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Geometry and Physics of Dirac Operators

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– Second lecture –

Content

1) Connections induced by Dirac (type) operators
2) The universal Dirac-Lagrangian and the Einstein-Hilbert Action

1 Connections induced by Dirac (type) operators

Let

$$(\mathcal{E}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\rm M}) \tag{1}$$

be a bundle of (complex) Clifford modules.

Every (even) connection on $\mathcal{E} \twoheadrightarrow M$ yields a Dirac operator:

$$\not{\nabla}^{\mathcal{E}} : \mathfrak{S}ec(M, \mathcal{E}) \xrightarrow{\nabla^{\mathcal{E}}} \mathfrak{S}ec(M, T^*M \otimes_M \mathcal{E}) \xrightarrow{\gamma_{\mathcal{E}}} \mathfrak{S}ec(M, \mathcal{E})\,. \tag{2}$$

Definition 1.1 *A connection on a Clifford module bundle is called a "Clifford connection", provided it fulfils:*

$$[\nabla^{\mathcal{E}}_X, \gamma_{\mathcal{E}}(\mathfrak{a})] = \gamma_{\mathcal{E}}(\nabla^{\rm Cl}_X(\mathfrak{a}))\,, \tag{3}$$

for all $\mathfrak{a} \in \mathfrak{S}ec(M, Cl_{\rm M})$ and $X \in \mathfrak{S}ec(M, TM)$.

The set of all Clifford connections on a Clifford module bundle $(\mathcal{E}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\rm M})$ is denoted by $\mathcal{A}_{\rm Cl}(\mathcal{E})$. It is an affine sub-space of the affine space $\mathcal{A}(\mathcal{E})$ of all (linear) connections on $\mathcal{E} \twoheadrightarrow M$.

Note that the underlying vector space of $\mathcal{A}_{\rm Cl}(\mathcal{E})$ is given by $\Omega^1(M, {\rm End}^+_\gamma(\mathcal{E}))$.

Definition 1.2 *The Dirac operator of a Clifford connection is called "a Dirac operator of Clifford type".*

Notation:

We denote a Clifford connection by $\partial_{\rm A} \in \mathcal{A}_{\rm Cl}(\mathcal{E})$, for it locally reads:

$$\partial_{\rm A} \overset{\rm loc.}{=} d + \omega + A\,. \tag{4}$$

Let $U \subset M$ be a local subset and $e_1, \ldots, e_n \in \mathfrak{S}ec(U, TM)$ be a locally defined (orthonormal) frame. Also, let $e^1, \ldots, e^n \in \mathfrak{S}ec(U, T^*M)$ be the corresponding dual frame. The locally defined one-form $\omega \in \Omega^1(U, {\rm End}^+(\mathcal{E}))$ is the "spin-connection form":

$$\omega \equiv -\tfrac{\epsilon}{8}\, g_{\rm M}(\nabla_k e_a, e_b)\, e^k \otimes [\gamma_{\mathcal{E}}(e^a), \gamma_{\mathcal{E}}(e^b)] \tag{5}$$

and $A \in \Omega^1(U, {\rm End}^+_\gamma(\mathcal{E}))$ is a local "gauge potential".

If $\mathcal{E} = \mathcal{S} \otimes_M E \twoheadrightarrow M$ is a twisted spinor bundle, than a Clifford connection is but a *twisted spin connection*:

$$\begin{aligned} \partial_{\rm A} &= \nabla^{\mathcal{S}\otimes E} \\ &= \nabla^{\mathcal{S}} \otimes {\rm id}_{\rm E} + {\rm id}_{\mathcal{S}} \otimes \nabla^{\rm E}\,. \end{aligned} \tag{6}$$

Whence,

$$\mathcal{A}_{\mathrm{Cl}}(\mathcal{E}) \simeq \mathcal{A}(E)\,. \tag{7}$$

In the case of twisted Grassmann bundles: $\mathcal{E} = \Lambda_{\mathrm{M}} \otimes_M E \twoheadrightarrow M$, the Clifford connections are locally parameterized by local gauge potentials:

$$A \in \Omega^1\Big(U, (Cl_{\mathrm{M}}^{\mathrm{op}})^{\mathbb{C}} \otimes_M \mathrm{End}(E)\Big)^+ \,. \tag{8}$$

