Geometry and Physics of Dirac Operators

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– Second lecture –

Content

1) Connections induced by Dirac (type) operators
2) The universal Dirac-Lagrangian and the Einstein-Hilbert Action
1 Connections induced by Dirac (type) operators

Let \((E, \gamma_E) \rightarrow (M, g_M)\) (1) be a bundle of (complex) Clifford modules.

Every (even) connection on \(E \rightarrow M\) yields a Dirac operator:

\[
\nabla^E : \text{Sec}(M, E) \xrightarrow{\nabla^E} \text{Sec}(M, T^*M \otimes M E) \xrightarrow{\gamma_E} \text{Sec}(M, E).
\]

(2)

**Definition 1.1** A connection on a Clifford module bundle is called a “Clifford connection”, provided it fulfils:

\[
\left[\nabla^E_X, \gamma_E(a)\right] = \gamma_E(\nabla^C_X(a)),
\]

for all \(a \in \text{Sec}(M, Cl_M)\) and \(X \in \text{Sec}(M, TM)\).

The set of all Clifford connections on a Clifford module bundle \((E, \gamma_E) \rightarrow (M, g_M)\) is denoted by \(\mathcal{A}_{Cl}(E)\). It is an affine sub-space of the affine space \(\mathcal{A}(E)\) of all (linear) connections on \(E \rightarrow M\).

Note that the underlying vector space of \(\mathcal{A}_{Cl}(E)\) is given by \(\Omega^1(M, \text{End}^+_+(E))\).

**Definition 1.2** The Dirac operator of a Clifford connection is called “a Dirac operator of Clifford type”.

**Notation:**
We denote a Clifford connection by \(\partial_A \in \mathcal{A}_{Cl}(E)\), for it locally reads:

\[
\partial_A \text{loc.} = d + \omega + A.
\]

(4)

Let \(U \subset M\) be a local subset and \(e_1, \ldots, e_n \in \text{Sec}(U, TM)\) be a locally defined (orthonormal) frame. Also, let \(e^1, \ldots, e^n \in \text{Sec}(U, T^*M)\) be the corresponding dual frame. The locally defined one-form \(\omega \in \Omega^1(U, \text{End}^+_+(E))\) is the “spin-connection form”:

\[
\omega = -\frac{i}{8} g_m(\nabla_k e_a, e_b) e^k \otimes [\gamma_E(e^a), \gamma_E(e^b)]
\]

and \(A \in \Omega^1(U, \text{End}^+_+(E))\) is a local “gauge potential”.

If \(E = S \otimes_M E \rightarrow M\) is a twisted spinor bundle, than a Clifford connection is but a **twisted spin connection**:

\[
\partial_A = \nabla^S \otimes \nabla^E
= \nabla^S \otimes \text{id}_E + \text{id}_S \otimes \nabla^E.
\]

(6)
Whence,
\[ A_{\text{Cl}}(E) \simeq A(E). \] (7)

In the case of twisted Grassmann bundles: \( E = \Lambda_M \otimes_M E \to M \), the Clifford connections are locally parameterized by local gauge potentials:
\[ A \in \Omega^1 \left(U, (Cl_{M}^{\text{op}})^C \otimes_M \text{End}(E)\right)^+. \] (8)

**Definition 1.3** Let \( (E, \gamma_E) \to (M, g_M) \) be a Clifford module bundle. The one-form \( \Theta \in \Omega^1(M, \text{End}^-(E)) \), which is defined by
\[ \Theta(v) := \frac{i}{n} \gamma_E(v^b) \] (9)
for all \( v \in TM \), is called the “canonical one-form” on the Clifford module.

