Jet Isomorphism

for

Conformal Geometry

Local invariants of conformal structures can be viewed as functions on the space of jets of metrics modulo diffeomorphism and conformal rescaling. We will define this space and show that it has a natural action of a parabolic subgroup P of the conformal group. A jet isomorphism theorem refers to a result asserting the existence of a P-equivariant isomorphism to a space described in terms of tensor representations of P.

Such results are central in the study of conformal invariant theory.

This approach is suggested by Fefferman's paper: *Parabolic Invariant Theory in Complex Analysis*, Adv. Math. 1979.

Conformal case:
Fefferman and Graham, in preparation
Graham and Hirachi, in preparation

Lecture 1

- Formulation of the space of conformal structures as a P-space.

- Statement of jet isomorphism theorem for n odd via ambient metric

Lecture 2

- Deformation complex and ambient realization

- Proof of jet isomorphism theorem for n odd via deformation complex

Lecture 3

- Formulation of jet isomorphism theorem for n even via inhomogeneous ambient metrics

Jet Isomorphism: Riemannian Geometry

Fix a quadratic form h of signature (p, q) on $\mathbb{R}^n$

$\mathcal{M} = \{\infty$-order jets of metrics g at $0 \in \mathbb{R}^n$
satisfying $g_{ij}(0) = h_{ij}\}$
Can identify $\mathcal{M} = \{(g_{ij,\alpha})_{|\alpha|\geq 1}\}$

$\mathrm{Diff} = \{\infty$-order jets of local diffeomorphisms φ
of $\mathbb{R}^n$ such that $\varphi(0) = 0\}$

$\mathrm{ODiff} = \{\varphi : \varphi'(0) \in O(h)\} \subset \mathrm{Diff}$
$\mathrm{Diff}_0 = \{\varphi : \varphi'(0) = Id\} \subset \mathrm{ODiff}$

ODiff acts on $\mathcal{M}$ on the left: $\varphi.g = (\varphi^{-1})^* g$.

Can view $O(h) \subset \mathrm{ODiff}$
$O(h)$ is the isotropy group of the flat metric.

Can factor $\mathrm{ODiff} = O(h) \cdot \mathrm{Diff}_0$.

$O(h)$ acts on $\mathcal{M}/\mathrm{Diff}_0$ and local invariants of Riemannian metrics correspond to $O(h)$-invariants of $\mathcal{M}/\mathrm{Diff}_0$.

The space $\mathcal{M}/\mathrm{Diff}_0$ can be $O(h)$-equivariantly parametrized in terms of curvature tensors and their covariant derivatives.

Definition: $\mathcal{R} = \{(R^{(0)}, R^{(1)}, R^{(2)}, \cdots)\}$ such that

$$R^{(r)} \in \Lambda^2\mathbb{R}^{n*} \otimes \Lambda^2\mathbb{R}^{n*} \otimes \otimes^r\mathbb{R}^{n*} \text{ and:}$$

1) $R_{i[jkl],m_1\cdots m_r} = 0$

2) $R_{ij[kl,m_1]m_2\cdots m_r} = 0$

3) $R_{ijkl,m_1\cdots[m_{s-1}m_s]\cdots m_r} = Q^{(s)}_{ijklm_1\cdots m_r}(R)$

Here $Q^{(s)}_{ijklm_1\cdots m_r}(R)$ is a quadratic expression in the $R^{(r')}$ with $r' \leq r-2$ arising from the Ricci identity.

$\mathcal{R}$ has a natural $O(h)$-action.

There is a polynomial map $\mathcal{M} \to \mathcal{R}$ which evaluates the covariant derivatives of curvature at the origin.

This map induces a map $\mathcal{M}/\mathrm{Diff}_0 \to \mathcal{R}$ which is $O(h)$-equivariant.

Riemannian Jet Isomorphism Theorem:
The map $\mathcal{M}/\mathrm{Diff}_0 \to \mathcal{R}$ is an $O(h)$-equivariant bijection with polynomial inverse.

Proof: use geodesic normal coordinates to define a slice for the Diff_0 action on $\mathcal{M}$. Then a linearization argument reduces the the theorem to showing that the linearized map restricted to metrics in normal form is an isomorphism. The linearized map can be explicitly identified as the direct sum of intertwining maps between two equivalent realizations corresponding to different Young projectors of irreducible representations of $GL(n,\mathbb{R})$.

Thus local Riemannian invariants correspond to $O(h)$-invariants of $\mathcal{R}$. Weyl's classical invariant theory for $O(h)$ completely describes such invariants.

Conformal Group

Define $\widetilde{h}_{IJ} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h_{ij} & 0 \\ 1 & 0 & 0 \end{pmatrix}$, signature $(p+1, q+1)$.

The conformal group is $G = O(\widetilde{h})/\{\pm I\}$.

G acts by conformal transformations on (a compactification of) $(\mathbb{R}^n, h_{ij})$.

The isotropy group of $0 \in \mathbb{R}^n$ can be identified with the subgroup $P = \{p \in O(\widetilde{h}) : pe_0 = ae_0, a > 0\}$

Then $P =$

$$\left\{ p = \begin{pmatrix} a & b_j & * \\ 0 & m^i{}_j & * \\ 0 & 0 & * \end{pmatrix} : a > 0,\ b_j \in \mathbb{R}^{n*},\ m^i{}_j \in O(h) \right\}$$

$$P = \mathbb{R}_+ \cdot \mathbb{R}^n \cdot O(h)$$

The corresponding conformal transformation on $\mathbb{R}^n$ is denoted φ_p, with conformal factor Ω_p:

$$\varphi_p^* h = \Omega_p^2 h.$$

These are given explicitly by:

$$(\varphi_p(x))^i = \frac{m^i{}_j x^j - \frac{1}{2}|x|^2 d^i}{a + b_j x^j - \frac{1}{2}c|x|^2}$$

$$\Omega_p = (a + b_j x^j - \frac{1}{2}c|x|^2)^{-1}$$

where

$$c = -\frac{1}{2a} b_j b^j, \qquad d^i = -\frac{1}{a} m^{ij} b_j$$

and

$$|x|^2 = h_{ij} x^i x^j$$

Note:

$$\varphi_p'(0) = a^{-1} m^i{}_j$$

$$\Omega_p(0) = a^{-1} \qquad d\Omega_p(0) = -a^{-2} b_j$$

Conformal Jets as P-space

$\mathcal{M} =$ same as before: jets of metrics such that

$$g_{ij}(0) = h_{ij}.$$

$C^\infty_+ =$ jets of functions Ω such that $\Omega(0) > 0$.

Consider Diff $\rtimes C^\infty_+$, with semidirect product such that $(\varphi, \Omega).g = (\varphi^{-1})^*(\Omega^2 g)$ is an action.

The product is given explicitly by:
$(\varphi_1, \Omega_1) \cdot (\varphi_2, \Omega_2) = (\varphi_1 \circ \varphi_2, (\Omega_1 \circ \varphi_2)\, \Omega_2)$

Define CDiff $\subset$ Diff $\rtimes C^\infty_+$ and CDiff$_0 \subset$ CDiff by:
CDiff $= \{(\varphi, \Omega) : (\Omega^{-1}\varphi')(0) \in O(h)\}$
CDiff$_0 = \{(\varphi, \Omega) : \varphi'(0) = Id,\ \Omega = 1 + O(|x|^2)\}$

CDiff acts on $\mathcal{M}$ by: $(\varphi, \Omega).g = (\varphi^{-1})^*(\Omega^2 g)$

Can view $P \subset$ CDiff by $p \mapsto (\varphi_p, \Omega_p)$.
Then P is the isotropy group of the flat metric under the CDiff-action.