Definition 1.3 *Let $(\mathcal{E}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\mathrm{M}})$ be a Clifford module bundle. The one-form $\Theta \in \Omega^1(M, \mathrm{End}^-(\mathcal{E}))$, which is defined by*

$$\Theta(v) := \tfrac{\epsilon}{n}\, \gamma_{\mathcal{E}}(v^\flat) \tag{9}$$

for all $v \in TM$, is called the "canonical one-form" on the Clifford module.

Let $U \subset M$ be an open subset and $e_1, \ldots, e_n \in \mathfrak{Sec}(U, TM)$ be a locally defined (orthonormal) frame with the dual frame $e^1, \ldots, e^n \in \mathfrak{Sec}(U, T^*M)$.

$$\begin{aligned} \Theta &\overset{\text{loc.}}{=} \tfrac{\epsilon}{n}\, g_{\mathrm{M}}(e_a, e_b)\, e^a \otimes \gamma_{\mathcal{E}}(e^b) \\ &\equiv \tfrac{\epsilon}{n}\, e^a \otimes \gamma_{\mathcal{E}}(e_a^\flat)\,. \end{aligned} \tag{10}$$

Lemma 1.1 *A connection on a Clifford module bundle is a Clifford connection if and only if the induced connection on $T^*M \otimes_M \mathrm{End}(\mathcal{E}) \twoheadrightarrow M$ fulfils:*

$$\nabla^{T^*M \otimes \mathrm{End}(\mathcal{E})}\Theta \equiv 0\,. \tag{11}$$

Proof: Nice exercise!

Definition 1.4 *Let $(\mathcal{E}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\mathrm{M}})$ be a Clifford bi-module bundle. A connection is called "S-reducible", provided its induced connection on $T^*M \otimes_M \mathrm{End}(\mathcal{E}) \twoheadrightarrow M$ fulfils:*

$$\nabla^{T^*M \otimes \mathrm{End}(\mathcal{E})}\Theta^{\mathrm{op}} \equiv 0\,. \tag{12}$$

Here, $\Theta^{\mathrm{op}}(v) := \frac{\epsilon}{n}\, \gamma_{\mathcal{E}}^{\mathrm{op}}(v^\flat)$, for all $v \in TM$ and $\gamma_{\mathcal{E}}^{\mathrm{op}} : Cl_{\mathrm{M}}^{\mathrm{op}} \to \mathrm{End}(\mathcal{E})$ is the representation of the algebra bundle of opposite Clifford algebras.

A connection on a twisted Grassmann bundle is S-reducible if and only if it is locally parameterized by a gauge potential

$$A \in \Omega^1(U, \mathrm{End}(E))\,. \tag{13}$$

Furthermore, a connection on the Grassmann bundle over a spin-manifold is S-reducible if and only if it coincides with the spin-connection.

Definition 1.5 *On a Clifford module bundle the (linear extension of the) map:*

$$\begin{array}{rcl} \delta_\gamma :\, \Omega^*(M, \mathrm{End}(\mathcal{E})) & \longrightarrow & \Omega^0(M, \mathrm{End}(\mathcal{E})) \\ \omega = \alpha \otimes B & \mapsto & \not{\omega} \equiv \gamma_{\mathcal{E}}(\sigma_{\mathrm{Ch}}^{-1}(\omega)) \circ B \end{array} \tag{14}$$

is called the "quantization map".

The restriction of the quantization map to $\Omega^1(M, \mathrm{End}(\mathcal{E}))$ has a canonical right-inverse that is given by the odd map:

$$\begin{array}{rcl} ext_\Theta :\, \Omega^0(M, \mathrm{End}^\pm(\mathcal{E})) & \longrightarrow & \Omega^1(M, \mathrm{End}^\mp(\mathcal{E})) \\ \Phi & \mapsto & \Theta \wedge \Phi \equiv \Theta\Phi\,, \end{array} \tag{15}$$

with $(\Theta \wedge \Phi)(v) := \Theta(v) \circ \Phi$, for all $v \in TM$.

