

SMALL UNITARY REPRESENTATIONS

G real reductive Lie group

Exs $GL(n, \mathbb{R})$ dim = n^2
rank = n $n \times n$ invertible
real matrices

(Problems discussed here are all solved.)

$Sp(2n, \mathbb{R})$ dim = $2n^2 + n$
rank = n $2n \times 2n$ real matrices
preserving symplectic form
 $\omega((p, q), (p', q')) = p \cdot q' - q p'$
($p, q \in \mathbb{R}^n$)

(Problems discussed here are
mostly open, for $n \geq 3$; necessary
unitary reps may be mostly known.)

$E_8(\mathbb{R})$ dim = 248
rank = 8 split real form of E_8

(Problems discussed here are almost
untouched.)

LARGE GOAL: determine \hat{G}_u = set
of equiv classes of irr unitary repns of G .

SMALLER GOAL: construct a few interesting
small unitary repns of G .

THIS IS NOT A LECTURE SERIES ON THE

ORBIT METHOD

PHILOSOPHY (KOSTANT, KIRILLOV)

G Lie group, \mathfrak{g}_0 = Lie algebra, $i\mathfrak{g}_0^*$

$i\mathbb{R}$ -valued real
linear functional:

G acts on \mathfrak{g}_0 ("Ad") and so on $i\mathfrak{g}_0^*$
("coadjoint action").

$i\mathfrak{g}_0^*$ = disjoint union of G -orbits
"coadjoint orbits"

PHILOSOPHY:

CLASSICAL

$\hat{G}_u \longleftrightarrow$ set of G orbits
on $i\mathfrak{g}_0^*$

QUANTUM

Com. is at
best several-to-several,
undefined on parts of \hat{G}_u
and of $i\mathfrak{g}_0^*$

If "philosophy" were "theorem," LARGE GOAL
would break in two parts:

- ① determine set of coadjoint orbits
for G reductive
- ② to each orbit, attach some interesting
representations.

Part ① is easy...

Ex. $G = GL_n(\mathbb{R})$ $i\mathfrak{o}_0^* \cong$ all real $n \times n$ matrices

Coadj. orbits = conjugacy classes of real $n \times n$ mat

Use JORDAN DECOMP to reduce to classifying $n \times n$ NILPOTENT matrices.

WORKS IN GENERAL...

THM: FOR G REDUCTIVE / \mathbb{R} , have detailed / complete description of coadj. orbits in terms of...

→ NILPOTENT COADJ. ORBITS

Following are equivalent... for $G \cdot \lambda = 0 \subseteq i\mathfrak{g}$:

a) for some [all] $\lambda \in \mathcal{O}$, and some [all] pos. $t \neq 1$
 $t \cdot \lambda \in \mathcal{O}$

