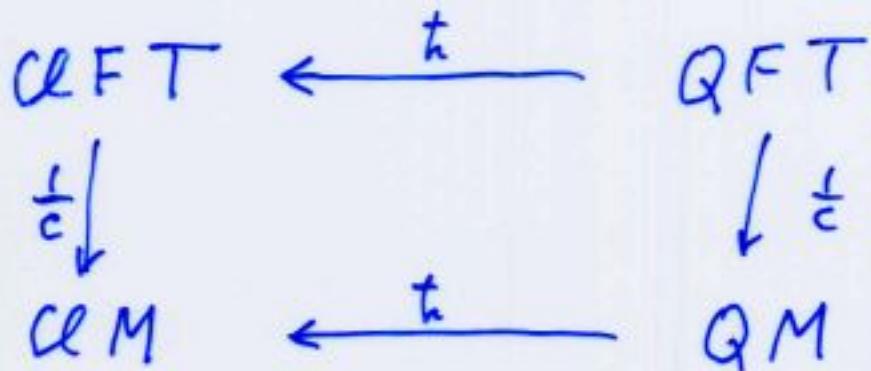


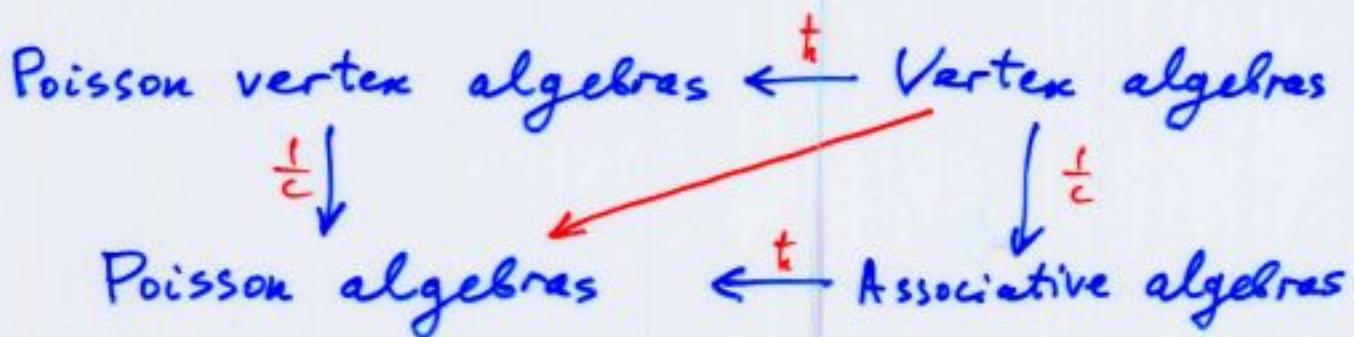
Introduction to vertex algebras

(1)

- ① Four fundamental theories:



Corresponding algebraic structures:



(2) Quasiclassical limit for associative algebras

Given a family of unital associative algebras (over \mathbb{C}) $A_{\frac{1}{k}}$, i.e. an ^{unital} _{associative} algebra over $\mathbb{C}[[\frac{1}{k}]]$ s.t. multiplication by $\frac{1}{k}$ is injective, and

$$[A_{\frac{1}{k}}, A_{\frac{1}{k}}] \subset \frac{1}{k} A_{\frac{1}{k}},$$

the quasiclassical limit is

$$A = A_{\frac{1}{k}} / \frac{1}{k} A_{\frac{1}{k}}$$

with the induced associative product, and the Poisson bracket

$$\{a, b\} = \{\tilde{a}, \tilde{b}\} \bmod \frac{1}{k} A_{\frac{1}{k}}, \text{ where}$$

$$[\tilde{a}, \tilde{b}] = \tilde{a}\tilde{b} - \tilde{b}\tilde{a} = \frac{1}{k} \{ \tilde{a}, \tilde{b} \} \in \frac{1}{k} A_{\frac{1}{k}} \subset A_{\frac{1}{k}},$$

\tilde{a}, \tilde{b} are some preimages of $a, b \in A$ in $A_{\frac{1}{k}}$.

We thus get a Poisson algebra A , i.e.

a unital commutative associative algebra with $\{, \}$, satisfying Lie algebra axioms + Leibniz rule:

$$\{a, bc\} = \{a, b\}c + b\{a, c\}$$

③ Wightman's axioms of QFT (algebraic version) ^{rough}.

Data:

- V space of states (vector space $/ \mathbb{C}$)
- $|0\rangle \in V$ vacuum vector
- a representation π of the Poincaré group
(= group of isometries of the Minkowski space M^D)
in the space V
- a collection of quantum fields (= End V -valued distributions $\{\Phi^\alpha(z)\}_{\alpha}$ on M^D)
such that $\Phi^\alpha(z)v$ is a V -valued function, $v \in V$)

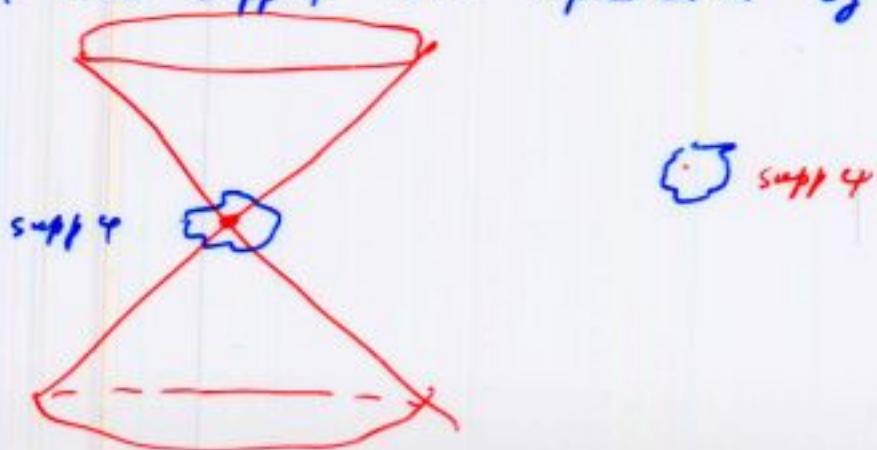
Axioms: (vacuum) $|0\rangle$ is fixed by the Poincaré group

(Poincaré covariance) $\pi(g)\phi^\alpha(z)\pi(g)^{-1} = \phi^{g\cdot\alpha}(g \cdot z)$

(completeness) $\phi^{\alpha_1}(\varphi_1) \dots \phi^{\alpha_s}(\varphi_s)|0\rangle$ span V