Whence,

$$\wp := ext_\Theta \circ \delta_\gamma :\, \Omega^1(M, \mathrm{End}(\mathcal{E})) \to \Omega^1(M, \mathrm{End}(\mathcal{E})) \tag{16}$$

is an idempotent. Its complement $\wp' := \mathrm{id}_{\Omega^1} - \wp$ sends $\mathcal{A}(\mathcal{E})$ into the set of *"twistor operators"* on the underlying Clifford module bundle:

$$\nabla^{\mathcal{E}} \;\mapsto\; \mathcal{T}(\nabla^{\mathcal{E}}) := \nabla^{\mathcal{E}} - \Theta \circ \not{\nabla}^{\mathcal{E}}\,. \tag{17}$$

Definition 1.6 *Two connections on a Clifford module bundle are said to be equivalent if they yield the same Dirac operator:*

$$\nabla^{\mathcal{E}} \sim \nabla'^{\mathcal{E}} \quad :\Leftrightarrow \quad \not{\nabla}^{\mathcal{E}} = \not{\nabla}'^{\mathcal{E}}\,. \tag{18}$$

Clearly,

$$\nabla^{\mathcal{E}} \sim \nabla'^{\mathcal{E}} \quad \Leftrightarrow \quad \nabla'^{\mathcal{E}} - \nabla^{\mathcal{E}} \in \mathrm{Ker}(\wp)\,. \tag{19}$$

Proposition 1.1 *Let $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ be a Dirac operator on $(\mathcal{E}, \gamma_\mathcal{E}) \twoheadrightarrow (M, g_{\rm M})$. The equivalence class of connections on $\mathcal{E} \twoheadrightarrow M$ that is defined by $\not{D}$ has a natural representative.*

Proof: Every Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ on a Clifford module bundle yields a unique connection, called the *"Bochner connection"* of $\not{D}$:

$$2\, ev_g(df, \partial_{\rm B}\psi) := \epsilon \Big([\not{D}^2, f] - \delta_g df\Big)\psi\,, \tag{20}$$

for all $f \in \mathcal{C}^\infty(M)$ and $\psi \in \mathfrak{S}ec(M, \mathcal{E})$.

This yields the **first order decomposition** of $\not{D}$:

$$\not{D} = \not{\partial}_{\rm B} + \Phi_{\rm D}\,, \tag{21}$$

with $\Phi_{\rm D} \in \mathfrak{S}ec(M, {\rm End}^-(\mathcal{E}))$ being uniquely defined by $\not{D}$.

The connection that corresponds to

$$\partial_{\rm D} := \partial_{\rm B} + ext_\Theta \wedge \Phi_{\rm D} \tag{22}$$

is thus uniquely defined by $\not{D}$. Furthermore,

$$\not{\partial}_{\rm D} = \not{D}\,. \tag{23}$$

□

Definition 1.7 *For given $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$, the even one-form:*

$$\omega_{\rm D} := ext_\Theta \wedge \Phi_{\rm D}\,, \tag{24}$$

is called the "Dirac form" of $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$.

The tangent vector field on M:

$$\xi_{\rm D} := {\rm tr}_\mathcal{E}(\omega_{\rm D}^\sharp)\,, \tag{25}$$

is called the "Dirac field" of $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$.

The connection on the underlying Clifford module bundle that corresponds to $\partial_{\rm D}$ is called the "Dirac connection" of $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$. Its curvature

$$curv(\not{D}) := \partial_{\rm D} \wedge \partial_{\rm D} \in \Omega^2(M, {\rm End}^+(\mathcal{E})) \tag{26}$$

is called the "Dirac curvature" of $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$.

Finally,

$$F_{\rm D} := curv(\not{D}) - \not{R}iem(g_{\rm M}) \in \Omega^2(M, {\rm End}^+(\mathcal{E})) \tag{27}$$

is called the "relative curvature" of $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$.