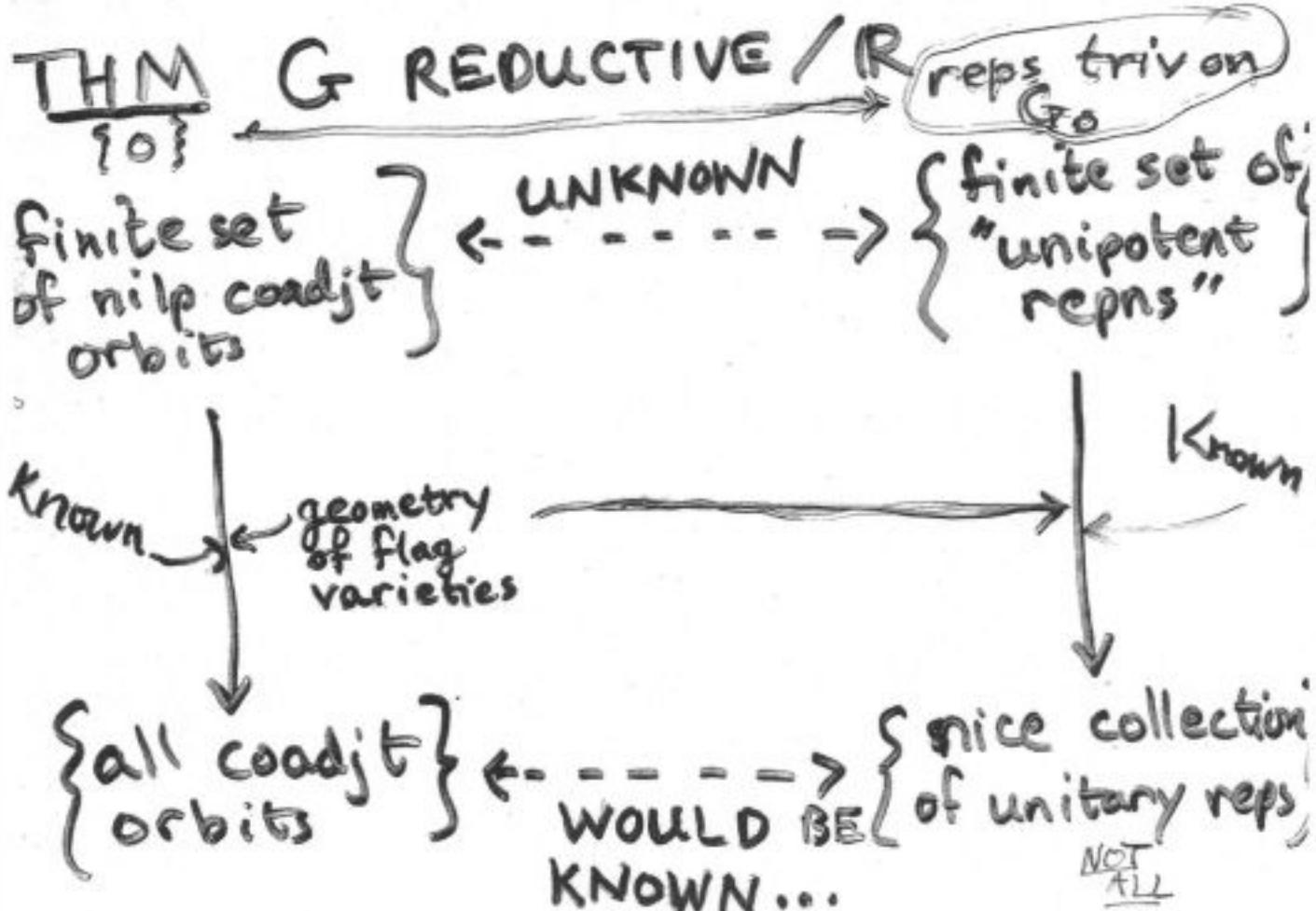
b) identify $\mathfrak{o}_{\lambda} \cong i\mathfrak{o}_0^*$ using G -inv't bil. form

$$\lambda \mapsto X_{\lambda}$$

Then for some [all] $\lambda \in \mathcal{O}$, X_{λ} is ad-nilpotent in \mathfrak{o}_0 .

THM FOR G REDUCTIVE / IR,
 set of nilpotent coadjoint orbits is
 explicitly known, FINITE

solves "parametrization of orbits" problem.
 or representations, have parallel reduction-to-
 nilp...



GOAL IS TO WORK ON TOP ARROW

(Extensive references in
 "Repr. Theory of Lie Groups", PCMI #8
 DV article)

HOW DO YOU KNOW YOU'VE WON?

- OR -

What does it mean for a repn to "correspond to a nilpotent orbit?"

$O \subset \text{ig}_0^*$ nilpotent orbit of G

$$G \cdot \lambda \simeq \underbrace{G/G_\lambda}_{\substack{\text{Subgroup of } G \\ \text{fixing } \lambda}}$$

HAMILTONIAN G -space
hence real symplectic, dim
 $= 2m$ EVEN.

To understand / describe homog space
relate it to PARABOLIC subgroups..

Fix nice non-degenerate $\text{Ad}(G)$ -
invariant symmetric bilinear form
 \langle , \rangle on \mathfrak{g}_0 .

$G = \text{GL}(n, \mathbb{R})$, $\mathfrak{g}_0 = n \times n$ matrices

$$\text{Ad}(g)(X) = g X g^{-1} \quad \langle X, Y \rangle = \text{tr}(XY)$$

PROP (Jacobson-Morozov) $\lambda \in \text{ig}_0^*$ is
nilpotent $\iff \exists H \in \mathfrak{g}_0$, $\text{ad}^*(H)(\lambda) = 2\lambda$,
with H hyperbolic. Choose such with

ROUGHLY: $\text{Ad}^*(H)$ diag/R,
real eigenvalues

$\langle H_\lambda, H_\lambda \rangle$ MINIMAL

STRUCTURE OF $G \cdot \lambda$ unique up to G_{λ}^{con}

λ nilp, $\text{Ad}^*(H_\lambda)(\lambda) = 2\lambda$, H_λ min. hyp.