Can factor CDiff $= P \cdot$ CDiff$_0$.

Then $\mathcal{M}/\mathrm{CDiff}_0$ inherits a natural P-action.

Local conformal invariants correspond precisely to P-invariants of $\mathcal{M}/\mathrm{CDiff}_0$.

Since $P \subset GL(n+2,\mathbb{R})$, P acts on tensor powers of $\mathbb{R}^{n+2}$, not $\mathbb{R}^n$. Want to describe $\mathcal{M}/\mathrm{CDiff}_0$ as a P-space in terms of tensors in $n+2$ dimensions.

Different ways to do this: ambient metric, tractors.

The structure of $\mathcal{M}/\mathrm{CDiff}_0$ depends crucially on whether n is even or odd. This is reflected by:

$$T(\mathcal{M}/\mathrm{CDiff}_0) \cong \mathcal{V}_\lambda^*/D\mathcal{V}_\mu^*,$$

where $\mathcal{V}_\lambda^*$, $\mathcal{V}_\mu^*$ are dual generalized Verma modules, and D an invariant differential operator.
If n is odd, $\mathcal{V}_\lambda^*/D\mathcal{V}_\mu^*$ is an irreducible $(\mathfrak{g},P)$-module.
If n is even, $\mathcal{V}_\lambda^*/D\mathcal{V}_\mu^*$ has a unique proper $(\mathfrak{g},P)$-submodule with irreducible quotient.

Jet Isomorphism: Conformal Geometry
n odd

Definition: $\widetilde{\mathcal{R}} = \{(\widetilde{R}^{(0)}, \widetilde{R}^{(1)}, \widetilde{R}^{(2)}, \cdots)\}$ such that

$$\widetilde{R}^{(r)} \in \Lambda^2\mathbb{R}^{n+2*} \otimes \Lambda^2\mathbb{R}^{n+2*} \otimes \otimes^r\mathbb{R}^{n+2*} \text{ and:}$$

1) $\widetilde{R}_{I[JKL],M_1\cdots M_r} = 0$

2) $\widetilde{R}_{IJ[KL,M_1]M_2\cdots M_r} = 0$

3) $\widetilde{R}_{IJKL,M_1\cdots[M_{s-1}M_s]\cdots M_r} = Q^{(s)}_{IJKLM_1\cdots M_r}(\widetilde{R})$

4) $\widetilde{h}^{IK}\widetilde{R}_{IJKL,M_1\cdots M_r} = 0$

5) $\widetilde{R}_{IJK0,M_1\cdots M_r} = -\sum_{s=1}^{r} \widetilde{R}_{IJKM_s,M_1\cdots\widehat{M_s}\cdots M_r}$

Here $Q^{(s)}_{IJKLM_1\cdots M_r}(\widetilde{R})$ is the same quadratic expression in the $\widetilde{R}^{(r')}$ with $r' \leq r-2$ arising from the Ricci identity.

The first 4 conditions are invariant under all of $O(\widetilde{h})$.

Recall that $p \in P$ satisfies $pe_0 = ae_0$, $a > 0$.

So if we define the character $\sigma_w(p) = a^{-w}$, and view

$$\widetilde{\mathcal{R}} \subset \Pi_{r=0}^{\infty}(\otimes^{4+r}\mathbb{R}^{n+2*} \otimes \sigma_{-2-r}),$$

then the last condition is invariant under P. Thus $\widetilde{\mathcal{R}}$ is a P-space.

Conformal Jet Isomorphism Theorem: If n is odd, then there is a P-equivariant polynomial bijection $c : \mathcal{M}/\mathrm{CDiff}_0 \to \widetilde{\mathcal{R}}$ with polynomial inverse.

If n is even, there is an analogous bijection from $(n-1)$-jets of metrics mod CDiff_0 to a correspondingly truncated version of the space $\widetilde{\mathcal{R}}$. An infinite order version of the Theorem for n even will be discussed in Lecture 3.

The Jet Isomorphism Theorem reduces the study of conformal invariants to the study of P-invariants of $\widetilde{\mathcal{R}}$. This is important because algebraic tensorial operations can be utilized to construct and study conformal invariants.

Next we discuss the construction of the space $\widetilde{\mathcal{R}}$ and the map c using the ambient metric. Lecture 2 will describe a proof of the bijectivity of c.

Ambient Metric: Flat Case

Define

$$\mathcal{N} = \{x \in \mathbb{R}^{n+2} \setminus \{0\} : \widetilde{h}_{IJ}x^I x^J = 0\}$$

$$\mathcal{Q} = \{[x] : x \in \mathcal{N}\} \subset \mathbb{P}^{n+1}$$

The metric $\widetilde{h}_{IJ}dx^I dx^J$ on $\mathbb{R}^{n+2}$ induces a conformal structure on $\mathcal{Q}$.

$G = O(\widetilde{h})$ acts conformally on $\mathcal{Q}$.

P is the isotropy group of a point in $\mathcal{Q}$, so

$$\mathcal{Q} \cong G/P.$$

Ambient metric for $\mathcal{Q}$ is the metric on $\mathbb{R}^{n+2}$:

$$\widetilde{h}_{IJ}dx^I dx^J$$

Ambient Metric for general $(M, [g])$

(Fefferman-Graham 1985)

$(M, [g])$: C^∞ conformal manifold, signature (p, q)

$\mathcal{G}$: Metric Bundle of $[g]$

$\mathcal{G} = \{(x, t^2 g(x)) : x \in M, t > 0\} \subset S^2 T^* M$

$$\begin{array}{c} \mathcal{G} \\ \pi \Big\downarrow \\ M \end{array} \quad \text{is an } \mathbb{R}_+\text{-bundle with dilations } \delta_s : \mathcal{G} \to \mathcal{G}$$

$$\delta_s(x, \bar{g}) = (x, s^2 \bar{g}). \qquad \text{Set } T = \tfrac{d}{ds}\delta_s|_{s=1}$$

Choice of metric g determines fiber coordinate t

and an identification $\mathcal{G} = \mathbb{R}_+ \times M \ni (t, x)$.

$\mathcal{G}$ has a tautological tensor $g_0 \in S^2 T^* \mathcal{G}$:

if $z = (x, \bar{g}) \in \mathcal{G}$ and X, $Y \in T_z \mathcal{G}$, then

$$g_0(X, Y) = \bar{g}(\pi_* X, \pi_* Y).$$

g_0 is degenerate: $g_0(T, X) = 0 \quad \forall\, X \in T\mathcal{G}$,

and homogeneous of degree 2: $\delta_s^* g_0 = s^2 g_0$.

The ambient space is $\tilde{\mathcal{G}} = \mathcal{G} \times \mathbb{R} \ni (z, \rho)$.

$\iota : \mathcal{G} \to \tilde{\mathcal{G}}$ is the inclusion $\iota(z) = (z, 0)$.

Dilations $\delta_s : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}$ act in the $\mathcal{G}$ factor.

Definition: An ambient metric $\tilde{g}$ for $(M, [g])$ is a smooth metric on $\tilde{\mathcal{G}}$ of signature $(p+1, q+1)$ satisfying:

1) $\delta_s^* \tilde{g} = s^2 \tilde{g}$

2) $\iota^* \tilde{g} = g_0$

3) $\mathrm{Ric}(\tilde{g}) = 0$ to infinite order along $\mathcal{G}$

Theorem: (n odd)

There exists an ambient metric in a homogeneous neighborhood of $\mathcal{G}$ in $\tilde{\mathcal{G}}$. It is unique up to:

(a) Pullback by a homogeneous diffeomorphism Φ of $\tilde{\mathcal{G}}$ satisfying $\Phi|_{\mathcal{G}} = Id$

(b) Homogeneous terms vanishing to infinite order along $\mathcal{G}$

Theorem: ($n \geq 4$ even)

There is an obstruction to existence at order $n/2$: the "ambient obstruction tensor".