Lemma 1.2 *Let $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ be a Dirac operator. Its induced equivalence class of connections on the underlying Clifford module bundle contains at most one Clifford connection. This is the case if and only if*

$$\partial_{\mathrm{D}}^{T^*M \otimes \mathrm{End}(\mathcal{E})}\Theta \equiv 0\,. \tag{28}$$

Proof: First, let the Dirac connection of $\not{D}$ be a Clifford connection. Any other connection $\nabla^{\mathcal{E}}$ whose quantization equals $\not{D}$ thus reads:

$$\nabla^{\mathcal{E}} = \partial_{\mathrm{D}} + \alpha\,, \quad \alpha \in \mathrm{Ker}(\wp)\,. \tag{29}$$

In particular, if $\nabla^{\mathcal{E}} = \partial_{\mathrm{A}}$ is also a Clifford connection, than $\alpha \in \Omega^1(M, \mathrm{End}^+_\gamma(\mathcal{E}))$. The map ext_Θ is injective. Hence, $\mathrm{Ker}(\wp) = \mathrm{Ker}(\delta_\gamma|_{\Omega^1})$. However, $\alpha \notin \mathrm{Ker}(\delta_\gamma|_{\Omega^1})$ since the restriction of the quantization map to $\Omega^*(M, \mathrm{End}_\gamma(\mathcal{E}))$ is an isomorphism.

Now, let $\partial_{\mathrm{A}} \sim \partial_{\mathrm{D}}$. Since $\not{\partial}_{\mathrm{A}} = \not{D}$, it follows that the Bochner connection of $\not{D}$ equals the Clifford connection: $\partial_{\mathrm{B}} = \partial_{\mathrm{A}}$. Therefore, $\partial_{\mathrm{D}} = \partial_{\mathrm{A}}$.

Whence, if the connection class of $\not{D}$ contains a Clifford connection, it must be unique and equal to the Dirac connection of $\not{D}$. Only in this case, one gets:

$$\partial_{\mathrm{D}} = \partial_{\mathrm{B}} = \partial_{\mathrm{A}}\,. \tag{30}$$

$\square$

Remark:
If a Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ is of Clifford type: $\not{D} = \not{\partial}_{\mathrm{A}}$, than its curvature reads:

$$curv(\not{\partial}_{\mathrm{A}}) = \not{R}iem(g_{\mathrm{M}}) + F_{\mathrm{A}}\,, \tag{31}$$

whereby the relative curvature F_{A} of $\not{\partial}_{\mathrm{A}}$ fulfils:

$$F_{\mathrm{A}} \in \Omega^2(M, \mathrm{End}_\gamma(\mathcal{E}))\,. \tag{32}$$

In the case of a twisted spinor bundle $\mathcal{E} = \mathcal{S} \otimes_M E \longrightarrow M$, the relative curvature of a Clifford type Dirac operator is given by

$$F_{\mathrm{A}} = \nabla^{\mathrm{E}} \wedge \nabla^{\mathrm{E}} \in \Omega^2(M, \mathrm{End}^+(E))\,. \tag{33}$$

In terms of *Yang-Mills gauge theories*, the relative curvature of a Clifford type Dirac operator thus plays the role of the *Yang-Mills curvature*.

Definition 1.8 *A Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ on a Clifford bi-module bundle is called "S-reducible", if its Dirac connection is S-reducible.*

On a twisted Grassmann bundle over a spin manifold, a Dirac operator is S-reducible if and only if it coincides with a twisted spin-Dirac operator.

Proposition 1.2 *Two Dirac operators $\not{D}', \not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ on a given Clifford module bundle yield the same Bochner connection if and only if*

$$\{(\not{D}' - \not{D}), \gamma_\mathcal{E}(\alpha)\} \equiv 0\,, \tag{34}$$

for all $\alpha \in T^*M$.