(locality) $[\phi^\alpha(\varphi), \phi^\beta(\psi)] = 0$ if

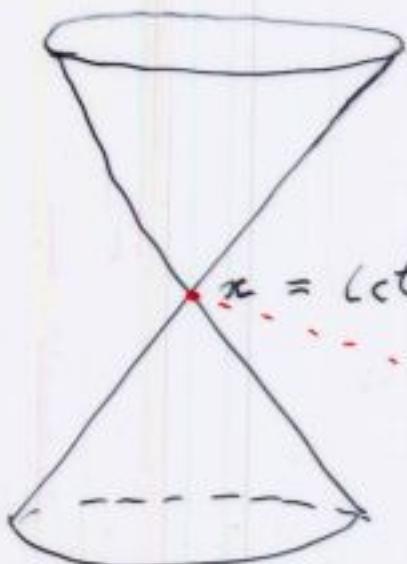
$\text{supp } \varphi$ and $\text{supp } \psi$ are separated by light cone:



Minkowski space-time $M^D = \mathbb{R}^D$

metric $|x|^2 = x_0^2 - x_1^2 - \dots - x_{D-1}^2$

$x_0 = ct$; x_1, \dots, x_{D-1} space coordinates



light cone $|x|^2 = 0$

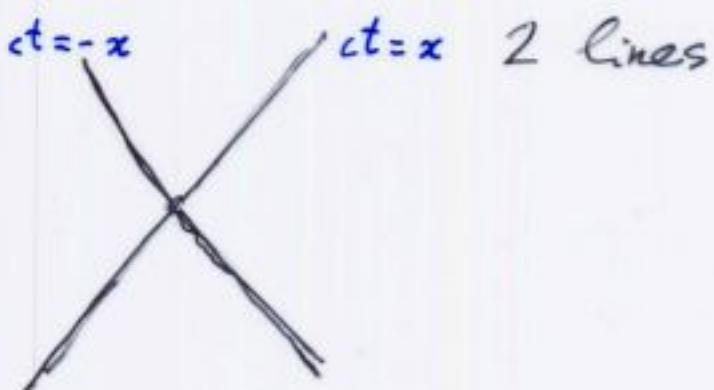
$$x = (ct, x_1, \dots, x_{D-1})$$

$$y = (ct', y_1, \dots, y_{D-1})$$

$$|x-y| < 0, \text{ i.e. } \sqrt{\sum_{i=1}^{D-1} (x_i - y_i)^2} / |t - t'| > c :$$

the speed of a signal travelling from x to y should be $> c \Rightarrow \phi^\alpha(x) \phi^\beta(y) = \phi^\beta(y) \phi^\alpha(x)$, i.e. measurements at x and y are independent

④ If $D = 2$, the light cone is:



Chiral part of 2D QFT =

take quantum fields supported on one
of the lines

Then Poincaré group = group of translations
of a 1-dim space + dilations

Ignore (but later bring back)

and representation of its Lie algebra
is a single operator $T \in \text{End } V$

Thus we arrive at the definition
of a vertex algebra (= chiral algebra)

"Vertex algebras for beginners" AMS
1996, 1998

Vertex algebra :

- Data:
- V space of states = vector space / \mathbb{C}
 - $|0\rangle \in V$ vacuum vector
 - $T \in \text{End } V$ translation operator
 - $\mathcal{F} = \{ \phi^\alpha(z) = \sum_{n \in \mathbb{Z}} \phi_{(n)}^\alpha z^{-n-1} \}_{\alpha}$

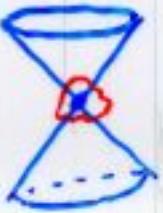
a collection of $\text{End } V$ -valued quantum fields
(i.e. $\phi_{(n)}^\alpha \in \text{End } V$, $\phi_{(n)}^\alpha v = 0$ for $n \gg 0$).

- Axioms:
- (vacuum) $T|0\rangle = 0$,
 - (translation covariance) $[T, \phi^\alpha(z)] = \frac{d}{dz} \phi^\alpha(z)$,
 - (completeness) $\phi_{(n_1)}^{\alpha_1} \dots \phi_{(n_s)}^{\alpha_s}|0\rangle$ span V ,
 - (locality) $(z-w)^N [\phi^\alpha(z), \phi^\beta(w)] = 0$ for some $N \in \mathbb{Z}_+$.

"Trivial" example. V = unital commut. assoc. alg.
 $|0\rangle = 1$, $T = 0$, $\mathcal{F} = \{L_a\}_{a \in V}$

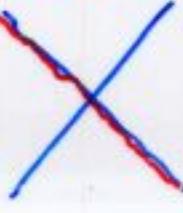
Exercise 1. VA with $T = 0$ are precisely unital commutative associative algebras

Comment: relation to Wightman axioms:
(VA for beginners)



\bullet

$D = 2$



chiral part

Remark One can't cancel $(z-w)^N$, e.g.

Ex 2. $\delta(z,w) = z^{-1} \sum_{n \in \mathbb{Z}} \left(\frac{w}{z}\right)^n$; $(z-w)\delta(z,w) = 0$

But this is basically all that can happen, namely, locality \Leftrightarrow

Ex 3 OPE $[\phi^\alpha(z), \phi^\beta(w)] = \sum_{j=0}^{N-1} \psi^j(w) \left(\frac{\partial}{\partial w} \right)^j \delta(z,w) / j!$

(interaction of fields ϕ^α and ϕ^β produces new fields ψ^j)

In order to keep track of this, introduce

λ -bracket:

$$[\phi^\alpha, \phi^\beta]_\lambda = \sum_{j=0}^{N-1} \frac{\lambda^j}{j!} \psi^j,$$

It encodes the "singular part" of OPE and satisfies axioms, similar to that of a Lie algebra. This new structure is called a **Lie conformal algebra**.

These correspond \sim bijectively to formal distribution Lie algebras.

affine Example 1. $\hat{g} = \mathbb{C}[t, t^{-1}] + \mathbb{C}K$ simple Lie alg. central

$$(*[at^m, bt^n] = [a, b]t^{m+n} + m\delta_{m,-n}(a, b)K)$$

is a formal distribution Lie algebra,

$$\mathcal{F} = \{ a(z) = \sum_n (at^n) z^{-n-1} \}_{a \in g}$$

$$*[a(z), b(w)] = [a, b](w) \delta(z-w) + (a, b)K \delta'_w(z-w)$$

$$\text{or } * [a, b] = [a, b] + \lambda(a, b)K \text{ is the}$$

λ -bracket of a Lie conf algebra $\mathbb{C}[T]g + \mathbb{C}K$

$$\text{Virasoro Example 2. } *[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C$$

$$\mathcal{F} = \{ L(z) = \sum_n L_n z^{-n-2} \}$$

$$! [L, L] = (T + 2\lambda)L + \frac{\lambda^3}{12}C$$

λ -bracket on the Vir Lie conf alg. $\mathbb{C}[T]L + \mathbb{C}C$

free fermions

Example 3. A vector superspace with a

non-degenerate skewsymmetric bilinear form $\langle \cdot, \cdot \rangle$

$$\hat{A} = A[t, t^{-1}] + \mathbb{C}I$$

$$[at^m, bt^n] = \langle a, b \rangle \delta_{m,-n-1} I$$

$$\mathcal{F} = \{ a(z) = \sum_n (at^n) z^{-n-1} \}_{a \in A}$$

$$[a, b] = \langle a, b \rangle I \text{ on } \mathbb{C}[T]A + \mathbb{C}I$$

non-negative powers of t

Correspondence : $R \leftrightarrow \text{Lie } R \supset (\text{Lie } R)_-$

Ex 4. ⑤ Lemma. $\phi^\alpha(z)|0\rangle \in V[[z]]$

(In general, by def., $\phi^\alpha(z)v \in V((z))$.)