Let \( U \subset M \) be an open subset and \( e_1, \ldots, e_n \in \mathcal{S}ec(U, TM) \) be a locally defined (orthonormal) frame with the dual frame \( e^1, \ldots, e^n \in \mathcal{S}ec(U, T^*M) \).
\[ \Theta \underset{\text{loc}}{=} \frac{i}{n} g_M(e_a, e_b) e^a \otimes \gamma_E(e^b) \equiv \frac{i}{n} e^a \otimes \gamma_E(e^b_a) \]. (10)

**Lemma 1.1** A connection on a Clifford module bundle is a Clifford connection if and only if the induced connection on \( T^*M \otimes_M \text{End}(E) \to M \) fulfills:
\[ \nabla T^*M \otimes \text{End}(E) \Theta = 0. \] (11)

**Proof:** Nice exercise!

**Definition 1.4** Let \( (E, \gamma_E) \to (M, g_M) \) be a Clifford bi-module bundle. A connection is called “\( S \)-reducible”, provided its induced connection on \( T^*M \otimes_M \text{End}(E) \to M \) fulfills:
\[ \nabla T^*M \otimes \text{End}(E) \Theta^{\text{op}} = 0. \] (12)

Here, \( \Theta^{\text{op}}(v) := \frac{i}{n} \gamma_E^{\text{op}}(v^b) \), for all \( v \in TM \) and \( \gamma_E^{\text{op}} : Cl_M^{\text{op}} \to \text{End}(E) \) is the representation of the algebra bundle of opposite Clifford algebras.
A connection on a twisted Grassmann bundle is $S$-reducible if and only if it is locally parameterized by a gauge potential

$$A \in \Omega^1(U, \text{End}(E)).$$

Furthermore, a connection on the Grassmann bundle over a spin-manifold is $S$-reducible if and only if it coincides with the spin-connection.

**Definition 1.5** On a Clifford module bundle the (linear extension of the) map:

$$\delta_\gamma : \Omega^0(M, \text{End}(E)) \to \Omega^0(M, \text{End}(E))$$

$$\omega = \alpha \otimes B \mapsto \gamma_\epsilon(\sigma^{-1}_\text{ch}(\omega)) \circ B$$

is called the "quantization map".

The restriction of the quantization map to $\Omega^1(M, \text{End}(E))$ has a canonical right-inverse that is given by the odd map:

$$\text{ext}_\Theta : \Omega^0(M, \text{End}^\pm(E)) \to \Omega^1(M, \text{End}^\mp(E))$$

$$\Phi \mapsto \Theta \wedge \Phi \equiv \Theta \Phi,$$

with $(\Theta \wedge \Phi)(v) := \Theta(v) \circ \Phi$, for all $v \in TM$.

Whence,

$$\varphi := \text{ext}_\Theta \circ \delta_\gamma : \Omega^1(M, \text{End}(E)) \to \Omega^1(M, \text{End}(E))$$

is an idempotent. Its complement $\varphi' := \text{id}_{\Omega^1} - \varphi$ sends $\mathcal{A}(E)$ into the set of "twistor operators" on the underlying Clifford module bundle:

$$\nabla^E \mapsto T(\nabla^E) := \nabla^E - \Theta \circ \nabla^E.$$

**Definition 1.6** Two connections on a Clifford module bundle are said to be equivalent if they yield the same Dirac operator:

$$\nabla^E \sim \nabla'^E :\iff \nabla'^E = \nabla^E.$$

Clearly,

$$\nabla^E \sim \nabla'^E \iff \nabla'^E - \nabla^E \in \text{Ker}(\varphi).$$
Proposition 1.1 Let $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$ be a Dirac operator on $(\mathcal{E}, \gamma_\mathcal{E}) \rightarrow (M, g_M)$. The equivalence class of connections on $\mathcal{E} \rightarrow M$ that is defined by $\mathcal{D}$ has a natural representative.

Proof: Every Dirac operator $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$ on a Clifford module bundle yields a unique connection, called the “Bochner connection” of $\mathcal{D}$:

$$2 \text{ ev}_\psi(df, \partial_b \psi) := \epsilon \left( [\mathcal{D}^2, f] - \delta_g df \right) \psi,$$  \hspace{1cm} (20)

for all $f \in C^\infty(M)$ and $\psi \in \text{Sec}(M, \mathcal{E})$.

