LECTURE I: THE LIE ALGEBRAS $sl(\infty)$, $o(\infty)$, $sp(\infty)$ AND THEIR SEMISIMPLE SUBALGEBRAS

$\mathbb{C}$

Ivan Penkov, Jacobs University Bremen
i.penkov@jacobs-university.de

1. EQUIVALENT DEFINITIONS OF $sl(\infty)$

a) UNION $\bigcup_n sl(n)$ UNDER ANY INCLUSIONS
$sl(n) \subset sl(n+1)$

b) INFINITE BUT FINITARY MATRICES OF TRACE 0

c) LOCALLY SIMPLE FINITARY LIE ALGEBRA OF TYPE A

D) LOCALLY SIMPLE LIE ALGEBRA OF TYPE A ADMITTING A ROOT DECOMPOSITION

e) $V \times W \to \mathbb{C}$ NON DEG. BILINEAR FORM
$V \otimes W$ – ASSOC. ALGEBRA AND LIE ALGEBRA
$sl(\infty) = [V \otimes W, V \otimes W]$

IDEA OF PROOF

a)

$$\begin{tikzcd}
\mathfrak{g}_1 \arrow[r] & \mathfrak{g}_2 \arrow[r] & \cdots \arrow[r] & \mathfrak{g}_i \arrow[r] & \mathfrak{g}_{i+1} \\
s_1 \arrow[r] \arrow[u, hook, "\varphi_1"] & s_2 \arrow[r] \arrow[u, hook, "\varphi_2"] & \cdots \arrow[r] & s_i \arrow[r] \arrow[u, hook, "\varphi_i"] & s_{i+1} \arrow[u, hook]
\end{tikzcd}$$

$$\boxed{\varphi_{i+1} = \text{conj} \circ \text{outer autorph}}$$

i) ⇒ d)   d) ⇒ a) : isomorphism via the root system A_∞

a) ⟺ b)   a) ⇒ c) Bazanov

a) ⟺ d) Mackey : $V \times W$ admit a pair of dual bases $v_1 \ldots v_n \ldots$ V

$v_1^* \ldots v_n^* \ldots$ W

Then $V \otimes W = \bigcup_n gl(n)$

Simple locally finite <u>finitary</u> Lie algebras:
A. Bazanov
Root-reductive Lie algebras: I. Dimitrov, I.P.
Infinite root systems : E. Neher & coll.
General locally finite Lie algebras:
Yu. Bahturin, H. Strade, G. Benkart, ...

Related results:

– equiv. definitions of $o(\infty)$ and

$o(\infty) = \bigcup_n o(2n) = \bigcup_n o(2n+1)$

$sp(\infty) = \bigcup_n sp(2n)$

$gl(\infty) = V \otimes W$, or infinite but finitary matrices

(2) I

A. Malcev, E. Dynkin : semisimple subalgebras of simple Lie algebras

$k \subset sl(n)$ $\quad$ k is characterized up to conjugacy by its action on $\mathbb{C}^n = V$

All representations are possible except the trivial one.

$k \subset sl(\infty)$

Case I : k – loc. simple, ie. $k = \bigcup_n k_n$ where k_n are simple.

$\Rightarrow k \simeq sl(\infty), so(\infty), sp(\infty)$

$$0 \to m_k V_k \oplus n_k W_k \oplus \mathbb{C}^{M_k} \to V \longrightarrow \mathbb{C}^{N_k} \to 0$$

$$0 \to m_k W_k \oplus n_k V_k \oplus \mathbb{C}^{R_k} \to W \longrightarrow \mathbb{C}^{S_k} \to 0$$

$$k \rightsquigarrow (m_k, n_k, M_k, N_k, R_k, S_k)$$

Case II : k – loc. semisimple $\Rightarrow k = \bigcup_n k_n$, where k_n are semisimple.

$\Rightarrow k \simeq \bigoplus sl(\infty), so(\infty), sp(\infty)$, f.dim. simple

$$0 \to \bigoplus (m_j V_{k^j} \oplus n_j W_{k^j}) \oplus \mathrm{Triv} \to V \to \mathrm{Triv} \to 0$$

and similarly for W.