If 3) is modified to:

3') $\mathrm{Ric}(\tilde{g}) = O(\rho^{n/2-1})$

then there is a solution $\tilde{g}$. It is unique up to homogeneous diffeomorphism and up to homogeneous terms which are $O(\rho^{n/2})$.

The solution $\tilde{g}$ has an extra geometric property: For each $p \in \tilde{\mathcal{G}}$, the parametrized dilation orbit $s \to \delta_s p$ is a geodesic for $\tilde{g}$.

The diffeomorphism invariance can be normalized by choosing a metric g in the conformal class. Given g, say $\tilde{g}$ is in normal form relative to g if:

i) $\tilde{g} = 2t\,dt \cdot d\rho + g_0$ at $\rho = 0$
(in identification $\tilde{\mathcal{G}} \cong \mathbb{R}_+ \times M \times \mathbb{R}$ induced by g).

ii) The lines $\rho \to (z, \rho)$ are geodesics for $\tilde{g}$.

$\widetilde{g}$ can be chosen to be in normal form relative to g, and is then uniquely determined by g up to $O(\rho^\infty)$ for n odd and up to $O(\rho^{n/2})$ for n even.

The Taylor expansion of $\widetilde{g}$ at $\rho = 0$ is given by polynomial expressions in derivatives of g.

The map $c : \mathcal{M} \to \widetilde{\mathcal{R}}$ is defined as follows. For $g \in \mathcal{M}$, extend g to a metric defined near $0 \in \mathbb{R}^n$. There is an ambient metric $\widetilde{g}$ on $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}$ in normal form relative to g, uniquely determined to infinite order. Evaluate the covariant derivatives of curvature of $\widetilde{g}$ at $t = 1$, $x = 0$ and $\rho = 0$. These satisfy the relations defining $\widetilde{\mathcal{R}}$.

Using the fact that the ambient curvature tensors are tensors on the ambient space, it can be shown that c induces a map $c : \mathcal{M}/\mathrm{CDiff}_0 \to \widetilde{\mathcal{R}}$, and that c is P-equivariant.

It is possible to show by direct analysis that c is bijective (Fefferman-Graham). A conceptual proof will be described in Lecture 2 (Graham-Hirachi).

If n is even, there is an analogous map from $(n-1)$-jets of metrics to a correspondingly truncated version of the space $\widetilde{\mathcal{R}}$. Lecture 3 will describe an infinite-order extension for n even.

Deformation Complex

and

Proof of

Jet Isomorphism Theorem

n odd

Recall:

h_{ij} = fixed quadratic form on $\mathbb{R}^n$, signature (p,q)

$$\widetilde{h}_{IJ} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & h_{ij} & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$G = O(\widetilde{h})/\{\pm I\}, \quad P = \text{isotropy group of } [e_0]$$

$$\mathcal{N} = \{x \in \mathbb{R}^{n+2} \setminus \{0\} : \widetilde{h}_{IJ} x^I x^J = 0\}$$

$$\mathcal{Q} = \{[x] \in \mathbb{P}^{n+1} : x \in \mathcal{N}\} \cong G/P$$

Have $\widetilde{h}_{IJ} x^I x^J = 2x^0 x^\infty + |x|^2$, so

$$\mathcal{Q} \cap \{[x^I] : x^0 \neq 0\} \cong \mathbb{R}^n \qquad \text{via}$$

$$\mathbb{R}^n \ni x^i \to \begin{bmatrix} 1 \\ x^i \\ -\frac{1}{2}|x|^2 \end{bmatrix} \in \mathcal{Q}$$

The conformal structure on $\mathcal{Q}$ is represented by the flat metric $h_{ij} dx^i dx^j$ on $\mathbb{R}^n$.

Notation: $X = x^I \partial_I$ on $\mathbb{R}^{n+2}$

Recall also that $P =$

$$\left\{ p = \begin{pmatrix} a & b_j & * \\ 0 & m^i{}_j & * \\ 0 & 0 & * \end{pmatrix} : a > 0,\, b_j \in \mathbb{R}^{n*},\, m^i{}_j \in O(h) \right\}$$

$$\sigma_w(p) = a^{-w}$$

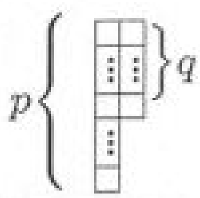
To each finite dimensional representation of P is associated a homogeneous vector bundle on $\mathcal{Q} = G/P$ and on $\mathbb{R}^n \hookrightarrow \mathcal{Q}$. Examples:

- $\mathcal{D}_w$: bundle of conformal densities of weight w, induced by σ_w $\qquad \mathcal{E}(w) = \Gamma(\mathcal{D}_w)$

- T: tangent bundle, induced by $p \mapsto a^{-1}m$

- Λ^p: p-forms, $\qquad \Lambda^p(w) = \Lambda^p \otimes \mathcal{D}_w$

- $\Lambda^{p,q}$ $(p \geq q)$: covariant tensors with Young symmetry



$f \in \Lambda^{p,q} \subset \Lambda^p \otimes \Lambda^q$ if

$$f_{i_1 \ldots i_p j_1 \ldots j_q} = f_{[i_1 \ldots i_p][j_1 \cdots j_q]}, \quad f_{[i_1 \ldots i_p j_1] j_2 \cdots j_q} = 0$$

Note $\Lambda^{1,1} = \odot^2$

$$\Lambda_0^{p,q} \subset \Lambda^{p,q} \qquad \text{trace-free tensors}$$

$\Lambda_0^{p,q}$ is irreducible

Differential operators (on $\mathbb{R}^n$):

$$d_1 : \Lambda^{p,q} \to \Lambda^{p+1,q}, \quad d_2 : \Lambda^{p,q} \to \Lambda^p \otimes \Lambda^{q+1}$$

$$\delta_1 : \Lambda^{p,q} \to \Lambda^{p-1} \otimes \Lambda^q, \quad \delta_2 : \Lambda^{p,q} \to \Lambda^{p,q-1}$$

Given by:

$$(d_1 f)_{i_0 i_1 \cdots i_p j_1 \cdots j_q} = \partial_{[i_0} f_{i_1 \cdots i_p] j_1 \cdots j_q},$$

$$(d_2 f)_{i_1 \cdots i_p j_1 \cdots j_{q+1}} = \partial_{[j_0} f_{|i_1 \cdots i_p| j_1 \cdots j_q]}.$$

$$(\delta_1 f)_{i_1 \cdots i_{p-1} j_1 \cdots j_q} = -\partial^k f_{i_1 \cdots i_{p-1} k j_1 \cdots j_q},$$

$$(\delta_2 f)_{i_1 \cdots i_p j_1 \cdots j_{q-1}} = -\partial^k f_{i_1 \cdots i_p j_1 \cdots j_{q-1} k}.$$

For $n \geq 4$, the deformation complex on $\mathbb{R}^n$ is

$$0 \to \mathfrak{g} \to \Lambda^1(2) \xrightarrow{D_0} \Lambda_0^{1,1}(2) \xrightarrow{D_1} \Lambda_0^{2,2}(2) \xrightarrow{D_2} \Lambda_0^{3,2}(2)$$

$$\to \cdots \to \Lambda_0^{n-2,2}(2) \xrightarrow{D_{n-2}} \Lambda_0^{n-1,1} \xrightarrow{D_{n-1}} \Lambda^{n-1}(-2) \to 0$$

where

$$\begin{aligned} D_0 &= \mathrm{tf}\,\mathrm{Sym}\, d_2 \\ D_1 &= \mathrm{tf}\, d_1 d_2 \\ D_p &= \mathrm{tf}\, d_1 \quad p = 2, 3, \ldots, n-3 \\ D_{n-2} &= \delta_2 d_1 \\ D_{n-1} &= \delta_2 \end{aligned}$$

It is a complex ($D^2 = 0$), analogous to deRham.
D_1, D_{n-2} are second order, others first order.
Gasqui-Goldschmidt (1984) "by hand"
Special case of gBGG complex.
Exact on jets.
Conformally invariant.
Alternate form when $n = 3$.