Hence have field-state correspondence:

$$\phi^\alpha(z) \mapsto \phi^\alpha := \phi^\alpha(z)|0\rangle \Big|_{z=0}.$$

Extension Theorem. Let $\overline{\mathcal{F}}$ be the collection of all translation covariant quantum fields, which are local to all fields from \mathcal{F} . Then

(a) all axioms of VA still hold;

(b) the field-state correspondence

$$\overline{\mathcal{F}} \rightarrow V, \phi(z) \mapsto \phi = \phi(z)|0\rangle \Big|_{z=0}$$

is bijective.

This leads, to the second, equivalent definition of a VA

8

Vertex algebra is $(V, |0\rangle, T, \phi \mapsto \phi(z))$
 subject to the following axioms:

$$(\text{vacuum}) \quad T|0\rangle = 0, \quad \phi(z)|0\rangle|_{z=0} = \phi$$

$$(\text{translation covariance}) \quad [T, \phi(z)] = \frac{d}{dz} \phi(z)$$

$$(\text{locality}) \quad (z-w)^N [\phi(z), \psi(w)] = 0, \text{ some } N \in \mathbb{Z}_+$$

Remark. Completeness automatic: $\phi_{(-1)}|0\rangle = \phi$.

Remark. Each quantum field $\phi(z) = \sum_{n \in \mathbb{Z}} \phi_{(n)} z^{-n}$,
 where $\phi_{(n)} \in \text{End } V$. Hence for each $n \in \mathbb{Z}$
 can define n -th product on V :

$$\phi_{(n)} \psi = \phi_{(n)}(\psi) \quad (\phi, \psi \in V)$$

Original Borcherds' definition (1986):
(use uniqueness thm)

V is an algebra with infinitely many
 $= 0 \text{ for } n > 0$
 products $a_{(n)} b$ and $|0\rangle \in V$ subject to axioms:

$$(\text{vacuum}) \quad |0\rangle_{(n)} a = \delta_{n,-1} a \quad (n \in \mathbb{Z}); \quad a_{(n)} |0\rangle = \delta_{n,-1} a \quad (n \geq -1)$$

$$(\text{Borcherds identity}) \quad \sum_{j \in \mathbb{Z}_+} \binom{m}{j} (a_{(n+j)} b)_{(m+k-j)} c = \sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} (a_{(m+n-j)} (b_{(k+j)} c)) - (-1)^k b_{(n+k-j)} (a_{(m+j)} c)$$

$a, b, c \in V; m, n, k \in \mathbb{Z}$

(6) Denote

$:ab: = a_{(-1)} b$ (normally ordered product)

$$[a_\lambda b] = \sum_{n \in \mathbb{Z}_+} \frac{\lambda^n}{n!} a_{(n)} b \quad (\lambda\text{-Bracket})$$

polynomial in λ

4th equivalent definition of a VA

(Bakalov - VK
math.QA/0204282)

Data : $(V, \langle \cdot, \cdot \rangle, T, :ab:, [a_\lambda b])$

Axioms :

(A) $(V, \langle \cdot, \cdot \rangle, T, :ab:)$ is a unital, differential algebra, (i.e. T is a derivation of all products)
"quasicommutative" and "quasiassociative":

(quasicommutativity) $:ab:-:ba:=\int^0 [a_\lambda b] d\lambda$

(quasiassociativity) $::abc:c:-:a:bcc::=$

= expression via λ -Bracket, symmetr. w.r.t.

(B) $(V, T, [a_\lambda b])$ is a Lie conformal algebra

(sesquilinearity) $[Ta_\lambda b] = -\lambda [a_\lambda b]$, T derivation

(skewsymmetry) $[a_\lambda b] = -[b_{-\lambda-T} a]$.

(Jacobi) $[a_\lambda [b_\mu c]] = [[a_\lambda b]_{\lambda+\mu} c] + [\delta_{\mu} [a_\lambda c]]$

(C) Quasi Leibniz rule :

$$[a_\lambda :bc:] = :[a_\lambda b]c:+:b[a_\lambda c]:+\int_0^\lambda [[a_\lambda b]_{\mu} c] d\mu$$

Poisson algebra like definition of a vertex algebra

Data: $(V, \langle \cdot, \cdot \rangle, T, :ab:, [\alpha_\lambda b])$

Axioms:

A $(V, \langle \cdot, \cdot \rangle, T, :ab:)$ is a unital differential algebra, "quasicomm." and "quasiassoc":

$$(\text{quasicomm.}) :ab:-:ba:=\int_{-T}^0 [\alpha_\lambda b] d\lambda$$

$$(\text{quasiassoc.}):ab:c - :a:b:c: = :(\int_0^T d\lambda a)[b_\lambda c]: + :(\int_0^T d\lambda b)[a_\lambda c]:$$

B $(V, T, [\alpha_\lambda b])$ is a Lie conformal algebra:

$$(\text{sesquilinearity}) [Ta_\lambda b] = -\lambda [\alpha_\lambda b], [a_\lambda Tb] = (T+\lambda)[\alpha_\lambda b]$$

$$(\text{skewsymmetry}) [\alpha_\lambda b] = -[\beta_{-\lambda-T} a]$$

$$(\text{Jacobi}) [\alpha_\lambda [\beta_\mu c]] - [\beta_\mu [\alpha_\lambda c]] = [[\alpha_\lambda \beta]_{\lambda+\mu} c]$$

C Quasi Leibniz rule:

$$[\alpha_\lambda :bc:] = :[\alpha_\lambda b]c:+:\beta[\alpha_\lambda c]: + \int_0^\lambda d\mu [[\alpha_\lambda \beta]_\mu c]$$

Poisson vertex algebra

Its definition is obtained from the 4th definition of a VA by deleting "quantum corrections in **(A)** and **(C)**:

Data: $(V, \langle \cdot, \cdot \rangle, T, \text{ab}, \{a, b\})$

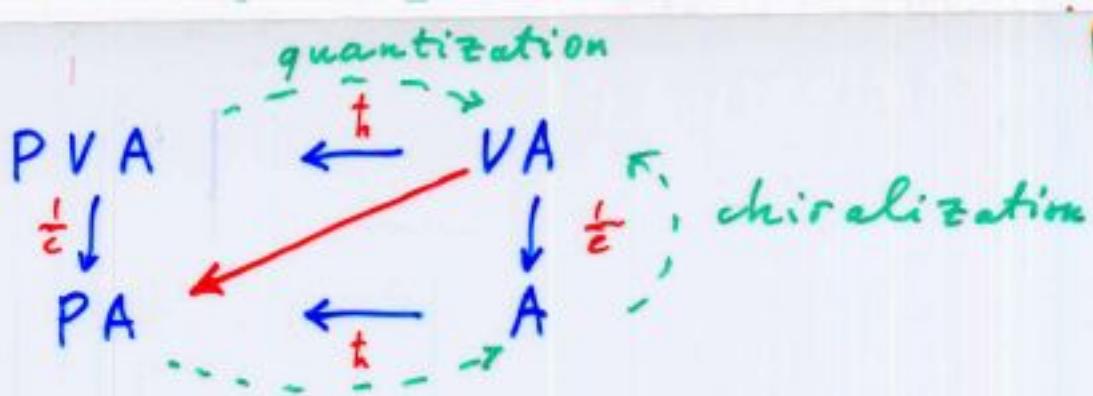
Axioms:

(A) $(V, \langle \cdot, \cdot \rangle, T, \text{ab})$ is a unital differential commutative associative algebra

(B) $(V, T, \{a, b\})$ is a Lie conformal algebra

(C) Leibniz rule:

$$\{a, bc\} = \{a, b\}c + b\{a, c\}.$$



Given a vertex algebra V , the associated Poisson algebra is

$$PA(V) = V / : (TV)V :$$

with the induced product by $: \cdot :$,
and the Poisson bracket:

$$\{a, b\} = [a, b]_{\lambda=0}$$

The diagonal map is the projection map.

Axioms of a PA follow immediately from the axioms of V .

⑧ The theory of infinite-diml integrable systems

Remark. If V is a Lie conformal algebra, then $\text{Lie } V := V/TV$ with bracket

$$[a, b] = [a, b]|_{\lambda=0}$$

is a Lie algebra, and V is a Lie V -mod:

$$a \cdot v = [a, v]|_{\lambda=0}.$$

Local functionals: are elements of $\text{Lie } V$.

Hamiltonian equation}: $u = [h, u]$, $u \in V$.
corr. to local functional \underline{h}

It is completely integrable if there are infinitely many integrals of motion $h_i \in \text{Lie } V$ in involution

$$\text{i.e. } [h, h_i] = 0 \quad (\Leftrightarrow h_i = 0) \text{ s.t. } [h_i, h_j] = 0$$

If V is a PVA ; get a classical Hamiltonian equation , if V is a VA , get a quantum Hamiltonian equation.

(11)

Example : KdV

Poisson VA $\mathcal{V} = \mathbb{C}[u, u', u'', \dots]$,

$\langle 0 \rangle = 1$, $Tu^{(n)} = u^{(n+1)}$ derivation

$\{u, u\} = \lambda$ Gardner-Faddeev-Zakharov Bracket GFZ
extended by Leibniz rule, (1971)
sesquilinearity and skewsym.

Consider the local functionals:

$$h_0 = \int u, h_1 = \int u^2, h_2 = \frac{1}{2} \int (u^3 - u'^2), \dots$$

Sign of \int means image in $\mathcal{V}/T\mathcal{V}$.

E.S. Check that they are in involution
and that the Hamiltonian eqn

$$\dot{u} = \{h_2, u\}$$

$$\text{is KdV : } \dot{u} = 3uu' + u'''.$$

⑨ The simplest non-trivial example of a vertex algebra is the quantization of the GFZ Poisson vertex algebra.

Definition. Let V_t be a family of VA, such that $[V_t, V_t] \subset t V_t [\lambda]$.

Then on $\mathcal{U} = V_t / t V_t$ get the canonical induced structure of a PVA, called the quasiclassical limit of V_t .

Example. Free boson VA = quantization of GFZ:

$$B_t = \mathbb{C}[u, u', u'', \dots][[t]]$$

$$|0\rangle = 1, T_u^{(n)} = u^{(n+1)}, \mathcal{F} = \{d(z) = \sum_n d_n z^{-n-1}\},$$

where $d_{-n} = n! u^{(n)}$, $d_n = \frac{t}{(n-1)!} \frac{\partial}{\partial u^{(n-1)}}$,

$$d_0 = 0, \text{ so that}$$

$$[d_m, d_n] = t m \delta_{m, -n}, \text{ or } [d(z), d(w)] = t \delta_w'(z-w)$$

$$\text{or } [d_\lambda, d] = t \lambda$$

In the quasiclassical limit:

$$\{d_\lambda, d\} = \lim_{t \rightarrow 0} \frac{[d_\lambda, d]}{t} = \lambda \quad \begin{matrix} \text{GFZ} \\ \text{crackt} \end{matrix}$$

More generally :

Universal enveloping VA $V(R)$
of a Lie conf. algebra R

Def 1. $V(R) = \text{Ind}_{(\text{Lie } R)_-}^{\text{Lie } R} \mathbb{C}$
 $\equiv U(\text{Lie } R)/U(\text{Lie } R)\text{Lie}_-$

$|0\rangle = 1$, T descends, $\mathcal{F} = R$

use def 1 of VA

Def 2. $V(R) = \text{span} : a_{i_1}, \dots a_{i_s} :$,
where $\{a_i\}$ basis of R $i_1 \leq \dots \leq i_s$
use def 4 of VA

Example : $R = \mathbb{C}[T]\alpha + \mathbb{C}$

$$[\alpha_\lambda, \alpha_\mu] = \lambda$$

$$\text{Lie } R = \{ [\alpha_m, \alpha_n] = m \delta_{m,-n} \} \quad m, n \in \mathbb{Z}$$

$$(\text{Lie } R)_- = \{ \alpha_m \mid m > 0 \}$$

$V(R)$ free boson

Family : $[\alpha_\lambda, \beta_\mu]_k = k [\alpha_\lambda, \beta_\mu]$

Quasiclassical limit : $\mathcal{V} = S_{\mathbb{C}}(R)$
PVA

The most important special cases:

Example 1. of simple Lie algebra, $k \in \mathbb{C}$.

$$V^k(g) = \text{Ind}_{g[t] + CK}^{\hat{g}} \mathbb{C}_k$$

$$\uparrow \quad = U(\hat{g})/U(\hat{g})(g[t] + C(K-t))$$

$|0\rangle = 1$, $T = -\frac{d}{dt}$

$$\mathcal{F} = \{a(z) = \sum_n a z^n / z^{-n-1}\}$$

Example 2. A superspace with a

skewsymmetric bilinear form $\langle ., . \rangle$

$$F(A) = \text{Ind}_{A[t] + CK}^{\hat{A}} \mathbb{C},$$

\uparrow
Free fermions based on A

To construct a family, just
replace in \hat{g} (resp \hat{A}) the bracket by

$$[a, b]_t = t [a, b]$$

⑩ How to construct the vertical arrows?

