The 30th Winter School GOMETRY AND PHYSICS
Czech Republic, Srni, January 16 – 23, 2010

Geometry and Physics of Dirac Operators

J. Tolksdorf
Max Planck Institute for Mathematics in the Sciences, Leipzig/Germany
University of Regensburg, Regensburg/Germany

– First lecture –

Content of the 1st lecture

1) A brief overview and reminder
2) General Clifford (bi-) modules and Dirac operators

Content of the 2nd lecture

1) Connections induced by Dirac (type) operators
2) The universal Dirac-Lagrangian and the Einstein-Hilbert Action

Content of the 3rd lecture

1) Dirac operators and spontaneous symmetry breaking
2) Real Dirac operators, the Standard Model and the mass of the Higgs
3) A bit more geometry concerning Dirac operators

1 A brief reminder

P. A. M. Dirac:

$$(i\not{\partial}_{\mathrm{A}} - m)\psi = 0 \qquad (m \geq 0)\,. \tag{1}$$

Geometrical interpretation:
Let $\mathbb{R}^{1,3} \equiv (\mathbb{R}^4, \eta)$ be the Minkowski space, where

$$\eta(\mathbf{e}^\mu, \mathbf{e}^\nu) := \begin{cases} +1\,, & \text{for all} \quad \mu = \nu = 0\,, \\ -1\,, & \text{for all} \quad 1 \leq \mu = \nu \leq 3\,, \\ 0\,, & \text{for all} \quad 0 \leq \mu \neq \nu \leq 3 \end{cases} \tag{2}$$

with respect to the standard basis $\mathbf{e}^0, \mathbf{e}^1 \ldots, \mathbf{e}^3 \in \mathbb{R}^4$.

Also, let $Cl_{1,3}$ be the real, associative algebra with unit that is generated by the standard basis of $\mathbb{R}^{1,3}$ according to the relations:

$$\mathbf{e}^\mu\mathbf{e}^\nu + \mathbf{e}^\nu\mathbf{e}^\mu = 2\,\eta(\mathbf{e}^\mu, \mathbf{e}^\nu) \qquad (0 \leq \mu, \nu \leq 3)\,. \tag{3}$$

Clearly, $Cl_{1,3} \simeq Cl(\mathbb{R}^4, \eta)$: the *universal Clifford algebra* of the quadratic space $(\mathbb{R}^4, \eta)$.

The *spin group*:

$$Spin(1,3) := \exp_{\mathrm{Cl}}(\sigma_{\mathrm{Ch}}^{-1}\Lambda^2\mathbb{R}^{1,3}) \subset Cl_{1,3} \tag{4}$$

is the two-fold covering group of $SO(1,3)$. Here,

$$\begin{array}{rcl} \sigma_{\mathrm{Ch}} : Cl_{1,3} & \xrightarrow{\simeq} & \Lambda\mathbb{R}^{1,3} \\ \mathbf{e}^{i_1}\mathbf{e}^{i_2}\cdots\mathbf{e}^{i_k} & \mapsto & \mathbf{e}^{i_1}\wedge\mathbf{e}^{i_2}\cdots\wedge\mathbf{e}^{i_k}\,, \end{array} \tag{5}$$

for all $0 \leq i_1 < i_2 < \cdots < i_k \leq 3$ and $k = 0, \ldots, 3$.

The *(geometrical) spinor module*:

$$\mathcal{S} := \Lambda\mathcal{W}\,, \tag{6}$$

where $\mathcal{W} \subset \mathbb{R}^{1,3} \otimes_{\mathbb{R}} \mathbb{C}$ is a maximal isotropic subspace with respect to $\eta^{\mathbb{C}}$, such that

$$\mathbb{R}^{1,3} \otimes_{\mathbb{R}} \mathbb{C} \;\simeq_{\mathbb{C}}\; \mathcal{W} \oplus \mathcal{W}^*\,. \tag{7}$$