Controls deformations of conformal structures:
Infinitesimally, have:

$D_0 V \leftrightarrow \mathrm{tf}\, \mathcal{L}_V h$, conformal Killing operator
$D_1 \leftrightarrow W =$ Weyl tensor, and
$\mathfrak{g} \leftrightarrow$ conformal symmetries of h

Ambient Lifts

Goal: give ambient description of spaces and maps
Different approaches: Čap-Slovak-Soucek,
Calderbank-Diemer

Definition: $\widetilde{\Lambda}^{p,q}$ = same symmetry, on $\mathbb{R}^{n+2}$
$\widetilde{\Lambda}_0^{p,q}$: trace-free with respect to $\widetilde{h}$
$\widetilde{\Lambda}^{p,q}(w)$: sections homog. deg. w: $\mathcal{L}_X \widetilde{f} = w\widetilde{f}$

$\widetilde{\mathcal{H}}^{p,q}(w) \subset \widetilde{\Lambda}_0^{p,q}(w)$: sections satisfying:

$$(*) \qquad \widetilde{\Delta}\widetilde{f} = 0, \qquad \widetilde{\delta}_1 \widetilde{f} = 0, \qquad X \lrcorner \widetilde{f} = 0$$

If $\mathcal{U} \subset \mathcal{Q}$, set $\widetilde{\mathcal{U}} = \pi^{-1}(\mathcal{U}) \subset \mathcal{N}$, and
$\widetilde{\mathcal{H}}_{\widetilde{\mathcal{U}}}^{p,q}(w)$ = jets along $\widetilde{\mathcal{U}}$ satisfying $(*)$
to infinite order along $\widetilde{\mathcal{U}}$

Can interpret $\widetilde{\mathcal{H}}^{p,q}(w)$ as a sheaf on $\mathcal{Q}$. Conditions defining $\widetilde{\mathcal{H}}^{p,q}(w)$ are G-invariant, so G "acts" on $\widetilde{\mathcal{H}}^{p,q}(w)$. ("Homogeneous sheaf")

Existence and Uniqueness of Ambient Lift:
Let $p \geq q \geq 0$, $w \in \mathbb{C}$.
If $p > q = 0$, assume that $w \neq 2p - n$.
If $q > 0$, assume that $w \neq p + 2q - n - 1,\ 2p + q - n$.

If $w + n/2 - p - q \notin \mathbb{N}$, then $\Lambda_0^{p,q}(w) \cong \widetilde{\mathcal{H}}^{p,q}(w)$
Equality as sheaves on $\mathcal{Q}$.
Isomorphism is G-equivariant.

A section f of $\Lambda_0^{p,q}(w)$ can be viewed as a covariant tensor (also denoted f) on $\mathcal{N}$, homogeneous of degree w, satisfying $X \lrcorner f = 0$. It is then clear that ι^* induces a map

$$\iota^* : \{\tilde{f} \in \tilde{\Lambda}_0^{p,q}(w) : X \lrcorner \tilde{f} = 0 \text{ on } \mathcal{Q}\} \to \Lambda_0^{p,q}(w)$$

where $\iota : \mathcal{N} \hookrightarrow \mathbb{R}^{n+2}$, so in particular this gives a map

$$\iota^* : \widetilde{\mathcal{H}}^{p,q}(w) \to \Lambda_0^{p,q}(w).$$

The Theorem asserts that this map is an isomorphism, i.e. each section of $\Lambda_0^{p,q}(w)$ has a unique extension (ambient lift) as a section of $\widetilde{\mathcal{H}}^{p,q}(w)$.

The disallowed values correspond to the existence of certain invariant differential operators acting on $\Lambda_0^{p,q}(w)$.

Main ingredients in proof:

1. Initial lift: "completeing" a tensor in $\Lambda_0^{p,q}(w)$ to $\tilde{\Lambda}_0^{p,q}(w)|_{\mathcal{N}}$ (differential splitting formulae, ideas go back to Tracy Thomas)

2. "Harmonic" extension off $\mathcal{N}$

Scalar case: $p = q = 0$
(Eastwood-Graham 1991)

Theorem says: if $w + n/2 \notin \mathbb{N}$, then $\mathcal{E}(w) \cong \widetilde{\mathcal{H}}(w)$.

Set $Q = \tilde{h}_{IJ}x^I x^J$. Given f homogeneous of degree w, need to find $\tilde{f}$ homogeneous degree w with $\widetilde{\Delta}\tilde{f} = O(Q^\infty)$ and $\tilde{f}|_{\mathcal{N}} = f$. Construct $\tilde{f}$ inductively to higher order. Suppose

$$\tilde{f}^{(k)} = O(Q^{k-1}).$$

Set

$$\tilde{f}^{(k+1)} = \tilde{f}^{(k)} + Q^k\eta \qquad \text{for } \eta \in \mathcal{E}(w-2k)$$

Then

$$\begin{aligned}\widetilde{\Delta}\tilde{f}^{(k+1)} &= \widetilde{\Delta}\tilde{f}^{(k)} + \widetilde{\Delta}(Q^k\eta)\\ &= \widetilde{\Delta}\tilde{f}^{(k)} + [\widetilde{\Delta}, Q^k]\eta + O(Q^k)\\ &= \widetilde{\Delta}\tilde{f}^{(k)} + 2kQ^{k-1}(2k+n+2X)\eta + O(Q^k)\\ &= \widetilde{\Delta}\tilde{f}^{(k)} + 2k(n+2w-2k)\eta Q^{k-1} + O(Q^k)\end{aligned}$$

so if $n + 2w \neq 2k$, can solve for η.

If $n/2 + w = k \in \mathbb{N}$, then harmonic extension is obstructed by the conformally invariant operator Δ^k.