FACT: $\text{Ad}^*(H_\lambda)$ [and hence $\text{Ad}(H_\lambda)$] have eigenvalues in \mathbb{Z} :

$$g_0 = \bigoplus_{m \in \mathbb{Z}} g_0(m, \lambda)$$

$$[g_0(m, \lambda), g_0(m', \lambda)] \subset g_0(m+m', \lambda)$$

$$g_0(\lambda) \stackrel{\text{def}}{=} \bigoplus_{m \geq 0} g_0(m, \lambda)$$

$$= \underbrace{g_0(0, \lambda)}_{l_0(\lambda)} \oplus \underbrace{u_0(\lambda)}_{\text{pos gradel}}$$

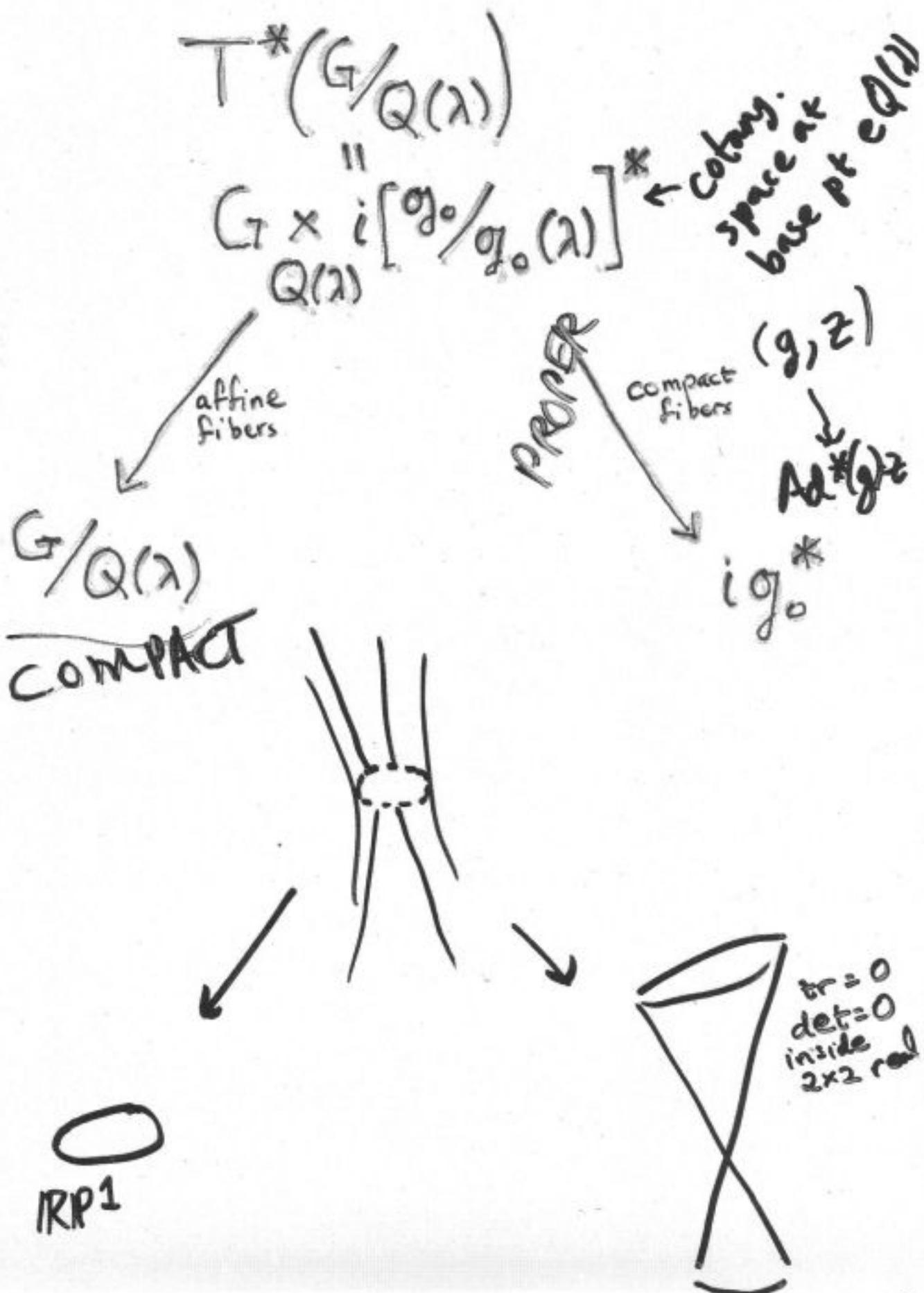
JACOBSON-MOROZOV PARABOLIC

2-eigenspace of $\text{Ad}^*(H_\lambda)$ $Q(\lambda)$

= annihilator of all $g_0(m, \lambda)$, $m \neq -2$

$$\subseteq \left[\frac{g_0}{g_0(\lambda)} \right]^* \rightsquigarrow G \times_{Q(\lambda)}^* \left[\frac{g_0}{g_0(\lambda)} \right]^*$$

= COTANGENT bdl of $G/Q(\lambda)$



CLASSIFICATION OF NILPOTENT COADS ORBITS

G real reductive $\rightarrow Q_{\min} = L_{\min} U_{\min}$
 $\min \text{ psgp}$

\rightarrow Dynkin diagram, aut of order 2,
subdiagram for L_{\min} (pres. by σ)

CONJ CLASS OF HYPERBOLIC ELTS

"

label vertices of diagram by
non-neg. real numbers so that

- vertices in L_{\min} labelled 0
- σ -inv.