This is the socle filtration of V as a k-module. It has depth at most 2.

Example 1. Consider $V' \subset V$ with $(V')^\perp = 0$.

Then $V' \otimes W \subset V \otimes W$.

$$gl(\infty) \subset gl(\infty)$$
$$sl(\infty) \subset sl(\infty)$$

For instance $V' = \mathrm{span}\{\underset{m_1}{v_1 - v_2}, \underset{m_2}{v_2 - v_3}, \underset{m_3}{v_3 - v_4} \dots\}$

DUAL BASIS: $\underset{m_1^*}{v_1^*}, \underset{m_2^*}{v_1^* + v_2^*}, \underset{m_3^*}{v_1^* + v_2^* + v_3^*}, \dots$

$$V \otimes W = \bigcup_n (\mathrm{span}\{v_1, \dots v_n\}) \otimes (\mathrm{span}\{v_1^*, \dots, v_n^*\}) =$$

$$= \bigcup_n (\mathrm{span}\{m_1, \dots, m_{n-1}, v_1\}) \otimes \mathrm{span}\{m_1^*, \dots, m_{n-1}^*\} \supset$$

$$V' \otimes W = \bigcup_n (\mathrm{span}\{m_1, \dots, m_{n-1}\}) \otimes \mathrm{span}\{m_1^*, \dots, m_{n-1}^*\}$$

Another choice of Levi components yields an exhaustion of $sl(V \otimes W)$:

$$\bigcup_n \left(\mathrm{span}\{v_1, \dots v_{n-1}\} \otimes \mathrm{span}\{v_1^*, \dots, v_{n-1}^*\} \right)$$

Note that the sequence of $V' \otimes W$-modules

$$0 \to V' \longrightarrow V \longrightarrow V/V' \to 0$$

does not split (both as a sequence of $V' \otimes W$-modules or $sl(V' \otimes W)$-modules).

This is the socle filtration of V as an $V' \otimes W$-module.

On the other hand, W is a simple $V \otimes W$-module and a simple $V' \otimes W$-module!

3 MAXIMAL SUBALGEBRAS

A. $sl(V \otimes W)$

1. $so(V)$, $sp(V)$
2. Stab V', $\mathrm{codim}\, V' = 1$, $(V')^\perp = 0$
   Stab W', $\mathrm{codim}\, W' = 1$, $(W')^\perp = 0$
3. Stab V'', $V'' \subset V$, $(V'')^{\perp\perp} = V''$

B. $o(V)$, $sp(V)$

1. $\operatorname{Stab} V'$, $V' \oplus (V')^{\perp} = V$ non-deg.

$\| $

$o(V') \oplus o(V'^{\perp})$

$sp(V') \oplus sp(V'^{\perp})$ $\dim V' \neq 2$, $\dim (V')^{\perp} \neq 2$ for $so(V)$

2. V' non-deg. $(V')^{\perp} = 0$, $\operatorname{codim} V' = 1$

$\operatorname{Stab} V'$

$\|$

$o(V')$, $sp(V')$

3. V' is isotropic and $(V')^{\perp\perp} = V'$

Furthermore $N_{\mathfrak{g}}(\mathfrak{p}) = \mathrm{Stab}_{\mathcal{F}} \cap \mathrm{Stab}_{\mathcal{G}}$, so not all parabolic subalgebras are self-normalizing.

For $\mathfrak{g} = \mathfrak{o}(\infty), \mathfrak{sp}(\infty)$ taut couples are replaced by self-taut generalized flags, see [D.-C., P.].