1-forms: $p = 1$, $q = 0$

Theorem says: if $w \neq 2 - n$ and $w + n/2 - 1 \notin \mathbb{N}$, then $\Lambda^1(w) \cong \widetilde{\mathcal{H}}^1(w)$. Recall $\tilde{f} \in \widetilde{\mathcal{H}}^1(w)$ means

$$\tilde{f} \in \tilde{\Lambda}^1(w), \quad \widetilde{\Delta}\tilde{f} = 0, \quad \tilde{\delta}\tilde{f} = 0, \quad X \lrcorner \tilde{f} = 0.$$

Can choose some $\tilde{f} \in \tilde{\Lambda}^1(w)$ such that $\iota^*\tilde{f} = f$. This determines $\tilde{f}$ up to addition of $\psi dQ + Q\phi$ with ψ a function and ϕ a 1-form, both of homogeneity $w - 2$. Can choose $\tilde{f}$ so that $X \lrcorner \tilde{f} = O(Q^2)$. Try to determine ψ, ϕ so that

$$X \lrcorner (\tilde{f} + \psi dQ + Q\phi) = O(Q^2)$$

and

$$\tilde{\delta}(\tilde{f} + \psi dQ + Q\phi) = O(Q).$$

First equation gives $\psi dQ(X) + Q\phi(X) = O(Q^2)$, so

$$2\psi + \phi(X) = O(Q).$$

Second equation gives

$$\tilde{\delta}\tilde{f} - 2(n + w)\psi - 2\phi(X) = O(Q),$$

so combining gives

$$\tilde{\delta}\tilde{f} - 2(n + w - 2)\psi = O(Q).$$

If $n + w \neq 2$, we can determine ψ mod Q.

Now all components of $f|_{\mathcal{N}}$ are determined mod $O(Q)$. Write $\tilde{f} = \tilde{f}_I dx^I$; all $\tilde{f}_I$ are homogeneous of degree $w-1$. By scalar case, can uniquely extend each $\tilde{f}_I$ harmonically to infinite order. Claim that the extension automatically satisfies $\tilde{\delta}\tilde{f} = 0$ and $X \lrcorner \tilde{f} = 0$ to infinite order. Check $\tilde{\delta}\tilde{f}|_{\mathcal{N}} = 0$. Commute $\widetilde{\Delta}$ through $\tilde{\delta}$ to see that $\widetilde{\Delta}\tilde{\delta}\tilde{f} = 0$. But $\tilde{\delta}\tilde{f}$ has homogeneity $w-2$ and $n+(w-2) \notin \mathbb{N}$, so uniqueness for scalar case gives $\tilde{\delta}\tilde{f} = 0$. Now $X \lrcorner \tilde{f} = 0$ is similar. Uses $\tilde{\delta}\tilde{f} = 0$ and that $X \lrcorner \tilde{f} = O(Q^2)$.

For higher p, q, the algebra of the "initial lift" is more complicated, but the basic idea is the same. For given p, q with $q > 0$, there are two invariant divergence operators obstructing the initial lift, giving rise to two excluded values of w. (If $p = q > 0$, one of the obstructing operators is second order–an iterated divergence.)

This ambient lift theorem implies a corresponding statement providing a $(\mathfrak{g}, P)$-equivariant isomorphism between jets at $[e_0] \in \mathcal{Q}$ of sections of $\Lambda_0^{p,q}(w)$ with jets at $e_0 \in \mathbb{R}^{n+2}$ of elements of $\widetilde{\mathcal{H}}^{p,q}(w)$. This "jet isomorphism theorem for $\Lambda_0^{p,q}(w)$" can be interpreted as providing an ambient description of the dual generalized Verma modules. The $(\mathfrak{g}, P)$-action in the ambient description is given in terms of tensor representations.

Lift of Deformation Complex, $n \geq 5$ odd

Observe that for p, q, $w \in \mathbb{Z}$, the condition

$$w + n/2 - p - q \notin \mathbb{N}$$

for harmonic lifting is automatic if n is odd. Can check that the lifting theorem applies to all $\widetilde{\Lambda}_0^{p,q}(w)$ which occur in the deformation complex except for the second-to-last one. This one fails because the last operator is δ_2, which is the divergence operator obstructing the initial lift. So we have an ambient description for each bundle in the deformation complex except the second to last one.

It is not difficult to identify the differential operators on $\mathbb{R}^{n+2}$ which correspond to the operators in the deformation complex in this realization. One obtains:

Theorem: Let $n \geq 5$ be odd. The deformation complex with last two spaces removed can be realized as:

$$0 \to \mathfrak{g} \to \widetilde{\mathcal{H}}^1(2) \xrightarrow{\widetilde{D}_0} \widetilde{\mathcal{H}}^{1,1}(2) \xrightarrow{\widetilde{D}_1} \widetilde{\mathcal{H}}^{2,2}(2) \xrightarrow{\widetilde{D}_2}$$

$$\widetilde{\mathcal{H}}^{3,2}(2) \to \cdots \xrightarrow{\widetilde{D}_{n-3}} \widetilde{\mathcal{H}}^{n-2,2}(2),$$

where the differential operators are:

$$\begin{aligned}
&\widetilde{D}_0 = \mathrm{Sym}\,\tilde{d}_2 \quad (\text{so } (\widetilde{D}_0\tilde{f})_{IJ} = \partial_{(I}\tilde{f}_{J)}) \\
&\widetilde{D}_1 = \tilde{d}_2\tilde{d}_1 \\
&\widetilde{D}_p = \tilde{d}_1 \quad p = 2, 3, \ldots, n-3.
\end{aligned}$$

Remarks:

1. Can extend this complex to include one more term lifting $\ker(D_{n-1}) \subset \Lambda_0^{n-1,1}(0)$.

2. There is a version for $n = 3$, including also the extra term lifting $\ker(D_2)$.

Jet Isomorphism Theorem Proof, n odd

Recall statement:

$\mathcal{M} = \{\infty$-order jets of metrics g at $0 \in \mathbb{R}^n$ satisfying $g_{ij}(0) = h_{ij}\}$

$\mathrm{CDiff}_0 = \{(\varphi, \Omega) : \varphi'(0) = Id,\, \Omega = 1 + O(|x|^2)\}$

$\widetilde{\mathcal{R}} = \{(\widetilde{R}^{(0)}, \widetilde{R}^{(1)}, \widetilde{R}^{(2)}, \cdots)\}$ such that

$\widetilde{R}^{(r)} \in \widetilde{\Lambda}^2 \otimes \widetilde{\Lambda}^2 \otimes \otimes^r \mathbb{R}^{n+2*}$ and:

1) $\widetilde{R}_{I[JKL],M_1\cdots M_r} = 0$

2) $\widetilde{R}_{IJ[KL,M_1]M_2\cdots M_r} = 0$

3) $\widetilde{R}_{IJKL,M_1\cdots[M_{s-1}M_s]\cdots M_r} = Q^{(s)}_{IJKLM_1\cdots M_r}(\widetilde{R})$

4) $\widetilde{h}^{IK}\widetilde{R}_{IJKL,M_1\cdots M_r} = 0$

5) $\widetilde{R}_{IJK0,M_1\cdots M_r} = -\sum_{s=1}^{r} \widetilde{R}_{IJKM_s,M_1\cdots\widehat{M_s}\cdots M_r}$

$c : \mathcal{M}/\mathrm{CDiff}_0 \to \widetilde{\mathcal{R}}$ giving curvature tensors of ambient metric is bijective.

First step: Linearization argument.
Suffices to prove that

$$dc : T\mathcal{M}/T\mathcal{O} \to T\widetilde{\mathcal{R}}$$

is an isomorphism, where
$\mathcal{O} = \mathrm{CDiff}_0$-orbit of flat metric h.
Involves truncation at finite jets to make everything finite-dimensional, constructing a slice via conformal normal form to show that $\mathcal{M}^N/\mathrm{CDiff}_0$ is a smooth manifold, and either the inverse function theorem or an algebraic induction argument on N.

Second step: Reformulate spaces and map dc.

Lemma: $T\mathcal{M}/T\mathcal{O} \cong \mathcal{J} \odot^2_0 / D_0\mathcal{J}\Lambda^1$, where
$\mathcal{J}$: infinite order jets at $0 \in \mathbb{R}^n$.