Proof: Making use of the definition of the Bochner connection of a Dirac operator, the proof follows from showing that

$$\partial'_{\mathrm{B}} = \partial_{\mathrm{B}} + \alpha_{\mathrm{B}}\,, \tag{35}$$

with the one-form $\alpha_{\mathrm{B}} \in \Omega^1(M, \mathrm{End}^+(\mathcal{E}))$ being defined by

$$\alpha_{\mathrm{B}}(v) = \tfrac{\epsilon}{2}\,\{(\not{D}' - \not{D}), \gamma_\mathcal{E}(v^\flat)\}\,, \tag{36}$$

for all $v \in TM$. □

Definition 1.9 *A Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ is called of "simple type" if its Bochner connection equals a Clifford connection.*

Proposition 1.3 *Let $(\mathcal{E}, \gamma_\mathcal{E}) \twoheadrightarrow (M, g_{\mathrm{M}})$ be a Clifford module bundle. A Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ is of simple type if and only if $\not{D} - \not{\partial}_{\mathrm{B}}$ anti-commutes with the Clifford action $\gamma_\mathcal{E}$.*

Proof: First, let the Bochner connection of $\not{D}$ be a Clifford connection: $\partial_{\mathrm{B}} = \partial_{\mathrm{A}}$. Since the Bochner connection of Clifford type Dirac operator $\not{\partial}_{\mathrm{A}}$ equals ∂_{A}, it follows that $\not{D}$ and $\not{\partial}_{\mathrm{B}}$ yield the same Bochner connection (namely ∂_{B}). Whence, according to the foregoing Proposition it follows that $\not{D} - \not{\partial}_{\mathrm{B}}$ anti-commute with the Clifford action.

Next, assume that the zero-order operator $\Phi_{\mathrm{D}} = \not{D} - \not{\partial}_{\mathrm{B}}$ anti-commute with the Clifford action. Hence, there is a unique zero-order operator $\phi_{\mathrm{D}} \in \mathfrak{Sec}(M, \mathrm{End}^-_\gamma(\mathcal{E}))$, such that

$$\Phi_{\mathrm{D}} = \tau_\mathcal{E} \circ \phi_{\mathrm{D}}\,. \tag{37}$$

Furthermore, $\not{D}$ and $\not{\partial}_{\mathrm{B}}$ have the same Bochner connection due to the foregoing Proposition. Hence, the Bochner connection of $\not{\partial}_{\mathrm{B}}$ coincides with ∂_{B}, which holds true if and only if ∂_{B} is a Clifford connection. □

Corollary 1.1 *Let $(\mathcal{E}, \gamma_\mathcal{E}) \twoheadrightarrow (M, g_{\mathrm{M}})$ be a Clifford module bundle. A Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ is of simple type if and only if there is Clifford connection and a $\phi \in \mathfrak{Sec}(M, \mathrm{End}^-_\gamma(\mathcal{E}))$, such that*

$$\not{D} = \not{\partial}_{\mathrm{A}} + \tau_\mathcal{E} \circ \phi\,. \tag{38}$$

□

The set of simple type Dirac operators is the largest class of Dirac operators whose Bochner connections are Clifford connections. Simple type Dirac operators thus build a natural generalization of Clifford type Dirac operators.

Definition 1.10 *A Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ on a Clifford module bundle is called of "Yang-Mills-Higgs type", if there is a Clifford connection such that*

$$\not{D} - \not{\partial}_{\mathrm{A}} \in \mathfrak{S}ec(M, \mathrm{End}^-_\gamma(\mathcal{E}))\,. \tag{39}$$

Since the Clifford connection is unique, there is a unique

$$\Phi_{\mathrm{H}} \in \mathfrak{S}ec(M, \mathrm{End}^-_\gamma(\mathcal{E}))\,, \tag{40}$$

such that

$$\not{D} = \not{\partial}_{\mathrm{A}} + \Phi_{\mathrm{H}}\,. \tag{41}$$

It follows that the Dirac connection of a Yang-Mills-Higgs type Dirac operator reads:

$$\begin{aligned} \partial_{\mathrm{D}} &\equiv \partial_{\mathrm{YMH}} \\ &= \partial_{\mathrm{A}} + H\,, \end{aligned} \tag{42}$$

with

$$H := \Phi_{\mathrm{H}}\Theta \in \Omega^1(M, \mathrm{End}^+(\mathcal{E})) \tag{43}$$

being the *"Higgs gauge potential"*.