LECTURE II: CARTAN, BOREL AND PARABOLIC SUBALGEBRAS OF $\mathfrak{sl}(\infty)$, $\mathfrak{so}(\infty)$, $\mathfrak{sp}(\infty)$

I. Penkov i.penkov@jacobs-university.de

DEF 1. A splitting Cartan subalgebra of $\mathfrak{g}$ is a subalgebra $\mathfrak{h} = \bigcup \mathfrak{h}_n$ for some exhaustion

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots \mathfrak{g}_n \subset \dots$$

of $\mathfrak{g}$ via root injections of simple Lie algebras (here $\mathfrak{h}_n \subset \mathfrak{g}_n$ are the corresponding Cartan subalgebras)

Up to conjugacy via "$G(V,W)$" $\mathfrak{sl}(\infty)$ and $\mathfrak{sp}(\infty)$ have one class of splitting Cartan subalgebras, while $\mathfrak{o}(\infty)$ has two.

A splitting Cartan subalgebra yields a root decomposition of $\mathfrak{g}$: $A_\infty = \{\varepsilon_i - \varepsilon_j \mid i,j \in \mathbb{Z}_{>0}\}$,

$C_\infty = \{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i,j \in \mathbb{Z}_{>0}\}$

$B_\infty = \{\pm \varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i,j \in \mathbb{Z}_{>0}\}$

$D_\infty = \{\pm\varepsilon_i \pm \varepsilon_j \mid i,j \in \mathbb{Z}_{>0}\}$.

GENERAL CARTAN SUBALGEBRAS.

TFAE for a subalgebra $\mathfrak{h} \subset \mathfrak{g} \simeq sl(\infty), o(\infty), sp(\infty)$

- $\mathfrak{h}$ is locally nilpotent and $\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{h}_{ss})$
- $\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{h}_{ss})$
- $\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{t})$, $\mathfrak{t}$ maximal toral
- $\mathfrak{h} = \overline{\mathfrak{g}^0(\mathfrak{h})} := \bigcap \mathfrak{g}^0(\mathfrak{h}_i)$ for $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$ for a fixed exhaustion $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots$ $\mathfrak{g}$.

(JA 308(2007), 583-611).

Theorem. Every $\mathfrak{h}$ as above is nilpotent of depth at most 2 (abelian if $\mathfrak{g} \not\simeq o(\infty)$) and admits a decomposition $\mathfrak{h} = \mathfrak{n}_{\mathfrak{h}} \oplus \mathfrak{t}$, $\mathfrak{n}_{\mathfrak{h}}$ being an ideal consisting of nilpotent elements.

Example: $V = \text{span}\{v_i\}$, $W = \text{span}\{v_i^*\}$

Consider the maximal dual system:

$v_3 + v_2,\ v_4 + v_2, \dots$

$v_3^* + v_1^*,\ v_4^* + v_1^*, \dots$

This is not a dual basis!

Define $\mathfrak{t} = sl(\infty) \cap \operatorname{span}\{(v_3+v_2)\otimes(v_3^*+v_1^*), \dots\}$

Then $Z_{sl(\infty)}(\mathfrak{t}) = (\mathbb{C}\cdot v_1 \otimes v_2^*) \oplus \mathfrak{t}$ and

$\mathfrak{t}$ is maximal toral

Any Cartan subalgebra yielding a root decomposition is splitting.

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A generalized flag $\mathcal{F}$ in V is a chain of subspaces ordered by any totally ordered set with the properties

- every space $F \in \mathcal{F}$ has an immediate successor or an immediate predecessor
- for every $v \in V$ there is pair $F' \subset F''$, $F', F'' \in \mathcal{F}$ such that $v \in F'' \setminus F'$.

Notation: $\mathcal{F} = \{F'_\alpha, F''_\alpha\}$, where α is the totally ordered set of pairs in $\mathcal{F}$.

Examples

- usual flags of three types

- let v_α be a basis of V ordered by any totally ordered set; put $F'_\alpha = \mathrm{span}\{v_\beta, \beta < \alpha\}$, $F''_\alpha = \mathrm{span}\{v_\beta, \beta \leq \alpha\}$. Then $\mathcal{F} = \{F'_\alpha, F''_\alpha\}$ is a maximal generalized flag.

The latter example shows any maximal generalized flag in a countable-dimensional space V.

Recall that we have $W \times V \to \mathbb{C}$. A maximal generalized flag $\mathcal{F}$ in V is splitting if it arises from the above construction for a W-preferred basis $\{v_\alpha\}$ of V (i.e such that $\{v_\alpha^*\}$ is a basis of W).