Proof: (Suppress writing $\mathcal{J}$ everywhere)
$T\mathcal{M} = \{s \in \odot^2 : s(0) = 0\}$.

$$\begin{aligned} T\mathcal{O} &= \{\mathcal{L}_V h : V = O(|x|^2)\} \oplus \{\Omega^2 h : \Omega = O(|x|^2)\} \\ &= D_0\mathcal{J}\Lambda^1 \oplus \{\Omega^2 h : \Omega = O(|x|^2)\}. \end{aligned}$$

Now $s(0)$ disappears in $\odot^2_0/D_0\Lambda^1$ anyhow, and trace parts cancel.

Proposition: $T\widetilde{\mathcal{R}} \cong \ker(\widetilde{D}_2) \subset \widetilde{\mathcal{J}}\widetilde{\mathcal{H}}^{2,2}$,
where $\widetilde{\mathcal{J}}$: infinite order jets at $e_0 \in \mathbb{R}^{n+2}$.

Proof: Have:

$T\widetilde{\mathcal{R}} = \{(\widetilde{R}^{(0)}, \widetilde{R}^{(1)}, \widetilde{R}^{(2)}, \cdots)\}$ such that

$\widetilde{R}^{(r)} \in \widetilde{\Lambda}^2 \otimes \widetilde{\Lambda}^2 \otimes \otimes^r \mathbb{R}^{n+2*}$ and:

1) $\widetilde{R}_{I[JKL],M_1 \cdots M_r} = 0$

2) $\widetilde{R}_{IJ[KL,M_1]M_2 \cdots M_r} = 0$

3) $\widetilde{R}_{IJKL,M_1 \cdots [M_{s-1} M_s] \cdots M_r} = 0$

4) $\widetilde{h}^{IK} \widetilde{R}_{IJKL,M_1 \cdots M_r} = 0$

5) $\widetilde{R}_{IJK0,M_1 \cdots M_r} = -\sum_{s=1}^{r} \widetilde{R}_{IJKM_s,M_1 \cdots \widehat{M_s} \cdots M_r}$

3) $\Leftrightarrow$ get a jet $\tilde{R}$ at e_0 of a section of $\tilde{\Lambda}^2 \otimes \tilde{\Lambda}^2$.

1) and 4) $\Leftrightarrow$ $\tilde{R}$ is a section of $\tilde{\Lambda}_0^{2,2}$.

5) $\Leftrightarrow$ $X \lrcorner \tilde{R} = 0$ to infinite order.

2) $\Leftrightarrow$ $\tilde{R} \in \ker(\widetilde{D}_2)$.

Now $\tilde{R}$ is a section of $\widetilde{\mathcal{H}}^{2,2} \Leftrightarrow$ in addition:

$\widetilde{\Delta}\tilde{R} = 0$ and $\tilde{\delta}\tilde{R} = 0$.

So it is clear that $\ker(\widetilde{D}_2|_{\widetilde{\mathcal{H}}^{2,2}}) \subset T\widetilde{\mathcal{R}}$.

But the conditions $\widetilde{\Delta}\tilde{R} = 0$ and $\tilde{\delta}\tilde{R} = 0$ easily follow from the fact that $\tilde{R}$ is trace-free using the second Bianchi identity.

So composing with these isomorphisms, the jet isomorphism theorem reduces to the statement on jets:

$$dc : \odot_0^2 / D_0 \Lambda^1 \to \ker(\widetilde{D}_2) \subset \widetilde{\mathcal{H}}^{2,2}$$

is an isomorphism.

The lift of the deformation complex contains:

$$\begin{array}{ccccccc}
\to \widetilde{\mathcal{H}}^1 & \to & \widetilde{\mathcal{H}}^{1,1} & \to & \widetilde{\mathcal{H}}^{2,2} & \xrightarrow{\widetilde{D}_2} \\
\updownarrow & & \updownarrow & & \updownarrow & \\
\to \Lambda^1 & \xrightarrow{D_0} & \odot_0^2 & \xrightarrow{D_1} & \Lambda_0^{2,2} & \xrightarrow{D_2}
\end{array}$$

Since the deformation complex is exact on jets, D_1 induces an isomorphism $\odot_0^2 / D_0 \Lambda^1 \cong \ker(D_2) \cong \ker(\widetilde{D}_2)$. One can show that this map agrees with dc, and the result follows.

When $n = 3$, the deformation complex looks a little different. But still have an ambient lift for $\ker(D_2) \subset \Lambda_0^{2,1}(0)$. The jet isomorphism follows in the same manner.

Inhomogeneous Ambient Metrics and Jet Isomorphism Theorem n even

(Joint with Kengo Hirachi)

When n is even, the construction of the ambient metric is obstructed at order $n/2$. So the map c evaluating the covariant derivatives of curvature of the ambient metric cannot be defined beyond this order. This is a reflection of a difference in the structure of $\mathcal{M}/\mathrm{CDiff}_0$ as a P-space when n is even.

The same phenomenon occurs when constructing the ambient lift for $\Lambda_0^{p,q}(w)$ when $w+n/2-p-q \in \mathbb{N}$.

In this lecture, an extension of the theory to these cases will be outlined.

Main Ingredients:

- Weaken homogeneity condition
- Solutions of ambient equations with log's
- Invariant smooth part
- Existence of ambiguity
- Jet isomorphism theorem for enlarged space

Topics

1. Ambient lift with log and jet isomorphism for scalars

2. Formulate jet isomorphism for conformal structures

3. Inhomogeneous ambient metrics

Ambient Lift with Log for Scalars

Recall: if $w + n/2 \notin \mathbb{N}$, then $\mathcal{E}(w) \cong \widetilde{\mathcal{H}}(w)$, where $\widetilde{\mathcal{H}}(w)=$ harmonic jets along $\mathcal{N}$ homogeneous of degree w. But if $w + n/2 = m \in \mathbb{N}$, then harmonic extension is obstructed at order m by the conformally invariant operator Δ^m.

Can always find a harmonic extension by including a log term:

<u>Proposition</u>. Suppose $w + n/2 = m \in \mathbb{N}$ and $f \in \mathcal{E}(w)$. There exists $\tilde{f}$ of the form

$$\tilde{f} = \tilde{s} + \tilde{l}\, Q^m \log Q$$

with $\tilde{s}$, $\tilde{l}$ smooth, $\tilde{s}$ homog. degree w, $\tilde{l}$ homog. deg. $w - 2m$, such that $\widetilde{\Delta}\ \tilde{f} = 0$ and $\tilde{f}|_{\mathcal{N}} = f$. Also, $\tilde{f}$ is unique modulo $Q^m \widetilde{\mathcal{H}}(w - 2m)$.

So the harmonic extension $\tilde{f}$ with log term is not unique: In addition to f, one must prescribe the Q^m coefficient of $\tilde{f}$ on $\mathcal{N}$; then $\tilde{f}$ is uniquely determined. Space of harmonic extensions with log is parametrized by $\mathcal{E}(w) \times \mathcal{E}(w-2m)$. The $\mathcal{E}(w-2m)$ factor is the ambiguity in the (non-smooth) harmonic lift.

Note that $\tilde{f}$ is no longer homogeneous.

Reformulation in terms of smooth part

It turns out that $\widetilde{l}$ can be written entirely in terms of $\widetilde{s}$, and the condition that $\widetilde{f}$ be harmonic can be written entirely in terms of $\widetilde{s}$. Thus one can reformulate the extension as a map
$\mathcal{E}(w)\times\mathcal{E}(w-2m) \to \widetilde{s} \ni \widetilde{\mathcal{E}}(w)$ staying entirely in the smooth category. Here $\widetilde{\mathcal{E}}(w) =$ jets along $\mathcal{N}$ of homogeneous functions of degree w, not necessarily harmonic.