It follows that

$$Cl^{\mathbb{C}}_{1,3} \;\simeq_{\mathbb{C}}\; \mathrm{End}(\mathcal{S})\,, \tag{8}$$

according to the Clifford map:

$$\begin{array}{rcl} \gamma_{\mathrm{W}} : \mathbb{R}^{1,3}\otimes\mathbb{C} & \longrightarrow & \mathrm{End}(\mathcal{S}) \\ \mathbf{v} = \mathbf{w} + \mathbf{u}^* & \mapsto & \left\{ \begin{array}{ccl} \mathcal{S} & \longrightarrow & \mathcal{S} \\ \mathbf{z} & \mapsto & \sqrt{2}\,(ext(\mathbf{w})\mathbf{z} + int(\mathbf{u}^*)\mathbf{z})\,. \end{array} \right. \end{array} \tag{9}$$

Furthermore,

$$\mathcal{S} = \mathcal{S}_{\mathrm{R}} \oplus \mathcal{S}_{\mathrm{L}}\,, \tag{10}$$

according to the *parity operator*:

$$\tau_{\mathrm{M}} := i\gamma_{\mathrm{W}}(\mathbf{e}^0 \cdots \mathbf{e}^3)\,, \tag{11}$$

with $\mathcal{S}_{\mathrm{R/L}}$ being the irreducible *Weyl modules* with respect to $Spin(1,3)$.

The (electromagnetically) charged Dirac spinor fields:

$$\psi \in \mathcal{C}^\infty(\mathbb{R}^{1,3}, \mathcal{E})\,, \tag{12}$$

where

$$\mathcal{E} := \mathcal{S} \otimes_{\mathbb{C}} \mathbb{C}\,. \tag{13}$$

"The" (gauge covariant) Dirac operator:

$$i\partial\!\!\!/_{\mathrm{A}} = i\gamma_{\mathrm{W}}(\mathbf{e}^\mu) \circ (\partial_\mu + iA_\mu)\,, \tag{14}$$

with $A \in \Omega^1(\mathbb{R}^{1,3})$ being the electromagnetic gauge potential.

Two basic features:

1) For $\tau_{\mathrm{M}}\psi_{\mathrm{R/L}} = \pm\psi_{\mathrm{R/L}}$, such that $\psi = \psi_{\mathrm{R}} + \psi_{\mathrm{L}}$:

$$i\partial\!\!\!/_{\mathrm{A}}\psi = m\psi \qquad \Leftrightarrow \qquad \left\{ \begin{array}{rcl} i\partial\!\!\!/_{\mathrm{A}}\psi_{\mathrm{R}} & = & m\psi_{\mathrm{L}}\,, \\ i\partial\!\!\!/_{\mathrm{A}}\psi_{\mathrm{L}} & = & m\psi_{\mathrm{R}}\,. \end{array} \right. \tag{15}$$

2) There exists an intertwiner:

$$\mathcal{C} : \mathcal{S} \quad \longrightarrow \quad \bar{\mathcal{S}} \,, \tag{16}$$

such that

$$\mathcal{C}^{-1} = \bar{\mathcal{C}} \,, \qquad \mathcal{C} \circ \tau_{\mathrm{M}} \circ \mathcal{C}^{-1} = -\bar{\tau}_{\mathrm{M}} \,, \tag{17}$$

The corresponding anti-linear involution:

$$\begin{array}{rcl} \mathcal{J} : \mathcal{E} & \longrightarrow & \mathcal{E} \\ \mathbf{z} & \mapsto & \mathbf{z}^{\mathrm{cc}} \equiv \mathcal{C}^{-1}(\bar{\mathbf{z}}) \end{array} \tag{18}$$

is called *charge conjugation*.

Charge conjugation *anti-commutes* with parity:

$$\mathcal{J} \circ \tau_{\mathrm{M}} \circ \mathcal{J} = -\tau_{\mathrm{M}} \,. \tag{19}$$

It follows that

$$(i\partial\!\!\!/ - m)\psi = \gamma_{\mathrm{W}}(A)\psi \qquad \Leftrightarrow \qquad (i\partial\!\!\!/ - m)\psi^{\mathrm{cc}} = -\gamma_{\mathrm{W}}(A)\psi^{\mathrm{cc}} \,. \tag{20}$$

Physical interpretation: When "quantized" the Dirac spinor
– ψ: state of a (quantum) particle of mass m and charge $+1$;
– ψ^{cc}: state of a (quantum) anti-particle of mass m and charge -1.