Def : A subalgebra $\mathfrak{b} \subset \mathfrak{g}$ is a splitting Borel subalgebra if there exists a root exhaustion $\dots \mathfrak{g}_i \subset \mathfrak{g}_{i+1} \subset \dots$ such that $\mathfrak{b} \cap \mathfrak{g}_i$ is a Borel subalgebra of $\mathfrak{g}_i$.

<u>Theorem</u>. Any splitting Borel subalgebra of $sl(\infty)$ is the stabilizer of a unique maximal splitting generalized flag in V. This correspondence is 1-1.

For $o(\infty)$, $sp(\infty)$ one has to consider isotropic generalized flags.

Example (important): Let $\{v_i\}$ be a W-preferred basis of V. Fix the following order

$$1 < 3 < 5 \dots \quad \dots 6 < 4 < < 2$$

This gives a Borel subalgebra $\mathfrak{b}$ with positive roots $\{\varepsilon_i - \varepsilon_j \mid i < j \text{ with resp. to the new order}\}$

The simple roots are $\varepsilon_1 - \varepsilon_3, \varepsilon_3 - \varepsilon_5, \dots \quad , \varepsilon_4 - \varepsilon_2$.

The root $\varepsilon_1 - \varepsilon_2$ is positive and is not a finite sum of simple roots.

<u>Def.</u> A Borel subalgebra of $\mathfrak{g}$ is a maximal locally solvable subalgebra.

Example. Let $V' \subset V$ be as in Lecture I. Then the upper triangular Borel subalgebra of $sl(W, V')$ is a splitting Borel subalgebra of $sl(W, V')$ but also a Borel subalgebra of $sl(W, V)$.

Def. A generalized flag $\mathcal{F} = \{F'_\varkappa, F''_\varkappa\}$ in V is <u>semiclosed</u> if

$$(F'_\varkappa)^{\perp\perp} = \begin{cases} F'_\varkappa \\ F''_\varkappa \end{cases}$$

for all $\varkappa$.

Theorem (Dimitrov, Dan-Cohen, P) Any Borel subalgebra of $sl(\infty)$ is the stabilizer of a unique maximal semiclosed generalized flag in V. The correspondence is 1-1.

(Any max. splitting $\mathcal{F}$ is maximal semiclosed!)

In the above example

$\mathcal{F} = \{V_1 \subset V_2 \subset \dots V' \subset V\}$.

Def. A parabolic subalgebra of $\mathfrak{g}$ is a subalgebra $\mathfrak{p}$ containing a Borel subalgebra $\mathfrak{b}$.

The stabilizers of (non-maximal) semiclosed generalized flags in V are clearly parabolic. But this is not all.

Def. Let $\mathcal{F}$ be a semiclosed generalized flag in V and $\mathcal{G}$ be a semiclosed generalized flag in W. $(\mathcal{F}, \mathcal{G})$ is a taut couple if $\mathcal{F}^{\perp}$ is stable under $St_{\mathcal{G}}$ and $\mathcal{G}^{\perp}$ is stable under $St_{\mathcal{F}}$.

E. Dan-Cohen

Theorem (D.-C., P) Parabolic subalgebras are precisely the subalgebras of the form

$$\mathfrak{n}_{\mathfrak{q}} + [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{p} \subset \mathfrak{q} = Stab_{\mathcal{F}} \cap Stab_{\mathcal{G}}.$$

$\mathfrak{p}$ determines the taut couple $\mathcal{F}, \mathcal{G}$ uniquely.

Lecture III: Subcategories of $\mathrm{Int}_{\mathfrak{g}}$

I. Penkov Jacobs University Bremen i.penkov@jacobs-university.de

$\mathfrak{g} = sl(\infty), o(\infty), sp(\infty)$

A $\mathfrak{g}$-module M is *integrable* if $\forall m \in M$ $\forall g \in \mathfrak{g}$ $\dim \mathrm{span}\{m, g\cdot m, g^2 \cdot m, \dots\} < \infty$.