This yields the first order decomposition of $\mathcal{D}$:

$$\mathcal{D} = \partial_b + \Phi_D,$$ \hspace{1cm} (21)

with $\Phi_D \in \text{Sec}(M, \text{End}^-(\mathcal{E}))$ being uniquely defined by $\mathcal{D}$.

The connection that corresponds to $\partial_b := \partial_b + \text{ext} \Theta \wedge \Phi_D$ (22)

is thus uniquely defined by $\mathcal{D}$. Furthermore,

$$\partial_b = \mathcal{D}.$$ \hspace{1cm} (23)

$\square$

Definition 1.7 For given $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$, the even one-form:

$$\omega_D := \text{ext} \Theta \wedge \Phi_D,$$ \hspace{1cm} (24)

is called the “Dirac form” of $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$.

The tangent vector field on $M$:

$$\xi_D := \text{tr}_\mathcal{E}(\omega_D),$$ \hspace{1cm} (25)

is called the “Dirac field” of $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$.

The connection on the underlying Clifford module bundle that corresponds to $\partial_b$ is called the “Dirac connection” of $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$. Its curvature

$$\text{curv}(\mathcal{D}) := \partial_b \wedge \partial_b \in \Omega^2(M, \text{End}^+(\mathcal{E}))$$ \hspace{1cm} (26)

is called the “Dirac curvature” of $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$.

Finally,

$$F_D := \text{curv}(\mathcal{D}) - \text{Riem}(g_M) \in \Omega^2(M, \text{End}^+(\mathcal{E}))$$ \hspace{1cm} (27)

is called the “relative curvature” of $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$. 

Lemma 1.2 Let $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$ be a Dirac operator. Its induced equivalence class of connections on the underlying Clifford module bundle contains at most one Clifford connection. This is the case if and only if

$$\partial^\Gamma_M \otimes \text{End}(\mathcal{E}) \Theta \equiv 0. \quad (28)$$

Proof: First, let the Dirac connection of $\mathcal{D}$ be a Clifford connection. Any other connection $\nabla^\gamma$ whose quantization equals $\mathcal{D}$ thus reads:

$$\nabla^\gamma = \partial + \alpha, \quad \alpha \in \text{Ker}(\varphi). \quad (29)$$

In particular, if $\nabla^\gamma = \partial_\lambda$ is also a Clifford connection, than $\alpha \in \Omega^1(M, \text{End}_+^\gamma(\mathcal{E}))$. The map $\text{ext}_\Theta$ is injective. Hence, $\text{Ker}(\varphi) = \text{Ker}(\delta_\gamma|_{\Omega^1})$. However, $\alpha \notin \text{Ker}(\delta_\gamma|_{\Omega^1})$ since the restriction of the quantization map to $\Omega^*(M, \text{End}_\gamma(\mathcal{E}))$ is an isomorphism.

Now, let $\partial_\lambda \sim \partial_\lambda$. Since $\partial_\lambda = \mathcal{D}$, it follows that the Bochner connection of $\mathcal{D}$ equals the Clifford connection: $\partial_\lambda = \partial_\lambda$. Therefore, $\partial_\lambda = \partial_\lambda$.

Whence, if the connection class of $\mathcal{D}$ contains a Clifford connection, it must be unique and equal to the Dirac connection of $\mathcal{D}$. Only in this case, one gets:

$$\partial_\lambda = \partial_\lambda = \partial_\lambda. \quad (30)$$

Remark:
If a Dirac operator $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$ is of Clifford type: $\mathcal{D} = \partial_\lambda$, than its curvature reads:

$$\text{curv}(\partial_\lambda) = \mathcal{R}\text{iem}(g_{\mathcal{M}}) + F_\lambda, \quad (31)$$

whereby the relative curvature $F_\lambda$ of $\partial_\lambda$ fulfills:

$$F_\lambda \in \Omega^2(M, \text{End}_\gamma(\mathcal{E})). \quad (32)$$

In the case of a twisted spinor bundle $\mathcal{E} = S \otimes_M E \rightarrow M$, the relative curvature of a Clifford type Dirac operator is given by

$$F_\lambda = \nabla^\kappa \wedge \nabla^\kappa \in \Omega^2(M, \text{End}^+(E)). \quad (33)$$

In terms of Yang-Mills gauge theories, the relative curvature of a Clifford type Dirac operator thus plays the role of the Yang-Mills curvature.