<u>E. Majorana:</u>

$$i\partial\!\!\!/\psi = m\psi^{\mathrm{cc}} \,, \tag{21}$$

where ψ carries the trivial representation of $U(1)$.

Basic feature:

$$i\partial\!\!\!/\psi = m\psi^{\mathrm{cc}} \qquad \Leftrightarrow \qquad \left\{ \begin{array}{rcl} i\partial\!\!\!/\psi_{\mathrm{R}} & = & m\psi^{\mathrm{cc}}_{\mathrm{R}} \,, \\ i\partial\!\!\!/\psi_{\mathrm{L}} & = & m\psi^{\mathrm{cc}}_{\mathrm{L}} \,. \end{array} \right. \tag{22}$$

Majorana module:

$$\mathcal{M} := \{ \mathbf{z} \in \mathcal{E} \,|\, \mathbf{z}^{\mathrm{cc}} = \mathbf{z} \} \,, \tag{23}$$

such that

$$\mathcal{E} = \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}\,. \tag{24}$$

Each *Majorana spinor field*: $\chi \in \mathcal{C}^\infty(\mathbb{R}^{1,3}, \mathcal{M})$, reads:

$$\chi = \psi + \psi^{\mathrm{cc}}\,, \qquad \tau_{\mathrm{M}}\psi = \pm\psi \in \mathcal{C}^\infty(M, \mathcal{E})\,. \tag{25}$$

2 General Clifford modules and Dirac operators

Let $(M, g_{\rm M})$ be an orientable (semi-)Riemannian manifold of even dimension $n = p + q \geq 2$ and signature $s = p - q \in \mathbb{Z}$.

Also, let $Cl_{\rm M} \twoheadrightarrow M$ be the induced *Clifford bundle*:

$$\begin{array}{rcl} Cl_{\rm M} := \mathcal{SO}_{\rm M} \times_{SO(p,q)} Cl_{p,q} & \longrightarrow & M \\ a = [(q, \mathbf{a})] & \mapsto & x = \pi_{\rm SO}(q)\,. \end{array} \tag{26}$$

Here,

$$\begin{array}{rcl} \pi_{\rm SO}(q) : \mathcal{SO}_{\rm M} & \stackrel{\iota_g}{\hookrightarrow} & \mathcal{F}_{\rm M} \twoheadrightarrow M \\ q & \mapsto & \pi(\iota_g(q)) \end{array} \tag{27}$$

is the $g_{\rm M}-$induced $SO(p,q)-$reduction of the *(oriented) frame bundle* $\pi : \mathcal{F}_{\rm M} \twoheadrightarrow M$ of M.

Proposition 2.1 *The set of all smooth $SO(p,q)-$reductions $(\mathcal{SO}_{\rm M}, \iota_g)$ of the (oriented) frame bundle of M is in one-to-one correspondence with the set of all smooth sections of the "Einstein-Hilbert bundle":*

$$\begin{array}{rcl} \mathcal{E}_{\rm EH} := \mathcal{F}_{\rm M} \times_{GL(n)} GL(n)/SO(p,q) & \twoheadrightarrow & M \\ [(p, [h])] & \mapsto & \pi(p)\,. \end{array} \tag{28}$$

Basically, this results from

$$\mathcal{F}_{\rm M} \twoheadrightarrow \mathcal{F}_{\rm M}/SO(p,q) \simeq \mathcal{F}_{\rm M} \times_{GL(n)} GL(n)/SO(p,q) \tag{29}$$

is an $SO(p,q)-$principal bundle.