DYNKIN: CONJ CLASS OF NILP ELTS / C

"
Special labelling of diagram by 0,1,2

Real nilp coadjt orbits = H ,

a) hyperbolic Dynkin element A AND

b) open orbit of G^H on +2 eig.

EXAMPLE $G = \mathbb{B}U(4,1)$
 (isom gp of Herm form on \mathbb{C}^5 , sign. $(4,1)$)

$$L_{\min} \cong U(3) \times GL(1, \mathbb{C})$$

σ = diagram aut.

COMPLEX NILP ORBITS

partition of 5	Dynkin diagram	defined in G ?
5	2-2-2-2	no - not triv on L_{\min}
41	2-1-1-2	no
32	1-1-1-1	no
311	2-0-0-2	YES
221	0-1-1-0	no
21 ³	1-0-0-1	YES
1 ⁵	0-0-0-0	YES

311: +2 eigenspace $\cong \mathbb{C}^3$ rep of $L_{\min} = U(3) \times \mathbb{C}$
one open orbit, one real nilp. $\xrightarrow{\text{acts by } l \geq 1}$

21³: +2 eigenspace \cong 1 diml rep of $U(3) \times \mathbb{C} \times$
two open orbit, one real nilp.

1⁵: +2 eig. = {0} \leadsto one real nilp.

Example $G = \mathrm{Sp}(2n, \mathbb{R})$

$\sigma = \text{trivial}$, $L_{\min} \cong (\mathbb{R}^*)^n$

All Dynkin diagrams are allowed

Example of diagram:

$$0 - 0 - 0 \dots - 0 = 2$$

$$G^H \cong GL(n, \mathbb{R})$$

orbits of $GL(n, \mathbb{R})$
classified by

signature (p, q, r) , $p+q+r=n$

#pos eigenv. #neg eigenv. #zero eigenv.

$$\begin{aligned} &+2 \text{ eig of } H \\ &\cong S^2(\mathbb{R}^n) \end{aligned}$$

\cong SYMM BIL FORM!

$n+1$ open orbits $\leftrightarrow n+1$ real
nilpotent orbits for this diagram

Orbits fit together subtly in quantization
sometimes forced to quantize several
together \rightarrow ref: Kashiwara/Vergne
Sahi ...

DIMENSIONS of nilp coadjt orbits

$\lambda \in i\mathfrak{g}_0^*$	nilp $\leadsto X_\lambda \in \mathfrak{g}_0$	
"three diml subalg"	TDS $\left\{ \begin{array}{l} H_\lambda \\ Y_\lambda \end{array} \right.$	hyperbolic Jacobson-Morozov el!
$\mathfrak{g}_0(6, \lambda)$	$[H_\lambda, X_\lambda] = 2X_\lambda$	
$\mathfrak{g}_0(4, \lambda)$	$[H_\lambda, Y_\lambda] = -2Y_\lambda$	
$\mathfrak{g}_0(2, \lambda)$	$[X_\lambda, Y_\lambda] = H_\lambda$	
$\mathfrak{g}_0(0, \lambda)$		$\mathfrak{g}_0 = \text{sum of all } (2) \text{ reps}$
$\mathfrak{g}_0(-2, \lambda)$		$\therefore \mathfrak{g}_0^\lambda = \mathfrak{g}_0^{X_\lambda}$ is graded, only non- κ degrees appear.
$\mathfrak{g}_0(-4, \lambda)$		
$\mathfrak{g}_0(-6, \lambda)$		

very easy

$$\dim_{\mathbb{R}} G \cdot \lambda = 2 \cdot [\# \text{ pos roots for } \mathfrak{g}_0^\lambda]$$

depends only on Dynkin diagrams
 - # pos roots for $\mathfrak{g}_0(0, \lambda)$
 - # roots in $\mathfrak{g}_0(1, \lambda)$

CONNECTING REPN'S TO ORBITS

"classical limit"

G real reductive (π, \mathcal{H}_π) irr. unitary
 $f =$ "test density" on G
 $\bar{\det}$ cptly supp smooth measure
 $\rightarrow \pi(f) = \int_G \pi(g) f(g) \quad$ bounded op.
 on \mathcal{H}_π

THM (Harish-Chandra) $\pi(f)$ is trace class,
 $\Theta_\pi(f) = \bar{\det} \text{tr } \pi(g)$ is a generalized function on G