We will restrict ourselves to discussing integrable modules. Some more general results see in DP, IMRN 99, P. Zuck: Transf. groups 2008

The category $\mathrm{Int}_{\mathfrak{g}}$ is vast (and "wild").

<u>Subcategories</u>: $\mathrm{Int}^{wt}_{\mathfrak{g},\mathfrak{h}}$: $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M^\lambda$

$\mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$: $\dim M^\lambda < \infty \ \forall \lambda$

$\mathrm{Int}^{\mathfrak{b}}_{\mathfrak{g}}$ – simple objects have a $\mathfrak{b}$-singular vector

$\overset{?}{\cap}$

$\widetilde{\mathrm{Tens}}_{\mathfrak{g}} \supset \mathrm{Tens}_{\mathfrak{g}}$

General remark: $\mathrm{Int}_{\mathfrak{g}}$ is not semisimple.

Two examples:

1) $V \otimes W$ as $sl(V, W)$-module

$$0 \to sl(V,W) \to V \otimes W \to \mathbb{C} \to 0$$

This sequence does not split $\Rightarrow \mathrm{Int}^{wt}_{\mathfrak{g},\mathfrak{h}}$ is not semisimple.

2) V as $sl(V', W)$-module (Here $(V')^{\perp} = 0$).

Exact sequence

$$0 \to V' \longrightarrow V \longrightarrow \underbrace{V/V'}_{T} \to 0$$

Does not split and V is not in $\mathrm{Int}^{wt}_{sl(V,W),\mathfrak{h}}$!

I. Highest weight modules

$$\mathfrak{g} = \mathfrak{n}^- \oplus \underbrace{\mathfrak{h} \oplus \mathfrak{n}^+}_{\mathfrak{b}} \quad \text{— for any splitting } \mathfrak{b}$$

$$\widetilde{V_{\mathfrak{b}}(\lambda)} = U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda$$

$V_{\mathfrak{b}}(\lambda)$ = unique simple quotient of $\widetilde{V_{\mathfrak{b}}(\lambda)}$.

$V_{\mathfrak{b}}(\lambda)$ is well-defined!

$V_{\mathfrak{b}}(\lambda)$ has one 1-dimensional weight space λ.

The others may or may not be infinite-dimensional.

Is the adjoint representation a highest weight module? Yes, but not with respect to a Borel subalgebra with "enough simple roots". In fact any $\mathfrak{b}$ for which $\varepsilon_1 - \varepsilon_2$ is the "longest" root works. Then the highest weight is $\varepsilon_1 - \varepsilon_2$.

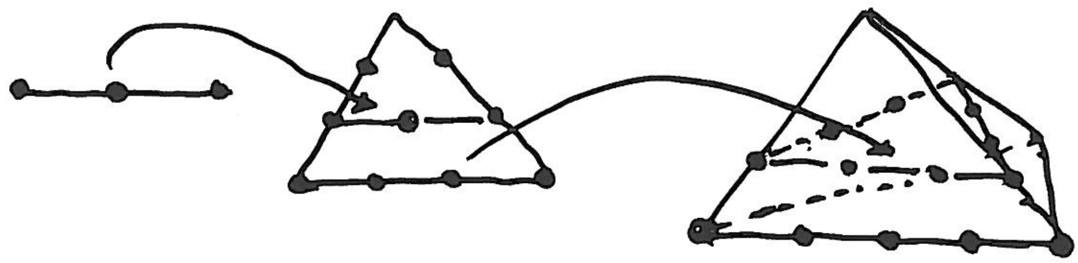
Note that $\mathfrak{g}$ has an infinite-dimensional weight space.

Character formulas for $V_{\mathfrak{b}}(\lambda)$ exist if $\mathfrak{b}$ has enough simple roots, IMRN 99

$\underline{\text{II}}$. $\mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$. Fix $\mathfrak{h}$ (diagonal)

Objects of $\mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$: $V_{\mathfrak{b}}(\lambda)$ if $\mathfrak{b}$ has enough simple roots.