The relative curvature of a Yang-Mills-Higgs type Dirac operator simply reads:

$$\begin{aligned} \mathrm{F}_{\mathrm{D}} &= \mathrm{F}_{\mathrm{A}} + \mathrm{d}_{\mathrm{A}}H + H \wedge H \\ &= \mathrm{F}_{\mathrm{A}} + (\,\mathrm{d}_{\mathrm{A}}\Phi_{\mathrm{H}} + \Phi_{\mathrm{H}} \wedge \Theta) \wedge \Theta\,. \end{aligned} \tag{44}$$

Remark:
Every Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ may be decomposed as

$$\not{D} = \not{\partial}_{\mathrm{A}} + \Phi\,. \tag{45}$$

However, this decomposition is not unique, in general, for $\Phi \in \mathfrak{S}ec(M, \mathrm{End}^-(\mathcal{E}))$ also depends on the choice of ∂_{A}.

Simple type Dirac operators generalize Dirac operators of Clifford type in the sense that

– $\partial_{\mathrm{B}} = \partial_{\mathrm{A}}$;
– Φ is uniquely determined by $D\!\!\!\!/$.

In contrast, Yang-Mills-Higgs type Dirac operators $\partial\!\!\!/_{\mathrm{YMH}}$ generalize Dirac operators of Clifford type in the sense that the decomposition

$$\partial\!\!\!/_{\mathrm{YMH}} = \partial\!\!\!/_{\mathrm{A}} + \Phi \tag{46}$$

is unique, though $\partial_{\mathrm{B}} \neq \partial_{\mathrm{A}}$.

2 The universal Dirac-Lagrangian and the Einstein-Hilbert Action

Definition 2.1 *Let $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ be an arbitrary Dirac operator on a Clifford module bundle $(\mathcal{E}, \gamma_\mathcal{E}) \twoheadrightarrow (M, g_{\rm M})$. The associated second order differential operator:*

$$\triangle_{\rm B} := \epsilon ev_g(\partial_{\rm B}^{T^*M\otimes\mathcal{E}} \circ \partial_{\rm B})\,, \tag{47}$$

is called the "Bochner (or connection/trace) Laplacian".

Proposition 2.1 *Every Dirac operator $\not{D} \in \mathcal{D}_\gamma(\mathcal{E})$ has a unique* **second order decomposition***:*

$$\not{D}^2 = \triangle_{\rm B} + V_{\rm D}\,, \tag{48}$$

with $V_{\rm D} \in \mathfrak{Sec}(M, {\rm End}^+(\mathcal{E}))$ being uniquely defined by $\not{D}$.

Furthermore, the "Dirac potential" explicitly reads:

$$V_{\rm D} = \delta_\gamma(curv(\not{D})) + \epsilon ev_g\Big(\partial_{\rm D}\omega_{\rm D}) - \omega_{\rm D}^2\Big)\ . \tag{49}$$

Basically, the proof follows from the very definition of the Bochner connection of a Dirac operator.

In the case where $\not{D} = \not{\partial}_{\rm A}$ is of Clifford type, it follows that

$$V_{\rm D} = \tfrac{\epsilon}{4}\, scal(g_{\rm M})\, {\rm id}_\mathcal{E} + \delta_\gamma(F_{\rm A}) \tag{50}$$

coincides with the well-known Schrödinger-Lichnerowicz formula of the zero-order operator of the square of a twisted spin-Dirac operator $\not{\nabla}^{\mathcal{S}\otimes E}$.

Note that the zero-order operator

$$\delta_\gamma(F_{\rm A}) \in \mathfrak{Sec}(M, {\rm End}^+(\mathcal{E})) \tag{51}$$

is always *trace-free*. This is because $F_{\rm A} \in \Omega^2(M, {\rm End}^+_\gamma(\mathcal{E}))$.