If $\dim \mathcal{H}_\pi < \infty$, then Θ_π is a smooth function on G , namely, $\text{tr } \pi(g)$. In particular,

$$\Theta_\pi(1) \stackrel{?}{=} \dim \pi$$

SINGULARITY OF Θ_π at 1 measures
 infinite-dimensionality of π

Truncate $\Theta_\pi \rightarrow$ compact support near 1
 Lift to $\Theta_0 \rightarrow$ cptly supp generalized fn
 near 0 in $\Theta_0 \Theta_\pi$

DILATE by small T pos. constant $\rightarrow \Theta_\pi^T$ genld fn. big supp

$$\Theta_\pi^T(h) = \sum_{k=-n}^{\infty} T^k \underbrace{\Theta_{\pi,k}(h)}_{\text{next distn}}$$

Taylor series:
 $0 \leq n \leq \lfloor \Delta^{-1} \rfloor$

Thm $\hat{\Theta}_{\pi, k}^{\text{FOURIER}} = G\text{-invt homog. distn.}$
(Barbasch - V) supp. on nilp cone in
 $i\mathfrak{g}_0^*$

Leading term $\hat{\Theta}_{\pi, -n}$ = lin comb of measures on nilp coadjt orbits, all in same $G_{\mathbb{C}}$ -orbit. $n = \frac{1}{2} \dim \Theta$.

GIVES

$$\pi \leadsto \sum a_i \cdot O_i$$

real scalars several real forms,
same complex orbit

$SL(2, \mathbb{R})$: 3 orbits O_0, O_+, O_-

$$\pi = m\text{-diml rep} \longrightarrow m \cdot O_0$$

$$\pi = \begin{array}{c} \text{sph/hol sph/compl} \\ \text{prime. series} \end{array} \longrightarrow 1 \cdot O_+ + 1 \cdot O_-$$

$$\pi = \begin{array}{c} \text{or limit} \\ \text{hol. dis. series} \end{array} \longrightarrow 1 \cdot O_+$$

$$\pi = \text{antihol d.s.} \longrightarrow 1 \cdot O_-$$

CORR. WE WANT:

$$O_0 \longleftrightarrow 1 \text{ diml}$$

$$O_+ \longleftrightarrow \text{hol lim d.s.}$$

$$O_- \longleftrightarrow \text{antihol lim d.s.}$$

$$\left. \begin{array}{l} O_+ + O_- \longleftrightarrow \text{2nd sph. p.s.} \\ \end{array} \right\}$$

GOAL: sharpen/refine corresp.

$$\pi \xrightarrow{\text{irr. adm}} \sum a_i \theta_i$$

↑ real ↑ K nilp const.

and make it ALGEBRAIC

G real reductive $\Rightarrow K$ maximal comp

$\mathfrak{g}_0 = \text{real Lie alg}$

$\mathfrak{g} = \mathfrak{g}_0 \otimes_R \mathbb{C}$

$K_{\mathbb{C}}$ complex reductive
algebraic

Harish-Chandra: studying irr. admissible
(includes unitary) reps of G up to
infl equiv

||

study irr. $(\mathfrak{g}, K_{\mathbb{C}})$ -modules X

$X|_{K_{\mathbb{C}}} = \text{direct sum of alg. irr. repns}$

\hookrightarrow so differentiates to Lie alg
rep of K

$X = \text{rep of Lie alg } \mathfrak{g} \hookrightarrow \text{restricts to } K$

REQUIRE: two actions of K agree

$$- k \cdot (Z \cdot v) = [\text{Ad}(k) Z] \cdot (k \cdot v)$$

($k \in K_{\mathbb{C}}, Z \in \mathfrak{g}, v \in X$)

THIS IS NOT A LECTURE SERIES ON

STANDARD LIMIT REPRESENTATIONS

$G = \text{SL}(2, \mathbb{R})$ $K = \text{SO}(2)$ $\hat{K} \cong \mathbb{Z}$

sph. princ. series $\pi_{\text{sph}}(\nu)$ irr for $\nu \notin 2\mathbb{Z} +$
cplx param

-8 -6 -4 -2 0 2 4 6 8 ...

non-sph. princ. series $\pi_n(\nu)$ REDUCIBLE at $\nu = 0$ (and other even ν)

-7 -5 -3 -1 + 1 3 5

holom (lim.) d.s. $\pi_+(m)$, $m \geq 0$

$m+1 \quad m+3 \quad m+5 \quad \dots$

$\pi_-(m)$ anti hol (lim.) d.s

$m+5-m-3-m-1$

NOTE $\pi_n(0) = \pi_+(0) \oplus \pi_-(0)$

Thm $\pi_{\text{sph}}(0)$, $\pi_+(m)$ ($m \geq 0$)
 $\pi_-(m)$

form a \mathbb{Z} -basis of res. to K of
virtual $(\mathfrak{g}, K_{\mathbb{C}})$ -modules of finite
length.