Example (module with no highest weight at all)



$$S^2(V_2) \subset S^3(V_3) \subset S^4(V_4) \subset \dots$$

Dimitrov module: it is a simple object of $\mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{z}}$, has 1-dimensional weight spaces (the weights are $\sum_i m_i \varepsilon_i$, $m_i \in \mathbb{Z}_{\geq 0}$, $m_i = 1$ for $i \gg 0$), and has no highest weight.

<u>Important fact</u>: $\mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) = 0$ for any $X, M \in \mathrm{Int}_{\mathfrak{g}}$.

Proof.
$\mathrm{Ext}^1_{\mathfrak{g}}(X, M^*) \simeq H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{C}}(X, M^*)) =$
$H^1(\mathfrak{g}, (X \otimes M)^*)$. Moreover the standard complex computing $H^*(\mathfrak{g}, (X \otimes M)^*)$ is dual to the standard complex computing the homology $H_*(\mathfrak{g}, X \otimes M)$. However, $H_1(\mathfrak{g}, X \otimes M) = 0$ being the direct limit of $H_1(\mathfrak{g}_i, X \otimes M)$. Hence

$$H^1(\mathfrak{g}, (X \otimes M)^*) = 0.$$

Define now $\Gamma_{\mathfrak{g}} : \mathfrak{g}\text{-mod} \rightsquigarrow \mathrm{Int}_{\mathfrak{g}}$

$\Gamma_{\mathfrak{g}}(Z) = \{ z \in Z \mid \dim \mathrm{span}\{z, g\cdot z, g^2\cdot z \dots\} < \infty \ \forall g \in \mathfrak{g} \}$.

$\Gamma_{\mathfrak{g}}$ is the functor of $\mathfrak{g}$-integrable vectors.

<u>Corollary</u> $\Gamma_{\mathfrak{g}}(M^*)$ is an injective object of $\mathrm{Int}_{\mathfrak{g}}$ for any $M \in \mathrm{Int}_{\mathfrak{g}}$.

<u>Example</u>. Let $M = V$ – finitary column vectors for $\mathfrak{g} = sl(\infty)$. Then $V^* =$ space of all row vectors. However $\mathfrak{g}\cdot V^* =$ space of finitary rows V_*.

Hence

$$0 \to V_* \to V^* \to T \to 0$$

trivial module of dim $2^{\mathbb{Z}}$.

Hence $\mathrm{Ext}^1_{\mathfrak{g}}(V_*, X) = 0$ if X is non-trivial

(V^* is the injective hull of V_*,

$(V_*)^*$ is the injective hull of V).

$$\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}} : \mathfrak{g}\text{-mod} \rightsquigarrow \mathrm{Int}^{wt}_{\mathfrak{g},\mathfrak{h}}$$

$$\| $$

$$\mathrm{Wt}\circ\Gamma_{\mathfrak{g}} \ , \text{ where } \mathrm{Wt}(M) = \bigoplus_{\lambda\in\mathfrak{h}^*} M^\lambda .$$

Moreover, if X is injective in $\mathrm{Int}_{\mathfrak{g}}$, then $\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(X)$ is injective in $\mathrm{Int}^{wt}_{\mathfrak{g},\mathfrak{h}}$.

<u>Example</u> $\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(V^*) = V_*$. Hence V_* (and V) are injective in $\mathrm{Int}^{wt}_{\mathfrak{g},\mathfrak{h}}$.

Let now $M \in \mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$. Then it is not difficult to check that $\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(\Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}(M^*)^*) \simeq M$. Hence M is injective, and $\mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$ is semisimple. (analog of Weyl's semisimplicity theorem).

III. Tensor modules.

A *simple tensor module* is a subquotient (submodule is enough) of $T^{p,q} := \underbrace{V \otimes \dots \otimes V}_{p} \otimes \underbrace{V_* \otimes \dots \otimes V_*}_{q}$.