Need, in addition, an energy operator H on V , satisfying:

$$[H, a(z)] = \left(z \frac{d}{dz} + \Delta_a \right) a(z)$$

$$\text{if } H\alpha = \Delta_a \alpha$$

(This amounts to enlarging the translation group to the whole Poincaré group in the covariance axiom)

Example. If one of the fields of V is the Virasoro field

$$L(z) = \sum_n L_n z^{-n-2},$$

$$\text{s.t. } L_{-1} = T, \text{ then } \underline{H = L_0}.$$

$$\text{In Example 1 : } L = \frac{1}{z^{(k+h)}} \sum_i :a_i b_i: \quad (a_i b_j) = \delta_{ij}$$

Sugawara construction

$$\text{In Example 2 : } L = \frac{1}{2} \sum_i : (T a_i) b_i :$$

$$\text{Or simply : } L_0 = -t \frac{d}{dt} \quad (L_{-1} = -\frac{d}{dt})$$

Introduce the ε -deformed quantum fields $a_\varepsilon(z) = (1 + \varepsilon z)^{\Delta_a} a(z)$

if $H a = \Delta_a a$.

$$= \sum_n a_{(n, \varepsilon)} z^{-n-1}$$

so that we have the ε -deformed n th products:

$$a_{(n, \varepsilon)} b = a_{(n, \varepsilon)}(b) = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_a}{j} \varepsilon^j a_{(n+j)} b.$$

Remark. The ε -deformed fields $a_\varepsilon(z)$ form a "gauged VA", satisfying the same axioms as VA, except that the translation covariance axiom changes:

$$[T, a_\varepsilon(z)] = \left(\frac{d}{dz} - \frac{\Delta_a \varepsilon}{1 + \varepsilon z} \right) a_\varepsilon(z).$$

Introduce the deformed normally ordered product on V :

$$a * b = a_{(-1, \epsilon=1)} b$$

Theorem. (Zhu, 1996) (a) $\mathcal{J} = \text{span} \left\{ ((T+H)a) * b \right\}_{a,b \in V}$
 (Be Sole, VK, 2005)

is a 2-sided ideal w.r.t. to the product $*$.

(b) $Z(V) := (V, *) / \mathcal{J}$ is a unital associative algebra, which "controls" (twisted) representation theory of V .

The right vertical arrow is just the canonical projection

$$\begin{matrix} V \\ \downarrow \\ Z(V) \end{matrix}$$

For the left vertical arrow, define
on a PVA \mathcal{V} , the bracket

$$\{a, b\} = \sum_{j \in \mathbb{Z}_+} (\Delta_a^{-1})_j a_{(j)} b.$$

Theorem. (De Sole - VK, 2005) (a) $\mathcal{I} = \text{span}\{((T+H)a)b\}$

is an ideal for both product and $\{\ , \}$.

(b) $\mathcal{Z}(\mathcal{V}) = \mathcal{V}/\mathcal{I}$ is a Poisson algebra.

(c) Defining the left arrow
as the canonical map,

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\quad} & \mathcal{Z}(\mathcal{V}) \\ \downarrow & & \downarrow \\ \mathcal{Z}(\mathcal{V}) & \xleftarrow{\quad} & \end{array}$$

we obtain the commutative diagramme:

$$\begin{array}{ccc} \mathcal{V} & \xleftarrow{\quad} & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{Z}(\mathcal{V}) & \xleftarrow{\quad} & \mathcal{Z}(\mathcal{V}) \end{array}$$

How the Zhu algebra $Z(V)$ "controls"
 $\text{Rep } V$? 15a

Def. A positive energy representation of V is a linear map $V \rightarrow \text{End } M$ -valued fields : $a \mapsto a^M(z) = \sum_n a_n^M z^{-n-1}$ where $a_n^M \in \text{End } M$, satisfying

(i) $\rightarrow I_M$ and Borcherds identity.

Also $M = \bigoplus_{j \geq 0} M_j$ such that

$a_n^M M_j \subset M_{j-n}$ for all n, j .

(Note: $[H, a_n] = -n a_n$ if $a(z) = \sum a_n z^{-n-1}$)

Theorem. The map $\overset{V_0}{\mapsto} a \mapsto a_0^M / M_0$ induces a representation of $Z(V)$ in M_0 .

This gives a functor :

$\text{PERep } V \rightarrow \text{Rep } Z(V),$

bijective on irreducibles.

Proof. BI says: $v \in M$

$$\sum_{j \in \mathbb{Z}_+} (-1)^j \binom{n}{j} (a_{m+n-j}^M b_{k+j-n}^M - (-1)^n b_{k-j}^M a_{m+j}^M) v \\ = \sum_{j \in \mathbb{Z}_+} \binom{m+\Delta_a-1}{j} (a_{(n+j)} b)^M_{m+k} v$$

Take $v \in M_0$, so that $a_n v = 0$ for $n > 0$.

Let $m=1, n=-1, k=-1$ to get:

$$a_0^M b_0^M v = \sum_{j \in \mathbb{Z}_+} \binom{\Delta_a}{j} (a_{(j-1)} b)_0^M v$$

We thus get homom $(V, *) \rightarrow \text{End } M_0$.

Similarly one shows that $J \rightarrow 0$.

This gives the functor.

Proof of the rest is similar,
but harder. □

Zh 1996, De Sole-VK 2005.

$\text{Span } H \subset \mathbb{Z}$ general twisted case
untwisted case

(1F)

2. Super affine VA

$$\hat{g}^{\text{super}} = g[t, t^{-1}, \theta] + CK \quad K=k$$

$$[at^m, bt^n] = [a, b]t^{m+n} + m(a, b)(k+h^\vee) \delta_{m,-n}$$

$$[\bar{a}t^m, \bar{b}t^n] = (k+h^\vee) \delta_{m,-n}, (a, b)$$

$$[at^m, \bar{b}t^n] = \overline{[a, b]} t^{m+n},$$

where $\bar{a} = a\theta$

$$V_{\text{super}}^k(g) = U(\hat{g}^{\text{super}})/U(\hat{g}^{\text{super}})(g[t, \theta] + C(K-h^\vee))$$

$$|0\rangle = 1, \quad T = -\frac{d}{dt}, \quad \mathcal{F} = \{a(z), \bar{a}(z)\}_{a \in g}$$

$$L = \frac{1}{2(k+h^\vee)} \left(\sum_i (:a^i a^i: + :T \bar{a}^i \bar{a}^i) + \frac{1}{k+h^\vee} \sum_{i,j} : \bar{a}^i [a^i, a^j] \bar{a}^j : \right)$$

$$G = \frac{1}{k+h^\vee} \left(\sum_i :a^i \bar{a}^i: + \frac{1}{3(k+h^\vee)} \sum_{i,j} : [\bar{a}^i, a^j] \bar{a}^i \bar{a}^j : \right)$$

$\{a^i\}$ orthonormal basis of g

VK - I. Todorov 1985

They form the Neveu-Schwarz algebra: (Lie conformal)