Definition 1.8 A Dirac operator $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$ on a Clifford bi-module bundle is called “S-reducible”, if its Dirac connection is S-reducible.
On a twisted Grassmann bundle over a spin manifold, a Dirac operator is S-reducible if and only if it coincides with a twisted spin-Dirac operator.

**Proposition 1.2** Two Dirac operators $\mathcal{D}'$, $\mathcal{D} \in \mathcal{D}_2(\mathcal{E})$ on a given Clifford module bundle yield the same Bochner connection if and only if

$$\{ (\mathcal{D}' - \mathcal{D}), \gamma_\mathcal{E}(\alpha) \} \equiv 0,$$

(34)

for all $\alpha \in T^*M$.

**Proof:** Making use of the definition of the Bochner connection of a Dirac operator, the proof follows from showing that

$$\partial_b' = \partial_b + \alpha_b,$$

(35)

with the one-form $\alpha_b \in \Omega^1(M, \text{End}^+(\mathcal{E}))$ being defined by

$$\alpha_b(v) = \frac{1}{2} \{ (\mathcal{D}' - \mathcal{D}), \gamma_\mathcal{E}(v^\flat) \},$$

(36)

for all $v \in TM$.

**Definition 1.9** A Dirac operator $\mathcal{D} \in \mathcal{D}_2(\mathcal{E})$ is called of “simple type” if its Bochner connection equals a Clifford connection.

**Proposition 1.3** Let $(\mathcal{E}, \gamma_\mathcal{E}) \rightarrow (M, g_M)$ be a Clifford module bundle. A Dirac operator $\mathcal{D} \in \mathcal{D}_2(\mathcal{E})$ is of simple type if and only if $\mathcal{D}' - \pi_\mathcal{E}$ anti-commutes with the Clifford action $\gamma_\mathcal{E}$.

**Proof:** First, let the Bochner connection of $\mathcal{D}$ be a Clifford connection: $\partial_b = \partial_b^\mathcal{E}$. Since the Bochner connection of Clifford type Dirac operator $\pi_\mathcal{E}$ equals $\partial_b^\mathcal{E}$, it follows that $\mathcal{D}$ and $\pi_\mathcal{E}$ yield the same Bochner connection (namely $\partial_b$). Whence, according to the foregoing Proposition it follows that $\mathcal{D}' - \pi_\mathcal{E}$ anti-commute with the Clifford action.

Next, assume that the zero-order operator $\Phi_b = \mathcal{D} - \pi_\mathcal{E}$ anti-commute with the Clifford action. Hence, there is a unique zero-order operator $\phi_b \in \text{Sec}(M, \text{End}^-(\mathcal{E}))$, such that

$$\Phi_b = \tau_\mathcal{E} \circ \phi_b.$$  

(37)

Furthermore, $\mathcal{D}$ and $\pi_\mathcal{E}$ have the same Bochner connection due to the foregoing Proposition. Hence, the Bochner connection of $\pi_\mathcal{E}$ coincides with $\partial_b$, which holds true if and only if $\partial_b$ is a Clifford connection. \hfill \Box

**Corollary 1.1** Let $(\mathcal{E}, \gamma_\mathcal{E}) \rightarrow (M, g_M)$ be a Clifford module bundle. A Dirac operator $\mathcal{D} \in \mathcal{D}_2(\mathcal{E})$ is of simple type if and only if there is Clifford connection and a $\phi \in \text{Sec}(M, \text{End}^-(\mathcal{E}))$, such that

$$\mathcal{D} = \pi_\mathcal{E} + \tau_\mathcal{E} \circ \phi.$$  

(38)
The set of simple type Dirac operators is the largest class of Dirac operators whose Bochner connections are Clifford connections. Simple type Dirac operators thus build a natural generalization of Clifford type Dirac operators.