GENERAL G

P = MAN cuspidal parabolic
 δ limit of discrete series
 for M

$$\rightsquigarrow \text{Ind}_P^G(\delta \otimes 1 \otimes 1) = \pi(\delta)$$

STANDARD LIMIT REPN with trivial continuous parameter. CALLED FINAL if it isn't like non-spherical $SL(2)$
 roughly \approx some intertwining ops are trivial.

Thm a) Each final standard limit repn $\pi(\delta)$ has unique lowest K-type $\mu(\delta)$ having mult. 1.

b) Gives bijection $\hat{K} \leftrightarrow$ [final std limit reps]

c) X any finite length $(\mathfrak{g}, K_{\mathbb{C}})$ -module

$$X|_K = \sum_{\delta \text{ final}} m_X(\delta) \cdot \pi(\delta)|_K$$

$m_X(\delta)$ coeffs from char formula
 X

Good things about formula

COMPUTABLE: $m_X(s)$ is more or less a coeff. in a character formula for X , or a sum of such coeffs; given by Kazhdan-Lusztig algorithm.

$\pi(s)|_K$ explicitly computable by Blattner formula, etc; so can use formula to compute $X|_K$

Bad things about formula

CANCELLATION?

trivial for $= \pi_{\text{spn}}(0) - \pi_+(1) - \pi_-(1)$

$SL(2, \mathbb{R})$ $\{2, 4, 6, \dots\}$ $\{-2, -4, -6, \dots\}$
 \downarrow on $SO(2)$

0 only $\{0, \pm 2, \pm 4, \dots\}$

CAN'T SEE UNDERLYING NILPOTENT ORBIT: typically

$\pi(s) \sim$ (principal nilp) + lower orbit terms
 ↑
 char expansion

after subtraction, leading terms disappear

INFINITESIMAL CHARACTERS

\mathfrak{g} complex reductive $\supset \mathfrak{g}$ Cartan subalg

$\Delta(\mathfrak{g}, \mathfrak{g})$ root system $\supset \Delta^+(\mathfrak{g}, \mathfrak{g})$ pos. roots.

\mathfrak{g}_{IR}^* = real span of (some chosen) algebraic character lattice

U

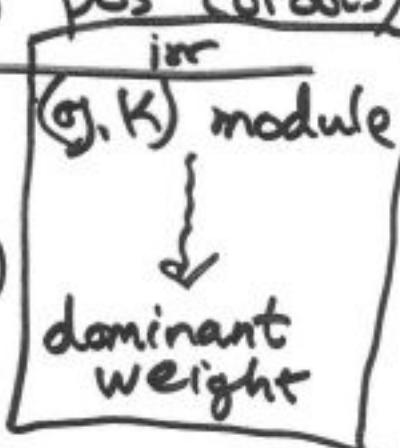
$\mathfrak{g}_{IR,+}^*$ = positive Weyl Chamber

(values ≥ 0 on all pos coroots)

$Z(\mathfrak{g})$ = center of $U(\mathfrak{g})$

$\cong S(\mathfrak{g})^{W(\mathfrak{g}, \mathfrak{g})}$

Harish-Chandra



X irr (\mathfrak{g}, K) -module $\Rightarrow Z(\mathfrak{g})$ acts

$\rightarrow \boxed{\lambda \in \mathfrak{g}^*/W(\mathfrak{g}, \mathfrak{g})}$ by scalars infinitesimal character

$\rightarrow \lambda_{IR} \in \mathfrak{g}_{IR}^*/W(\mathfrak{g}, \mathfrak{g})$ real part

$\rightarrow \boxed{\lambda_-(X) \in \mathfrak{h}_{IR}^*}$ UNIQUE DOMINANT REP

notes (→)

STANDARD LIMIT MODULE

$$\pi(\delta) = \text{Ind}_{MAN}^G (\delta \otimes 1 \otimes 1)$$

$\rightarrow \lambda(\delta)$ HC parameter of discrete series δ

THIS IS INF'L CHAR OF $\pi(\delta)$.

(ALSO $\lambda(\delta) \approx$ highest weight of lowest K -type $\mu(\delta)$; exactly = if G is a complex group.)