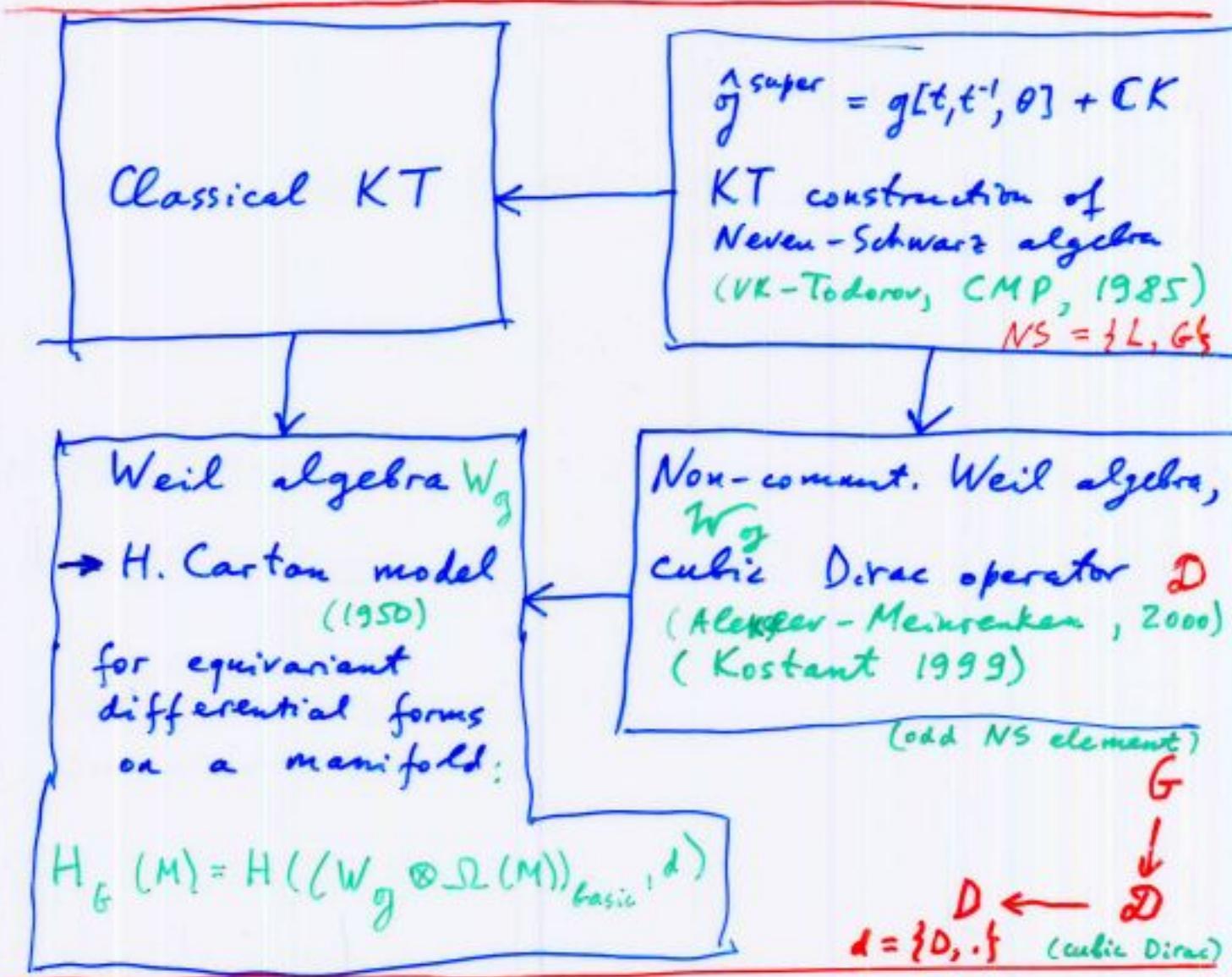
$$[L_\lambda L] = (T+2\lambda)L + \frac{c}{12}\lambda^3,$$

$$[L_\lambda G] = (T + \frac{3}{2}\lambda)G, \quad [G_\lambda G] = 2L + \frac{\lambda^2}{3}C$$

$$(c = \frac{k \dim g}{k+h^\vee} + \frac{\dim g}{2})$$

Remark. Guiding principle in finding formulae
CONFORMAL WEIGHT

Example 2. (De Sole-VK. DB)



$$W_g = (S(g[\theta] + CK) /_{(k=1)}, \quad d = \frac{d}{d\theta}) \quad (\theta = 0)$$

$$\bar{a} = \theta a, \quad [\bar{a}, \bar{b}] = (a, b) K$$

$$g^{\text{super}} = g[\theta] + CK$$

Quantization \hat{W}_g : replace $S(g^{\text{super}})$ by $\mathcal{U}(g^{\text{super}})$

Chiralization: replace g^{super} by $\hat{g}^{\text{super}} \rightarrow V^{\text{super}}(g)$

De Sole-VF
2006

(18a)

$$U_{\text{sup}}^k(g) \leftarrow V_{\text{sup}}^k(g) \quad L \quad G$$

$$\begin{matrix} \downarrow & \downarrow \\ \text{Weil algebra}^{(\text{complex})} & \leftarrow \text{non commut. Weil algebra}^{(\text{complex})} \end{matrix}$$

$$W(g)$$

$$W(g)$$

C

D

g^{super} ~~magical~~

$$W(g) = \underbrace{(S(g[\theta] + CK))}_{K=1} / , d = \frac{d}{d\theta}$$

where the central extension $g[\theta] + CK$ is defined by $[\bar{a}, \bar{b}] = (a, b)K$, ~~and since~~
 $[a, b]_S, [\bar{a}, \bar{b}]$ as usual, $\bar{a} = a\theta$ ~~are different~~

$$W(g) = \underbrace{(U(g[\theta] + CK))}_{K=1} / , d = \frac{d}{d\theta}$$

$$d = [D, \cdot] \quad \text{resp} \quad d = [\mathcal{D}, \cdot]$$

where \mathcal{D} and D is the

cubic Dirac operator (classical and quantum)

$$\mathcal{D}^2 = C \quad D^2 = C + \left(\frac{h}{24} - \frac{l}{16} \right) \dim g$$

H. Cartan ~1950

Alexeev-Meinrenken, Kostant ~1999

Explicit formulas:

$$\left. \begin{array}{l}
 \text{classical} \\
 \text{quantum}
 \end{array} \right\} \begin{array}{l}
 D = \sum_i a^i \bar{a}^i + \frac{1}{3} \sum_{i,j} \overline{[a^i, a^j]} \bar{a}^i \bar{a}^j \\
 C = \frac{1}{2} \sum_i a^{i^2} + \frac{1}{2} \sum_{i,j} \overline{\bar{a}^i [a^i, a^j]} \bar{a}^j \\
 D = \sum_i a^i \bar{a}^i + \frac{1}{3} \sum_{i,j} \overline{[a^i, a^j]} \bar{a}^i \bar{a}^j \\
 C = \frac{1}{2} \sum_i a^{i^2} + \frac{1}{2} \sum_{i,j} [a^i, a^j] \bar{a}^i \bar{a}^j \\
 \qquad \qquad \qquad + \frac{(-2L)}{16} \text{diag}
 \end{array}$$

