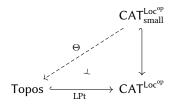
Disclaimer

The following slides are work in progress.

Small prestacks

As expected, there is the issue of size: $CAT^{Loc^{op}}$ is too large.

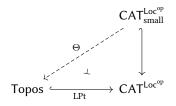
But one can still try: in Di Liberti 2022 the author provides a *relative* left biadjoint on *small* prestacks.



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The existence of Θ follows from the theory of biKan extension, and is realised by the formula

$$\Theta(F) \simeq \mathsf{CAT}^{\mathsf{Loc}^{\mathsf{op}}}(F, \mathsf{LPt}\ \mathsf{Set}[\mathbb{O}]).$$

Maybe Lpt : Topos \hookrightarrow CAT $^{Loc^{op}}$ factors through CAT $^{Loc^{op}}$, and makes Θ is a genuine biadjoint.

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No. Small bicolimits of representables live in $Cat^{Loc^{op}} \hookrightarrow CAT^{Loc^{op}}$, but in general the image of LPt does not.

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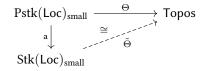
There is still hope: Lpt lands in small stacks.

$$\overbrace{\text{Topos} \overset{\text{LPt}}{\longleftrightarrow} \text{Stk}(\text{Loc})_{\text{small}} \overset{\text{LPt}}{\longleftrightarrow} \text{Stk}(\text{Loc}) \overset{\text{LPt}}{\longleftrightarrow} \text{CAT}^{\text{Loc}^{\text{op}}}}$$

Why? We saw that $\tilde{p}: [\ker p] \to \mathrm{LPt}\; \mathcal{E}$ made LPt \mathcal{E} the stackification of $[\ker p]$. So LPt \mathcal{E} is a small bicolimit of representables in $\mathrm{Stk}(\mathsf{Loc})$.

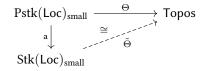
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Thank you!

References



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