Definition 2.1 *A "Clifford module bundle"*

$$(\mathcal{E}, \gamma_\mathcal{E}) \twoheadrightarrow (M, g_{\rm M}) \tag{30}$$

is a $\mathbb{Z}_2-$graded (complex) vector bundle $\pi_\mathcal{E} : \mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \rightarrow M$, together with an odd Clifford map:

$$\begin{array}{rcl} \gamma_\mathcal{E} : T^*M & \longrightarrow & {\rm End}(\mathcal{E}) \\ \alpha & \mapsto & \left\{ \begin{array}{rcl} \mathcal{E} & \longrightarrow & \mathcal{E} \\ z & \mapsto & \gamma_\mathcal{E}(\alpha) z\,, \end{array} \right. \end{array} \tag{31}$$

whereby

$$\{\tau_\mathcal{E}, \gamma_\mathcal{E}(\alpha)\} \equiv 0\,, \qquad \gamma_\mathcal{E}(\alpha)^2 = \epsilon g_{\rm M}(\alpha, \alpha)\, {\rm id}_\mathcal{E} \quad (\epsilon = \pm 1)\,. \tag{32}$$

The sub-algebra

$$\mathrm{End}_\gamma(\mathcal{E}) := \{B \in \mathrm{End}(\mathcal{E}) \,|\, [\gamma_\mathcal{E}(\alpha), B] \equiv 0\} \tag{33}$$

denotes the "commutant" with respect to the induced "Clifford action"

$$\begin{array}{rcl} \gamma_\mathcal{E} : Cl_{\mathrm{M}} & \longrightarrow & \mathrm{End}(\mathcal{E}) \\ a & \mapsto & \gamma_\mathcal{E}(a)\,. \end{array} \tag{34}$$

$$\tag{35}$$

Definition 2.2 *A "Clifford bi-module bundle" is a $\mathbb{Z}_2-$graded vector bundle, which carries a representation of both the algebra bundle of Clifford algebras and of the opposite Clifford algebras.*

Proposition 2.2 *The mapping*

$$\begin{array}{rcl} Cl^{\mathbb{C}}_{\mathrm{M}} \otimes_M \mathrm{End}_\gamma(\mathcal{E}) & \longrightarrow & \mathrm{End}(\mathcal{E}) \\ a \otimes B & \mapsto & \gamma_\mathcal{E}(a) \circ B \end{array} \tag{36}$$

is a (bundle) isomorphism (over the identity on M).

This is a consequence of the Wedderburn Theorems about equivariant (linear) mappings.

Corollary 2.1 *It follows that*

$$\mathfrak{S}ec(M, \mathrm{End}(\mathcal{E})) \simeq \Omega^*(M, \mathrm{End}_\gamma(\mathcal{E})) \equiv \mathfrak{S}ec(M, \bigoplus_{k\in\mathbb{Z}} \Lambda^k_{\mathrm{M}} \otimes_M \mathrm{End}_\gamma(\mathcal{E}))\,, \tag{37}$$

This is mainly due to Chevalley's *linear* isomorphism between the Clifford and the Grassmann bundle:

$$\begin{array}{rcl} \sigma_{\mathrm{Ch}} : Cl_{\mathrm{M}} & \longrightarrow & \Lambda_{\mathrm{M}} \\ a & \mapsto & \gamma_{\mathrm{Ch}}(a) 1_\Lambda\,. \end{array} \tag{38}$$

Here,

$$\begin{array}{rcl} \gamma_{\mathrm{Ch}} : T^*M & \longrightarrow & \mathrm{End}(\Lambda_{\mathrm{M}}) \\ \alpha & \mapsto & \left\{ \begin{array}{rcl} \Lambda_{\mathrm{M}} & \longrightarrow & \Lambda_{\mathrm{M}} \\ \omega & \mapsto & ext(\alpha)\omega + \epsilon int(\alpha)\omega \end{array} \right. \end{array} \tag{39}$$

denotes the canonical Clifford action on the *Grassmann bundle* of M:

$$\begin{array}{rcl} \Lambda_{\mathrm{M}} := \mathcal{SO}_{\mathrm{M}} \times_{SO(p,q)} \Lambda_{p,q} & \twoheadrightarrow & M \\ \omega = [(q, \mathbf{z})] & \mapsto & \pi(\iota_g(q))\,. \end{array} \tag{40}$$

Definition 2.3 *Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \twoheadrightarrow M$ be a (complex) vector bundle. A first order differential operator $\not{D}$, acting on $\mathfrak{Sec}(M,\mathcal{E})$, is called of "Dirac type", provided the principal symbol of $\not{D}^2$ defines an $SO(p,q)-$reduction $g_{\rm M} \in \mathfrak{Sec}(M,\mathcal{E}_{\rm EH})$ of the frame bundle of M.*