THINK OF STANDARD LIMIT REPS

$\pi(\delta)$ as partially ordered by $|\lambda(\delta)|$

Since $\lambda(\delta) \in$ lattice in $\mathfrak{g}_{\text{IR}}^*$, have only finitely many δ with $|\lambda(\delta)| \leq \text{const.}$

THM If X irr (G, K) module of infl char λ ,

then

$$X|_K = \sum_{\delta} m_X(\delta) \pi(\delta)$$

$$|\lambda(\delta)| \leq |\text{Re } \lambda|$$

$X =$ finite length $(\mathfrak{g}, K_{\mathbb{C}})$ module
 \rightarrow compatible actions of alg gp $K_{\mathbb{C}}$,
algebra $\underline{U(\mathfrak{g})}$
FILTERED!

pick fin. diml. $K_{\mathbb{C}}$ -invt $X_0 \subset X$ so

$$X = U(\mathfrak{g}) \cdot X_0$$

DEFINE

$$X_k = U_k(\mathfrak{g}) X_0$$

$$X_0 \subset X_1 \subset X_2 \subset \dots \quad \bigcup X_k = X$$

fin diml $K_{\mathbb{C}}$ -invt

\rightarrow $\boxed{\text{gr } X = \text{finitely generated graded } S(\mathfrak{g}/\mathfrak{n}_k) \text{-module with compatible } K_{\mathbb{C}} \text{ action}}$

\rightarrow well-defined map of Grothendieck groups

$$K(M(\mathfrak{g}, K_{\mathbb{C}})) \xrightarrow{\text{gr}} K(S(\mathfrak{g}/\mathfrak{n}_k^*), K_{\mathbb{C}})$$

$\mathcal{N}_{\mathfrak{g}}^* = \text{nilpotent elts in } (\mathfrak{g}/\mathfrak{n}_k)^*$

THM Map gr is surjective, kernel equal to virtual (\mathfrak{g}, K_C) modules whose restriction to K_C is zero. Hence

$$\{\text{gr } \pi(\delta) \mid \delta \text{ final std limit}\}$$

is a \mathbb{Z} -basis for

$$K(S(\mathfrak{g}/\mathfrak{n}), K_C)$$

$\underbrace{n_g^*}_{n_\theta^*}$

- finitely generated modules for polynomial ring $S(\mathfrak{g}/\mathfrak{n})$
- supported on nilpotent cone n_θ^*
- compatible algebraic action of K_C

This basis has good connections to repn theory, infinitesimal character, etc.
Need a different basis with good connections

to geometry of nilpotent cone; and
then NEED TO CONTROL CHANGE OF BASIS.

THM G real reductive $\Rightarrow K$ maxl cpt
ostant
jetiguchi
 \mathfrak{g}_0 real Lie alg. $K_C = \text{cplx re}$
 $i\mathfrak{g}_0^* \supset \mathcal{N}_{IR}^*$ alg.
real nilp cone $\mathfrak{g}^* \supset (\mathfrak{g}/\mathbb{R})^* \supset \mathcal{N}_0^*$
theta-nilp. cone

There is a natural bijection

$$G/G^\lambda \approx G \cdot \lambda \longleftrightarrow K_C \cdot \xi \approx \frac{K_C}{K_\lambda}$$

between orbits of G on \mathcal{N}_{IR}^* and orbits of K_C on \mathcal{N}_0^* . The maximal compact subgroups

$$K^\lambda \subset G^\lambda, \quad K^\xi \subset K_C$$

are naturally isomorphic: for $\lambda \leftrightarrow \xi$,

$$K^\lambda \approx K^\xi$$

FIX A PAIR (ξ, ε) $\xi \in \mathcal{N}_0^*, \varepsilon \in (K_C^\xi)^\wedge$
 \rightarrow alg vector bundle $\mathcal{V}_{\xi, \varepsilon}$ over $K_C \cdot \xi$

FIX A f.g. graded K_C -equiv S(\mathfrak{g}/\mathbb{R})-module
 $N(\xi, \varepsilon)$ supported on $\overline{K_C \cdot \xi}$, agreeing
 with $\mathcal{V}_{\xi, \varepsilon}$ over $K_C \cdot \xi$.

unique modulo something supported on $\partial \overline{K_C \cdot \xi}$ SPANI (20)

COR The images

$$[N(\xi, \tau)] \in K(S_{\lambda_\theta^*}(g_k), K_C)$$

(as ξ runs over orbit representatives of
 K_C on \mathcal{N}_θ^* , $\tau \in (K_C^\theta)^\wedge$)
form a \mathbb{Z} -basis.

COR π irr. admissible rep of G

$\rightarrow X$ irr (\mathfrak{o}, K_C) module

$\rightarrow \text{gr } X$ assoc. graded module for
 $S(g_k), K_C$

Write $K_C \cdot \xi_1, \dots, K_C \cdot \xi_r$ for orbits
of maximal dim in $\text{supp}(\text{gr } X)$,

$K_C \cdot \xi_{r+1}, \dots, K_C \cdot \xi_{r+s}$ for remaining
orbits in closure. THEN there
are repns τ_j of $K_C^{\xi_j}$ (possibly ~~irred.~~)

$$[\text{gr } X] = \sum_{j=1}^{r+s} [N(\xi_j, \tau_j)]$$

so basis $\{\pi(s)\}$ is easy to compute

basis $\{[N(\xi, \tau)]\}$ controls assoc.
variety

BIG PROBLEM: Compute change-of-basis matrix.