Calculate: $\widetilde{\Delta}\widetilde{f} = \widetilde{\Delta}(\widetilde{s} + \widetilde{l}\,Q^m \log Q)$

$$= (\widetilde{\Delta}\widetilde{s} + 2m\widetilde{l}Q^{m-1}) + \widetilde{\Delta}\widetilde{l}\,Q^m \log Q.$$

So $\widetilde{\Delta}\widetilde{f} = 0 \Leftrightarrow \widetilde{\Delta}\widetilde{l} = 0$ and $\widetilde{\Delta}\widetilde{s} = -2m\widetilde{l}Q^{m-1}$.
Apply $\widetilde{\Delta}^{m-1}$ to second equation to get

$$-2m\widetilde{l} = c_m\widetilde{\Delta}^m\widetilde{s}.$$

This gives $\widetilde{l}$ in terms of $\widetilde{s}$. Substituting back, can write both equations in terms of $\widetilde{s}$:

$$\widetilde{\Delta}\widetilde{s} = c_mQ^{m-1}\widetilde{\Delta}^m\widetilde{s} \text{ and } \widetilde{\Delta}^{m+1}\widetilde{s} = 0.$$

This gives the substitute ambient lift theorem:

<u>Theorem</u>. Suppose $w + n/2 = m \in \mathbb{N}$. Then $\mathcal{E}(w) \times \mathcal{E}(w-2m) \cong \widetilde{\mathcal{H}}_s(w)$, where $\widetilde{\mathcal{H}}_s(w) =$

$$\{\widetilde{s} \in \widetilde{\mathcal{E}}(w) : \widetilde{\Delta}\widetilde{s} = c_mQ^{m-1}\widetilde{\Delta}^m\widetilde{s} \text{ and } \widetilde{\Delta}^{m+1}\widetilde{s} = 0\}.$$

The G-structure on $\mathcal{E}(w) \times \mathcal{E}(w-2m)$ is not the product; one must break the G-invariance to define the $\mathcal{E}(w-2m)$ factor. The correct statement respecting the G-structure is that there is a G-equivariant exact sequence

$$0 \to \mathcal{E}(w-2m) \to \widetilde{\mathcal{H}}_s(w) \to \mathcal{E}(w) \to 0$$

Summary: the space that has the ambient representation is not the initial space $\mathcal{E}(w)$, but an enlargement: $\mathcal{E}(w) \times \mathcal{E}(w-2m)$.

The ambient space is similar to that in the case n odd: $\{\tilde{f} \in \tilde{\mathcal{E}}(w) : \widetilde{\Delta}\tilde{s} = 0\}$; now the harmonic equation has an inhomogeneous term, the coefficient of which solves a second equation.

The corresponding statement on jets at a point is the substitute jet isomorphism theorem for scalars in the obstructed cases.

For the deformation complex for n even, the spaces in the first half of the sequence have obstructed ambient lifts. There are analogous substitute jet isomorphism theorems for these spaces which are used in the proof of the jet isomorphism for conformal structures for n even.

Jet Isomorphism Theorem, n even

The jet isomorphism theorem for conformal structures involves similar features as above. This time the space $\mathcal{M}$ of jets of metrics is augmented by the space of jets of trace-free symmetric 2-tensors. There is a map from the product to a space of ambient curvature tensors which induces a bijection from the quotient by CDiff_0. Set

$$\widetilde{\mathcal{T}} = \Pi_{r=0}^{\infty} \widetilde{\Lambda}^{2,2} \otimes \otimes^r \mathbb{R}^{n+2*} \otimes \sigma_{-r-2}$$

Then $\widetilde{\mathcal{T}}$ has a natural P-action. When n was odd, the space $\widetilde{\mathcal{R}}$ of lists of ambient curvature tensors was a P-invariant subset of $\widetilde{\mathcal{T}}$.

Theorem: n even. There is a P-equivariant polynomial injection $c : (\mathcal{M} \times \mathcal{J}\odot_0^2)/\mathrm{CDiff}_0 \to \widetilde{\mathcal{T}}$, whose image $\widetilde{\mathcal{R}}$ is a submanifold of $\widetilde{\mathcal{T}}$ whose tangent space $T\widetilde{\mathcal{R}}$ is the space of jets $\tilde{R} \in \widetilde{\mathcal{J}}\widetilde{\Lambda}^{2,2}(2)$ which are solutions to the following equations:

1) $\tilde{R}_{IJ[KL,M]} = 0$

2) $X \lrcorner \tilde{R} = 0$

3) $\widetilde{\mathrm{tr}} \tilde{R} = c_n Q^{n/2-1} \widetilde{\Delta}^{n/2-1} \widetilde{\mathrm{tr}} \tilde{R}$

4) $\widetilde{\Delta}^{n/2} \widetilde{\mathrm{tr}} \tilde{R} = 0$

Also, $c^{-1} : \widetilde{\mathcal{R}} \to (\mathcal{M} \times \mathcal{J}\odot_0^2)/\mathrm{CDiff}_0$ is polynomial.

Here $\widetilde{\operatorname{tr}}\widetilde{R}_{IK} = \widetilde{h}^{JL}\widetilde{R}_{IJKL}$ corresponds to the Ricci tensor.

Comparison with n odd

For n odd, $T\widetilde{R}$ is defined by the same equations except that 3) and 4) are replaced by the single equation $\widetilde{\operatorname{tr}}\widetilde{R} = 0$. This is analogous to the situation for the scalar problem. Observe that $\widetilde{\operatorname{tr}}\widetilde{R}$ does vanish to order $n/2 - 1$.

For n odd, the nonlinear space $\widetilde{\mathcal{R}}$ was identified explicitly in terms of the Ricci identity. For n even, the same Ricci-identity relation holds on $\widetilde{\mathcal{R}}$. However, there are nonlinear equations giving rise to 3) and 4) which are not written explicitly. For n odd, the equation that the Ricci tensor vanish to all orders is linear in the derivatives of curvature so this was not an issue. It is not a problem that the nonlinear terms in the equations defining $\widetilde{\mathcal{R}}$ are not explicit.

The rest of the talk will describe the construction of the map c. Once c has been constructed, the proof of the jet isomorphism theorem uses the same idea as for n odd: lift the deformation complex. However, the algebra is much more complicated, as there is an ambiguity for the lift of each term in the deformation complex.

Inhomogeneous Ambient Metrics

$= n$ $\qquad\qquad\qquad\qquad = \mathbb{R}^{n+2}$

Recall: $\mathcal{G}$: metric bundle of $[g]$, $\quad \widetilde{\mathcal{G}} = \mathcal{G} \times \mathbb{R}$

$g_0 =$ tautological 2-tensor on $\mathcal{G}$

For n odd, ambient metric $\widetilde{g}$ on $\widetilde{\mathcal{G}}$ satisfies:

1) $\delta_s^* \widetilde{g} = s^2 \widetilde{g}$

2) $\iota^* \widetilde{g} = g_0$

3) $\mathrm{Ric}(\widetilde{g}) = 0$ to infinite order along $\mathcal{G}$

Existence is obstructed for n even, analogous to obstruction to ambient lift of a density when $w + n/2 \in \mathbb{N}$. Construct ambient metrics with log's. But log of ???. No canonical Q for the nonlinear problem.

r: defining function for $\mathcal{G} \subset \tilde{\mathcal{G}}$ homog. deg. 0

$r_\#$: defining function homog. degree 2

" $r_\# \longleftrightarrow Q$ "

<u>Definition</u>. $\mathcal{A}_{\log}$ = asymptotic expansions of metrics on $\tilde{\mathcal{G}}$ of signature $(p+1, q+1)$ of the form:

$$\tilde{g} \sim \tilde{g}^{(0)} + \sum_{N \geq 1} \tilde{g}^{(N)} r (r^{n/2-1} \log |r_\#|)^N$$

where each $\tilde{g}^{(N)}$, $N \geq 0$, is a smooth symmetric 2-tensor field on $\tilde{\mathcal{G}}$ satisfying $\delta_s^* \tilde{g}^{(N)} = s^2 \tilde{g}^{(N)}$, and such that $\iota^* \tilde{g} = g_0$.