These are parametrized as V_λ, where

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k, 0 \dots 0, -\mu_\ell, -\mu_{\ell-1}, \dots -\mu_1)$$

These weights are precisely the highest weights of V_λ with resp. to the Borel subalgebra $\mathfrak{b}$ introduced above. It corresponded to the order $(1 < 3 < 5 \dots \quad \dots 6 < 4 < 2)$.

$V_\lambda \in \mathrm{Int}^{fin}_{\mathfrak{g},\mathfrak{h}}$ if $\mu_\ell = 0$ for all ℓ or $\lambda_k = 0$ for all k.

The adjoint representation corresponds to the weight $(1, 0 \dots 0, -1)$.

In [PStyr], we have computed the

socle filtration of $T^{p,q}$. We have also computed the socle filtrations of the indecomposable direct summands of $T^{p,q}$. These indecomposable direct summand have simple socles.

Example.

—

$$V \otimes V = S^2(V) \oplus \Lambda^2(V)$$

$$V \otimes V_* = \begin{array}{c} \mathbb{C} \\ \hline \mathfrak{g} = V_{1\underline{0}-1} \end{array}$$

—

$$V \otimes V \otimes V = S^3(V) \oplus 2\, V_{21\underline{0}} \oplus \Lambda^3(V)$$

$$V \otimes V \otimes V_* = \begin{array}{c} V_{1\underline{0}} = V \\ \hline V_{2\underline{0}-1} \end{array} \oplus \begin{array}{c} V_{1\underline{0}} = V \\ \hline V_{11\underline{0}-1} \end{array}$$

—

$$V \otimes V \otimes V^* \otimes V^* =$$

$$\begin{array}{c} \mathbb{C} \\ \hline V_{1\underline{0}-1} = \mathfrak{g} \\ \hline V_{2\underline{0}-2} \end{array} \oplus \begin{array}{c} V_{1\underline{0}-1} = \mathfrak{g} \\ \hline V_{2\underline{0}-1-1} \end{array} \oplus \begin{array}{c} V_{1\underline{0}-1} = \mathfrak{g} \\ \hline V_{11\underline{0}-2} \end{array} \oplus \begin{array}{c} \mathbb{C} \\ \hline V_{1\underline{0}-1} = \mathfrak{g} \\ \hline V_{11\underline{0}-1-1} \end{array}$$

—

In [PSer] we define the category $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ as the full subcategory of $\mathrm{Int}_{\mathfrak{g}}$ such that

- any M has finite Loewy length
- $M^* \in \mathrm{Int}_{\mathfrak{g}}$ and also has finite Loewy length.

<u>Theorem</u>. The simple objects of $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ are precisely the simple tensor modules.

Example

socle filtration

$$0 \subset Sc \oplus \underset{\mathfrak{g}}{\underset{\shortparallel}{F}} \subset F_2 + F_c + J \subset (V \otimes V_*)^*$$

$$2^{\mathbb{Z}} V_* \oplus 2^{\mathbb{Z}} V \oplus 2^{\mathbb{Z}} \mathbb{C} \qquad\qquad 2^{\mathbb{Z}} \mathbb{C}$$

$\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ has enough injectives:

$V_\lambda \longrightarrow I_\lambda$

Properties : I_λ are bricks

: any module in $\widetilde{\mathrm{Tens}}_{\mathfrak{g}}$ has finitely many non-isomorphic simple subquotients

$$-\ \mathrm{Ext}^1_{\mathfrak{g}}(V_\lambda, V_\mu) = \begin{cases} 0 & \\ \mathbb{C}^{2^{\#}} & \text{if } \mathrm{Hom}(V_\lambda^i, V_\mu^j) \text{ for } j > i \gg 0 \end{cases}$$

***.

<u>Conj.</u> Define $\Gamma^{wt} = \bigcap_{\mathfrak{h}} \Gamma^{wt}_{\mathfrak{g},\mathfrak{h}}$. Then $\Gamma^{wt} \circ \Gamma_{\mathfrak{g}}(\widetilde{\mathrm{Tens}}_{\mathfrak{g}})$ is a category in which each indecomposable has finite length and the injective hull of V_λ is the indecomposable direct summand of $T^{\cdot}$ with socle V_λ.