**Definition 1.10** A Dirac operator $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$ on a Clifford module bundle is called of “Yang-Mills-Higgs type”, if there is a Clifford connection such that

$$\mathcal{D} - \tilde{\partial}_\lambda \in \text{Sec}(M, \text{End}_\gamma(\mathcal{E})).$$

(39)

Since the Clifford connection is unique, there is a unique

$$\Phi_\mathcal{H} \in \text{Sec}(M, \text{End}_\gamma(\mathcal{E})), \quad (40)$$

such that

$$\mathcal{D} = \tilde{\partial}_\lambda + \Phi_\mathcal{H}. \quad (41)$$

It follows that the Dirac connection of a Yang-Mills-Higgs type Dirac operator reads:

$$\partial_\mathcal{D} \equiv \partial_{\text{YM-H}} = \partial_\lambda + H, \quad (42)$$

with

$$H := \Phi_\mathcal{H} \Theta \in \Omega^1(M, \text{End}^+(\mathcal{E})) \quad (43)$$

being the “Higgs gauge potential”.

The relative curvature of a Yang-Mills-Higgs type Dirac operator simply reads:

$$F_\mathcal{D} = F_\lambda + d_\lambda H + H \wedge H = F_\lambda + (d_\lambda \Phi_\mathcal{H} + \Phi_\mathcal{H} \wedge \Theta) \wedge \Theta. \quad (44)$$

**Remark:**
Every Dirac operator $\mathcal{D} \in \mathcal{D}_\gamma(\mathcal{E})$ may be decomposed as

$$\mathcal{D} = \tilde{\partial}_\lambda + \Phi. \quad (45)$$
However, this decomposition is not unique, in general, for $\Phi \in \text{Sec}(M, \text{End}^{-}(\mathcal{E}))$ also depends on the choice of $\partial_\lambda$.

Simple type Dirac operators generalize Dirac operators of Clifford type in the sense that
- $\partial_B = \partial_A$;
- $\Phi$ is uniquely determined by $\mathcal{D}$.

In contrast, Yang-Mills-Higgs type Dirac operators $\partial_{\text{YMH}}$ generalize Dirac operators of Clifford type in the sense that the decomposition

$$\partial_{\text{YMH}} = \partial_\lambda + \Phi$$

(46)

is unique, though $\partial_B \neq \partial_\lambda$. 


2 The universal Dirac-Lagrangian and the Einstein-Hilbert Action

Definition 2.1 Let $\mathcal{D} \in \mathcal{D}_{\gamma}(\mathcal{E})$ be an arbitrary Dirac operator on a Clifford module bundle $(\mathcal{E}, \gamma_{\mathcal{E}}) \rightarrow (M, g_{\mathcal{M}})$. The associated second order differential operator:

$$\triangle_{\mathfrak{n}} := \epsilon ev_g (\partial_{\mathfrak{n}}^T \mathcal{M} \otimes \mathcal{E} \circ \partial_{\mathfrak{n}}),$$

(47)
is called the “Bochner (or connection/trace) Laplacian”.

Proposition 2.1 Every Dirac operator $\mathcal{D} \in \mathcal{D}_{\gamma}(\mathcal{E})$ has a unique second order decomposition:

$$\mathcal{D}^2 = \triangle_{\mathfrak{n}} + V_{\mathcal{D}},$$

(48)

with $V_{\mathcal{D}} \in \text{Sec}(M, \text{End}^{+}(\mathcal{E}))$ being uniquely defined by $\mathcal{D}$.

Furthermore, the “Dirac potential” explicitly reads:

$$V_{\mathcal{D}} = \delta_{\gamma}(\text{curv}(\mathcal{D})) + \epsilon ev_g \left( \partial_{\mathfrak{n}} \omega_{\mathcal{D}} - \omega_{\mathfrak{n}}^2 \right).$$

(49)

Basically, the proof follows from the very definition of the Bochner connection of a Dirac operator.