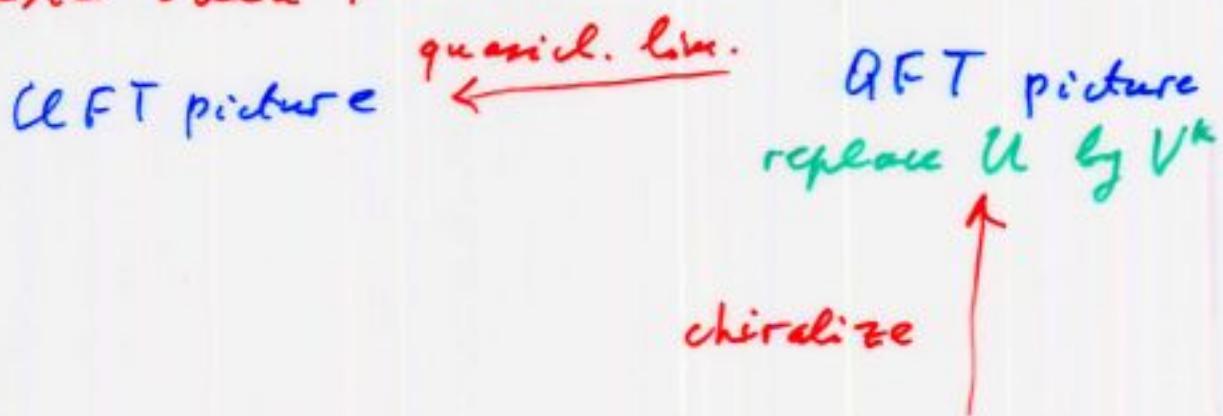
Remarks: 1) H. Cartan's theorem for equivariant cohomology:

$$H_G(M) = H((W(g) \otimes \mathcal{R}(M))_{\text{basic}}, d)$$

2) $(W(g), D)$ is used in representation theory of Lie groups (multiplet decoups, Vogan conjecture)

3) $V_{\text{sup}}^k(g)$, L , G used in representation theory of the NS algebra. VK-Todorov
Currently in repr. theory of \hat{g} VK - Mousleyer - Trojia - Rapci

Basic idea :



Classical-mech picture $\xrightarrow{\text{quantize}}$ Quantum-mech picture

e.g. $S(g \text{ or } g^{\text{super}})$ replace S by U

More complicated situations reduced to this one if we are able to represent our Poisson algebra

as $PA = H(S(\dots), d_A)$

Then $A = H(U(\dots), d_A)$

$V_A = H(V^k(\dots), d_{VA})$

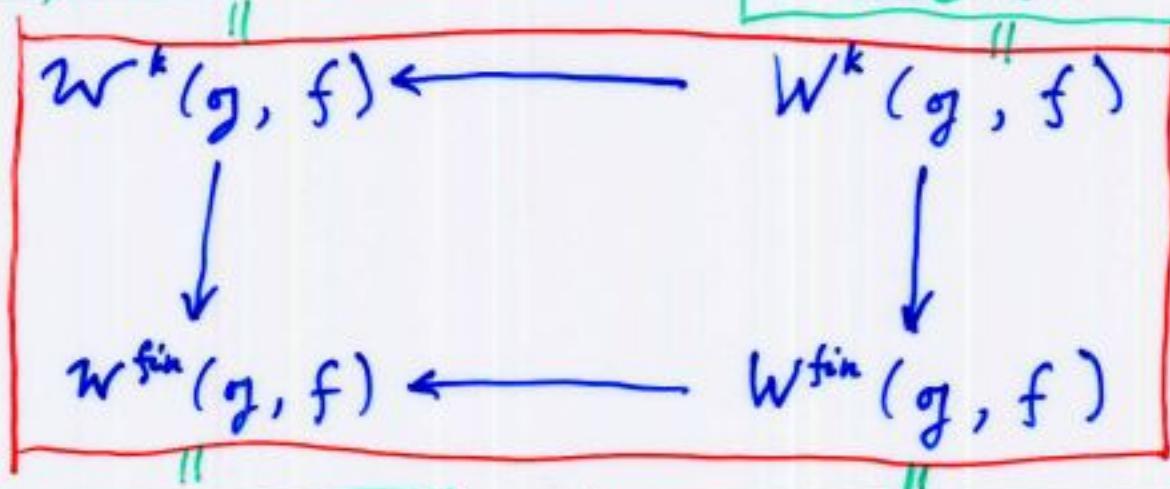
(20)

\mathfrak{g} = a finite-dimensional simple Lie (super) algebra and f a nilpotent (even) element of \mathfrak{g} , $k \in \mathbb{C}$

simple Lie (super) algebra and f a nilpotent (even) element of \mathfrak{g} , $k \in \mathbb{C}$

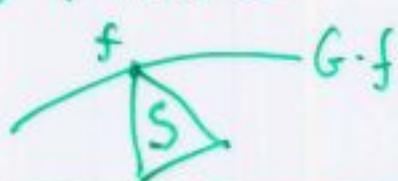
Classical affine W -algebra
= generalized Drinfeld-Sokolov reduction
1980s, KdV

Affine W -algebra
= generalized quantum Drinfeld-Sokolov reduction



Shadowy slices

$f + \text{Ker } e$



Finite W -algebra

$$\cong (\mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(m)} \mathbb{C}_f)^n$$

$$\mathfrak{g} \supset \mathfrak{sl}_2 = \langle e, h, f \rangle$$

$$[e, f] = h$$

$$n = \mathfrak{g}_{\geq 1/2} \supset m = \mathfrak{g}_{\geq 0}$$

$$[h, e] = e$$

$$\text{for } \text{ad } h$$

$$[h, f] = -f$$

for superregular f

$$S \rightarrow \mathfrak{h}/W$$

is semiuniversal deformation of simple singularity

Kostant 78, Lynch 79, ...
Premet, Gan-Ginzburg 2002, ...

Whittaker models

primitive ideals in $\mathfrak{U}(\mathfrak{g})$

Applications :

Integrable KdV
type hierarchies
of evolution PDE

Representation theory
of Virasoro like Lie algebras

$$\text{KdV: } \mathfrak{g} = \mathfrak{se}_2, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad = \quad \text{Virasoro algebra example}$$

Theory of singularities

Primitive ideals
in \mathfrak{g}

Construction of $W^k(g, f)$:

Quantize and after that chiralize the Slodowy slice:

$$\text{Slodowy slice} = H(S(g) \otimes \Lambda, d)$$

(Hamiltonian reduction)

Quantization: replace $S(g)$ by $U(g)$ and Λ by $C\ell$.