Furthermore, a Dirac type operator $\not{D}$ on $\mathfrak{Sec}(M,\mathcal{E})$ is called a "Dirac operator", if it is odd with respect to the $\mathbb{Z}_2-$grading of $\mathcal{E} \twoheadrightarrow M$:

$$\{\tau_{\mathcal{E}}, \not{D}\} \equiv 0\,. \tag{41}$$

The set of all Dirac type operators on $\mathcal{E} \twoheadrightarrow M$ is denoted by $\mathfrak{D}(\mathcal{E})$. The set of all Dirac operators is denoted by $\mathcal{D}(\mathcal{E})$.

Remark:
Every Dirac type operator $\not{D}$ turns the vector bundle $\mathcal{E} \twoheadrightarrow M$ into a Clifford module bundle, for

$$\begin{array}{rcl} T^*M \times_M \mathcal{E} & \longrightarrow & \mathcal{E} \\ (df, z) & \mapsto & [\not{D}, f]z \end{array} \tag{42}$$

yields a Clifford map.

Moreover, the set of all Dirac type operators, which yield the *same* Clifford action on $\mathcal{E} \twoheadrightarrow M$, is an affine space with the underlying vector space being given by $\Omega^0(M, \mathrm{End}(\mathcal{E}))$. Likewise, the set of all Dirac operators, which give rise to the same Clifford action, is an affine space modeled over $\Omega^0(M, \mathrm{End}^-(\mathcal{E}))$. The affine space of all Dirac type operators with the same Clifford action is denoted by $\mathfrak{D}_\gamma(\mathcal{E})$. Accordingly, the affine space of all Dirac operators with the same Clifford action is given by $\mathcal{D}_\gamma(\mathcal{E})$.

Proposition 2.3 *Let $(M, g_{\rm M})$ be an orientable, even-dimensional (semi-)Riemannian spin-manifold. Also, let $\mathcal{S} \twoheadrightarrow M$ be a spinor bundle associated with a chosen spin structure. The mapping*

$$\begin{array}{rcl} \mathcal{S} \otimes_M \mathrm{Hom}_\gamma(\mathcal{S},\mathcal{E}) & \longrightarrow & \mathcal{E} \\ z \otimes \phi & \mapsto & \phi(z) \end{array} \tag{43}$$

is a (bundle) isomorphism (over the identity on M).

Consequently, ever Clifford module bundle over a spin-manifold is equivalent to a "twisted spinor bundle":

$$\mathcal{E} \simeq_{\mathbb{C}} \mathcal{S} \otimes_M \mathcal{W}\,, \tag{44}$$

where $\mathcal{W} := \mathrm{Hom}_\gamma(\mathcal{S},\mathcal{E})$.

Again, this follows from the Wedderburn Theorems about equivariant (linear) mappings.

Note that in the case of a spin-manifold: $\mathrm{End}_\gamma(\mathcal{E}) \simeq \mathrm{End}(\mathcal{W})$.

Two (basic) Examples:

- *Twisted spinor bundles:* $\mathcal{E} := \mathcal{S} \otimes_M E \twoheadrightarrow M$, with $E \twoheadrightarrow M$ being a (maybe trivially) $\mathbb{Z}_2-$graded vector bundle.

$$\mathrm{End}_\gamma(\mathcal{E}) = \mathrm{End}(E)\,. \tag{45}$$

- *Twisted Grassmann bundles:* $\mathcal{E} := \Lambda_{\mathrm{M}} \otimes_M E \twoheadrightarrow M$.

$$\begin{aligned} \mathrm{End}_\gamma(\mathcal{E}) &= \mathrm{End}(\mathcal{S}^*) \otimes_M \mathrm{End}(E) \\ &\simeq_{\mathbb{C}} (Cl_{\mathrm{M}}^{\mathrm{op}})^{\mathbb{C}} \otimes_M \mathrm{End}(E)\,. \end{aligned} \tag{46}$$