In the case where $\mathcal{D} = \partial_{\mathfrak{n}}$ is of Clifford type, it follows that

$$V_{\mathcal{D}} = \frac{1}{4} \text{scal}(g_{\mathcal{M}}) \text{id}_{\mathcal{E}} + \delta_{\gamma}(F_{\mathfrak{A}})$$

(50)

coincides with the well-known Schrödinger-Lichnerowicz formula of the zero-order operator of the square of a twisted spin-Dirac operator $\nabla^{s \otimes \mathcal{E}}$.

Note that the zero-order operator

$$\delta_{\gamma}(F_{\mathfrak{A}}) \in \text{Sec}(M, \text{End}^{+}(\mathcal{E}))$$

(51)
is always trace-free. This is because $F_{\mathfrak{A}} \in \Omega^2(M, \text{End}^{+}_{\mathfrak{A}}(\mathcal{E}))$.

Definition 2.2 Let $\mathcal{E} = \mathcal{E}^{+} \oplus \mathcal{E}^{-} \rightarrow M$ be a Hermitian vector bundle with the Hermitian product being denoted by $\langle \cdot, \cdot \rangle_{\mathcal{E}}$.

The map:

$$\mathcal{L}_{\mathcal{D}} : \mathcal{D}(\mathcal{E}) \rightarrow \Omega^{n}(M, C)$$

$$\mathcal{D} \mapsto \star tr_{\mathcal{E}} V_{\mathcal{D}},$$

(52)
is called the “universal Dirac-Lagrangian”.
Likewise, the map:

\[ L_{D,\text{tot}} : \mathcal{D}(\mathcal{E}) \times \text{Sec}(M, \mathcal{E}) \to \Omega^n(M, \mathcal{C}) \]

\[ (\varphi, \psi) \mapsto *\langle \psi, D\psi \rangle + \text{tr}_D V_0, \]

is called the “total Dirac-Lagrangian”.

**Proposition 2.2** The universal Dirac-Lagrangian is equivariant with respect to the action of the “affine gauge group”:

\[ \mathcal{G}_D = \mathcal{G}_{D,\text{tot}} \ltimes \mathcal{T}_D, \]

where, respectively,

\[ \mathcal{G}_{D,\text{tot}} := \text{Diff}(M) \ltimes \text{Aut}(\mathcal{E}), \]

\[ \mathcal{T}_D := \Omega^1(M, \text{End}^+_\gamma(\mathcal{E})), \]

is the gauge group of the total Dirac-Lagrangian and the “translation group”.

The proof needs some (home)work! Indeed, it can be shown that the universal Dirac-Lagrangian is actually invariant with respect to the (linear extension of the) map:

\[ \mathcal{D}(\mathcal{E}) \times \mathcal{T}_D \to \mathcal{D}(\mathcal{E}) \]

\[ (\varphi, df) \mapsto \varphi + [\varphi, f]. \]

Note that the gauge group of the total Dirac-Lagrangian is only a (proper) subgroup of the gauge group of the universal Dirac-Lagrangian.

Up to the boundary term \(*\text{div}\xi_D \in \Omega^n(M, \mathcal{C})\), the universal Dirac-Lagrangian explicitly reads:

\[ L_D(\varphi) = *\text{tr}_\gamma(\text{curv}(\varphi) - \epsilon \text{ev}_g(\omega_D^2)), \]

with

\[ \text{tr}_\gamma \equiv \text{tr}_\varepsilon \circ \delta_\gamma : \Omega^*(M, \text{End}(\mathcal{E})) \to \mathcal{C}^\infty(M, \mathcal{C}). \]

being the “quantized trace”.

It follows that when restricted to the subset of Clifford type Dirac operators, the universal Dirac-Lagrangian coincides with the Lagrangian density of General Relativity:

\[ L_D(\varphi) = \frac{\epsilon \text{rank}(\mathcal{E})}{4} * \text{scal}(g_{\text{st}}). \]