Chiralization: replace $U(g)$ by the vertex algebra $V^k(g)$
 (= a correct completion of $U(\hat{g}_k)$)
 and $C\ell$ by F (= a correct completion
 of $\hat{C}\ell$)
 to get:

$$W^k(g, f) = H(V^k(g) \otimes F, d)$$

(late 80's, Feigin-Frenkel 91, Frenkel-VK-Wakimoto 92)

VK-Roan-Wakimoto 2003, VK-Wakimoto 2004)
 math-ph/0302015

math-ph/0304011
 math-ph/0404049

(11)

Construction of $W^k(g, f)$

- Data:
- simple Lie (super)algebra g with a (super)symmetric non-deg. invariant bilinear form (\cdot, \cdot)
 - nilpotent element f in g_{even}
 - $k \in \mathbb{C}$

Take e, h, f with

$$[e, f] = h, [h, e] = e, [h, f] = -f,$$

ad h - eigenspace decomposition:

$$g = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} g_j ; \quad g = g_- + g_0 + g_+$$

$$A_{ch} = \prod (g_- + g_+) \quad \text{with} \quad \langle \cdot, \cdot \rangle_{ch} = (\cdot, \cdot)$$

$$A_{ne} = g_{\frac{1}{2}} \quad \text{with} \quad \langle a, b \rangle_{ne} = (f, [a, b])$$

~~For each have a super Lie
category structure on $A + G + t$~~

~~$t = \{t\} = \langle a, b \rangle$~~

Let $F^{gl} = F(A_{ch}) \otimes F(A_{ne})$

and introduce the complex

$$(C^k(g, f) = V^k(g) \otimes F^{gk}, d_0)$$

where $d_{(0)} = \text{Res}_{z=0} d(z)$, and

$\phi(z)$ is the quantum field, corr. to

$$d = \sum_{\alpha} (- u_\alpha e^\alpha + (f, u_\alpha) e^\alpha)$$

basis of g_+

$$+ \sum_{\text{basis of } g_{\frac{1}{2}}} : \Phi_{\alpha} e^{\alpha} : + \frac{1}{2} \sum_{\alpha, \beta} : \varphi_{[\alpha, \beta]} e^{\alpha} e^{\beta} :$$

Here $\{u_\alpha\}$ is a basis of \mathfrak{g}_+ ,

$\{e_k\}$ corr basis of $\pi g_f \subset A_{dk}$,

$\{e^a\}$ dual basis of $T\mathfrak{g} - \sum A_{ab}$

$\{\phi_\lambda\}$ corr basis of A_{he} .

Lemma. $[d, d] = 0 \Rightarrow d_{(0)}^2 = 0$

Also $d_{(0)}$ is a derivation of all products of $C^k(g, f)$, hence homology is again a VA.

$$W^k(\sigma, f) = H(C^k(\sigma, f), d_{(0)})$$

$$C^k(\sigma, f) = \bigoplus_{m \in \mathbb{Z}} C^k(\sigma, f)_{(m)} \quad \text{(charge decom)}$$

charge $\sigma = \text{charge } A_{ne} = 0$,

charge $\Pi \sigma_{\pm} = \pm 1$

$d_{(0)}$ decreases charge by 1

$$W^k(\sigma, f) = \bigoplus_{m \in \mathbb{Z}} W^k(\sigma, f)_{(m)}$$

Th. (i) $W^k(\sigma, f)_{(m)} = 0$ if $m \neq 0$

(ii) $W^k(\sigma, f) = W^k(\sigma, f)_{(0)}$ is

freely generated by fields, labeled

by $\sigma^f = \bigoplus_{j \leq 0} \sigma_j^f$ have conformal weight $= 1-j$

labeled $\overset{\text{green}}{\sigma}^f$ w.r. to $L = L^{sg} + T_x + L^{ch} + L^{ne}$,

$$c = \text{sdim } \sigma_0 - \frac{1}{2} \text{sdim } \sigma_{\frac{1}{2}} - \frac{12}{k+h^r} |p - \pi(k+h^r)|^2$$

Remark: Feigin-Frenkel thm.

Functor H

Let F^{gl} be the universal enveloping VA of the Lie conformal superalgebra

$$\mathbb{C}[t](A_{ch} + A_{AE}) + \mathbb{C}1 \quad (\text{ghosts})$$

Given a $\hat{\mathfrak{g}}^*$ -module M of level k , have the complex $(M \otimes F^{gl}, d_{(0)}^M)$, which gives the functor

$$\begin{aligned} H : \text{Rep } \hat{\mathfrak{g}}_k &\longrightarrow \text{Rep } W^k(\mathfrak{o}, f) \\ M &\mapsto H(M \otimes F^{gl}, d_{(0)}^M) \end{aligned}$$

Using this functor, can construct a unified representation theory of all superconformal algebras:
 free field realizations (quantum Miura)
 determinant formulas,
 character formulas.

Functor:

$$\text{RRep } \hat{\mathfrak{g}}_k \xrightarrow{H} \text{Rep } W^k(g, f)$$

$$M \longmapsto H(M \otimes F^{\otimes k}, d^M)$$

It is exact and irr \mapsto irr or 0

(FKM, KRW conjectured, Arakawa 04, 05
proved in some most important cases)

Simplest example:

$\mathfrak{g} = sl_2$, modular invariant \hat{sl}_2 -modules: (KW88)

$$M = L(k = -2 + \frac{p}{q}, \lambda = \frac{nq - mp}{q}) \quad \begin{array}{l} (p, q) = 1 \\ p > q \geq 1 \\ 0 \leq n \leq q-1 \\ 0 \leq m \leq p-2 \end{array}$$

$$H(M, d) = 0 \text{ iff } m = q-1 \text{ (integrable)} \quad (\text{integrable} \Rightarrow 0)$$

Otherwise get Virasoro minimal series:

$$c = 1 - \frac{6(p-q)^2}{pq}, \quad h_{m', n'}^{(p, q)} = \frac{(pm' - qn')^2 - (p-q)^2}{4pq}$$

$$2 \leq q < p, \quad m' = m+1, n' = n+1, \quad \begin{array}{l} 1 \leq m' \leq q-1 \\ 1 \leq n' \leq p-1 \end{array}$$

Derive characters, det formulae, etc., for $W^k(g, f)$ if known for $\hat{\mathfrak{g}}$.

But for $\hat{\mathfrak{g}}$ super little is known about characters. There are conjectures.

Applications to representation theory

g	f	$W^k(g, f)$
sl_2	f	Virasoro
$g = \text{Lie algebra}$	principal	quantum D S geometric Langlands
sl_3	minimal	Bershadsky-Polyakov
$spo(2, 1)$	minimal	Neveu-Schwarz $N=1$
$spo(2, m)$	minimal	Bershadsky-Kazhdan
$sl(2, 1)$	minimal	$N=2$
$spo(2, 3)$	minimal	$N=3$
$sl(2, 2)/CI$	minimal	$N=4$
$D(2, 1, \alpha)$	minimal	big $N=4$
$spo(2, 3)$	principal	Shatashvili-Vafa spin \pm holonomy
$D(2, 1, \alpha)$	principal	Shatashvili-Vafa G_2 holonomy
sl_n	subprincipal	Feigin-Semikhatov 2004