Solved by Pramod Achar for $GL(n, \mathbb{C})$
(Proved by Achar for $GL(n, \mathbb{C})$)

(FALSE) CONJECTURE There is a bijection

$$\left\{ \text{pairs } (\xi, \tau) \text{ up to } K_C \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{final std. limit} \\ \text{reprs } \pi(s) \end{array} \right\}$$
$$(\xi, \tau) \longmapsto s(\xi, \tau)$$

SO THAT

$$[N(\xi, \tau)] = \pm \text{gr } \pi(s(\xi, \tau)) + \sum_{\substack{\text{mg. gr } \pi(s) \\ |\lambda(s')| < \\ \lambda(s(\xi, \tau))}}$$

That is, change of basis is lower triangular

it would be true whenever no nilpotent K_C
orbit boundary has ^{cpt}codim = 1.
(Possibly proved by Bezrukavnikov)

EXAMPLE $SL(2, \mathbb{C})$ $K = SU(2)$

Final std limit reps = $\underbrace{\pi(m)}$, $m=0, 1, 2, \dots$

principal series repn
 $\leftrightarrow C^\infty$ sections of line bundle
 $\mathcal{O}(m)$ on \mathbb{CP}^1

$\boxed{\pi(m)|_K = su(2) \text{ reps of dim } m+1, m+3, m+5, \dots}$

Two nilpotent K_C orbits on $(\mathfrak{g}/\mathfrak{k})^*$...

$\xi = 0$ $K_C^\xi = K_C$; irr reps τ_d of dim $d=1, 2, 3, \dots$

$N(\xi, \tau_d) = \text{copy of } \tau_d \text{ at } 0$ $\delta(\xi, \tau_d) = d+1$
 $= \pi(d-1) - \pi(d+1)$ $d=1, 2, 3, \dots$

$\xi = \text{non-zero}$, $K_C^\xi = \left\{ \begin{pmatrix} \varepsilon & z \\ 0 & \varepsilon \end{pmatrix} \mid \varepsilon = \pm 1, z \in \mathfrak{s} \right\}$

Two irr chars $\tau_{\text{triv}}, \tau_{\text{sgn}}$

$N(\xi, \tau_{\text{triv}}) = \text{functions on nilpotent cone in } (\mathfrak{g}/\mathfrak{k})^*$
 $= \pi(0) \quad \delta(\xi, \tau_{\text{triv}}) = 0$

$N(\xi, \tau_{\text{sgn}}) = \text{odd functions on double cover}$
 $= \pi(1) \quad \delta(\xi, \tau_{\text{sgn}}) = 1$

ATTACHING A WEIGHT TO A UNION OF ORBITS...

$S \subset \mathcal{N}_\theta^*$ closed $K_{\mathbb{C}}$ -inut subvariety
of nilpotent cone

M ^{NON ZERO}_{compatible} $(S(g/k), K_{\mathbb{C}})$ -module,
 $\text{Supp}(M) \subset S$.

Write $M|_K = \sum_S m_S(M) \cdot \pi(\delta)|_K$

Define $|M| = \max_{m_S(M) \neq 0} |\lambda(\delta)|$
 ↑ biggest
 "infl char"
 appearing

Define

$$|S| = \min_{M \text{ supp on } S} |M|$$

Means: anything supported inside S has to have infl char $\geq |S|$

Can show:

$$|\{\circ\}| = |\wp|$$

half sum
of pos roots

Achar: $GL(n, \mathbb{C})$ | Richardson
orbit for $P = LU$ | = $|\beta_L|$

Prop Suppose X is (g, K) module

with

- a) $\text{ass var}(X) = \bar{O}$ (closure of one orbit)
- b) corr. vector bdlle on \bar{O} irr
- c) X admits non-deg int Henn form
- d) X has infl char λ satisfying

$$|\lambda| < |\partial\bar{O}|$$

← something of infl char
of count line
↑ or
↓ or

Then X is unitary.

This is why we care about $|S|$;
information about change of basis
matrix $\pi(\delta) \rightsquigarrow N(\xi, z)$ is intended
only to compute $|S|$.

Ex. X = ladder repn (reps of K are in a
single line, mult = 1).

Then $\text{ass var}(X) = \bar{O}_{\min}$, and (b) automatic.
 $\partial\bar{O} = \{0\}$, and $|\{\delta_0\}| = p$; so need only infl char
smaller than p