Independent of choice of r, $r_\#$, invariant under smooth homogeneous diffeomorphisms such that $\Phi|_{\mathcal{G}} = I$.

<u>Definition</u>: A metric $\tilde{g} \in \mathcal{A}_{\log}$ is straight if for each $p \in \tilde{\mathcal{G}}$, the dilation orbit $s \to \delta_s p$ is a geodesic for $\tilde{g}$.

<u>Definition</u>: An inhomogeneous ambient metric for $(M, [g])$ is a straight metric $\tilde{g} \in \mathcal{A}_{\log}$ satisfying $\mathrm{Ric}(\tilde{g}) = 0$.

$$\tilde{g} \sim \tilde{g}^{(0)} + \tilde{g}^{(1)} r^{\frac{n}{2}} \log |r_\#| + \sum_{N \geq 2}$$

If $r \longrightarrow \lambda r$, then

$\tilde{g}^{(0)} \longrightarrow \tilde{g}^{(0)}$ and

$\tilde{g}^{(1)} \longrightarrow \lambda^{\frac{n}{2}} \tilde{g}^{(1)}$

If $r_\# \longrightarrow \lambda r_\#$, then

$\tilde{g}^{(0)} \longrightarrow \tilde{g}^{(0)} + r^{\frac{n}{2}} \log \lambda \, \tilde{g}^{(1)}$ and

$\tilde{g}^{(1)} \longrightarrow \tilde{g}^{(1)}$

Important property: $T \Longleftrightarrow X$

Proposition. Let $\tilde{g} \in \mathcal{A}_{\log}$ be straight. Then [redacted] $\tilde{g}(T,T)$ is a smooth defining function for $\mathcal{G}$ homogeneous of degree 2.

$\tilde{g}(T,T) = Q$.

Smooth part. If $\tilde{g} \in \mathcal{A}_{\log}$ is straight, take $r_{\#} = \tilde{g}(T,T)$ in asymptotic expansion. Then the smooth part of $\tilde{g}$ is $\tilde{g}^{(0)}$. Invariantly determined: If Φ is a smooth homogeneous diffeomorphism such that $\Phi|_{\mathcal{G}} = I$, then $(\Phi^*\tilde{g})^{(0)} = \Phi^*(\tilde{g}^{(0)})$.

Recall from Lecture 1: A metric g in the conformal class determines an identification $\mathcal{G} \cong \mathbb{R}_+ \times M$, so also an identification $\tilde{\mathcal{G}} \cong \mathbb{R}_+ \times M \times \mathbb{R}$. Given g, say smooth metric $\tilde{g}$ is in normal form relative to g if:

i) $\tilde{g} = 2t\,dt \cdot d\rho + g_0$ at $\rho = 0$
(identification $\tilde{\mathcal{G}} \cong \mathbb{R}_+ \times M \times \mathbb{R}$ induced by g).

ii) The lines $\rho \to (z, \rho)$ are geodesics for $\tilde{g}$.

An inhomogeneous ambient metric $\tilde{g}$ is said to be in normal form relative to g if its smooth part is in normal form relative to g.

Main existence, uniqueness theorem for inhomogeneous ambient metrics:

Theorem. Up to pull-back by a smooth homogeneous diffeomorphism which restricts to the identity on $\mathcal{G}$, the inhomogeneous ambient metrics for $(M,[g])$ are parametrized by the choice of an arbitrary trace-free symmetric 2-tensor field A_{ij} (the ambiguity tensor) on M.

Parametrization: Choose g. Write $\widetilde{g}$ in normal form in decomposition $\widetilde{\mathcal{G}} = \mathbb{R}_+ \times M \times \mathbb{R}$ induced by g. Let $\widetilde{g}^{(0)} =$ smooth part. Consider M component. Homogeneous of degree 2, so write $\widetilde{g}^{(0)}_{ij} = t^2 g^{(0)}_{ij}$. Then $g^{(0)}_{ij}(x,\rho) =$ smooth 1-parameter family of metrics on M with $g^{(0)}_{ij}(x,0) =$ given metric g. The ambiguity tensor A_{ij} is:

$$\mathrm{tf}\left((\partial_\rho)^{n/2} g^{(0)}_{ij}|_{\rho=0} \right) = A_{ij}.$$

The choice of g and A uniquely determine $\widetilde{g}$ in normal form, and therefore also the smooth part $\widetilde{g}^{(0)}$. The inhomogeneous ambient metric $\widetilde{g}$ is a device to determine the smooth homogeneous metric $\widetilde{g}^{(0)}$. Use $\widetilde{g}^{(0)}$ to construct ambient curvature tensors, not $\widetilde{g}$. Cannot differentiate $\widetilde{g}$ to order beyond $n/2$ in any case. All analogous to scalar case.

Note that $\widetilde{g}^{(0)}$ is not Ricci-flat in general, only up to order $n/2-1$.

Consider the curvature tensor $\widetilde{R}^{(0)}$ of $\widetilde{g}^{(0)}$ and its covariant derivatives. Pass to jets at a point. Obtain a map

$$\mathcal{M} \times \mathcal{J}\odot_0^2 \to \Pi_{r=0}^{\infty} \widetilde{\Lambda}^{2,2} \otimes \otimes^r \mathbb{R}^{n+2*} = \widetilde{\mathcal{T}}$$

by

$$(g_{ij}, A_{ij}) \to (\widetilde{R}^{(0)}, \nabla \widetilde{R}^{(0)}, \nabla^2 \widetilde{R}^{(0)}, \cdots)|_{\substack{\rho=0,\, t=1, \\ x=0}}$$

Under a conformal change $\widehat{g} = \Omega^2 g$, $\widetilde{g}$ and $\widetilde{g}^{(0)}$ change by pullback by a smooth, homogeneous diffeomorphism Φ. Can identify the Jacobian of Φ along $\mathcal{G}$ to derive the corresponding change in $\widetilde{\mathcal{R}}^{(0)}$. This shows that there is an induced map

$$c : (\mathcal{M} \times \mathcal{J}\odot^2_0)/\mathrm{CDiff}_0 \to \widetilde{\mathcal{T}}$$

which is P-equivariant, as claimed in the jet isomorphism theorem.

The curvature tensor $\widetilde{R}^{(0)}$ satisfies:

1) $\widetilde{R}^{(0)}_{IJ[KL,M]} = 0$

2) $X \lrcorner \widetilde{R}^{(0)} = 0$

But it is not Ricci-flat, giving rise to the substitute equations for the linearization, again analogous to the scalar case.

__Application.__ Characterization of scalar conformal invariants.

Can extend existing results in parabolic invariant theory to characterize P-invariants of $T\tilde{R}$ for n even.

Applies to invariants of conformal structures via jet isomorphism theorem.

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