Definition 2.2 *Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \twoheadrightarrow M$ be a Hermitian vector bundle with the Hermitian product being denoted by $\langle\cdot,\cdot\rangle_\mathcal{E}$.*

The map:

$$\begin{array}{rccl} \mathcal{L}_{\rm D} : & \mathcal{D}(\mathcal{E}) & \longrightarrow & \Omega^n(M, \mathbb{C}) \\ & \not{D} & \mapsto & *{\rm tr}_\mathcal{E} V_{\rm D}\,, \end{array} \tag{52}$$

is called the "universal Dirac-Lagrangian".

Likewise, the map:

$$
\begin{array}{rcl}
\mathcal{L}_{\mathrm{D,tot}} : \mathcal{D}(\mathcal{E}) \times \mathfrak{S}ec(M,\mathcal{E}) & \longrightarrow & \Omega^n(M,\mathbb{C}) \\
(\not{D},\psi) & \mapsto & *(\langle\psi,\not{D}\psi\rangle_{\mathcal{E}} + \mathrm{tr}_{\mathcal{E}} V_{\mathrm{D}})\ ,
\end{array} \tag{53}
$$

is called the "total Dirac-Lagrangian".

Proposition 2.2 *The universal Dirac-Lagrangian is equivariant with respect to the action of the "affine gauge group":*

$$\mathcal{G}_{\mathrm{D}} = \mathcal{G}_{\mathrm{D,tot}} \ltimes \mathcal{T}_{\mathrm{D}}\,, \tag{54}$$

where, respectively,

$$\mathcal{G}_{\mathrm{D,tot}} := \mathrm{Diff}(M) \ltimes \mathrm{Aut}(\mathcal{E})\,, \tag{55}$$

$$\mathcal{T}_{\mathrm{D}} := \Omega^1(M,\mathrm{End}^+_\gamma(\mathcal{E})) \tag{56}$$

is the gauge group of the total Dirac-Lagrangian and the "translation group".

The proof needs some (home)work! Indeed, it can be shown that the universal Dirac-Lagrangian is actually invariant with respect to the (linear extension of the) map:

$$
\begin{array}{rcl}
\mathcal{D}(\mathcal{E}) \times \mathcal{T}_{\mathrm{D}} & \longrightarrow & \mathcal{D}(\mathcal{E}) \\
(\not{D}, df) & \mapsto & \not{D} + [\not{D}, f]\,.
\end{array} \tag{57}
$$

$$\tag{58}$$

Note that the gauge group of the total Dirac-Lagrangian is only a (proper) subgroup of the gauge group of the universal Dirac-Lagrangian.

Up to the boundary term $*div\xi_{\mathrm{D}} \in \Omega^n(M,\mathbb{C})$, the universal Dirac-Lagrangian explicitly reads:

$$\mathcal{L}_{\mathrm{D}}(\not{D}) = *\mathrm{tr}_\gamma(curv(\not{D}) - \epsilon ev_g(\omega^2_{\mathrm{D}}))\,, \tag{59}$$

with

$$\mathrm{tr}_\gamma \equiv \mathrm{tr}_{\mathcal{E}} \circ \delta_\gamma : \Omega^*(M,\mathrm{End}(\mathcal{E})) \to \mathcal{C}^\infty(M,\mathbb{C})\,. \tag{60}$$

being the "quantized trace".

It follows that when restricted to the subset of Clifford type Dirac operators, the universal Dirac-Lagrangian coincides with the *Lagrangian density of General Relativity*:

$$
\begin{array}{rcl}
\mathcal{L}_{\mathrm{D}}(\not{\partial}_{\mathrm{A}}) & = & *\mathrm{tr}_\gamma(curv(\not{\partial}_{\mathrm{A}}) \\
& = & \frac{\epsilon\,\mathrm{rank}(\mathcal{E})}{4} * scal(g_{\mathrm{M}})\,.
\end{array} \tag{61}
$$