The 30th Winter School GOMETRY AND PHYSICS
Czech Republic, Srni, January 16 – 23, 2010

Geometry and Physics of Dirac Operators

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– Third lecture –

Content

1) Dirac operators of YMH type and spontaneous symmetry breaking
2) Real Dirac operators, the Standard Model and the mass of the Higgs
3) A bit more geometry concerning Dirac operators

1 Dirac operators of simple type and spontaneous symmetry breaking

Two seemingly unrelated topics:

1) The *"issue of fermionic mass"*:

$$i\partial\!\!\!/_{\rm A}\psi = m\psi \qquad \Leftrightarrow \qquad \begin{cases} i\partial\!\!\!/_{\rm A}\psi_{\rm R} &= m\psi_{\rm L}\,, \\ i\partial\!\!\!/_{\rm A}\psi_{\rm L} &= m\psi_{\rm R}\,. \end{cases} \tag{1}$$

These coupled Weyl equations are gauge invariant if and only if the Weyl spinors: $\psi_{\rm R}$, $\psi_{\rm L}$ carry the *same representation* of the underlying gauge group:

Way out:
Since the action of the gauge group is supposed to commute with the parity involution, one necessarily has to replace the constant mass matrix: $m \in {\rm End}^+_\gamma(\mathcal{E})$, by the *"Higgs field"*:

$$\Phi_{\rm H} \in \mathfrak{S}ec(M, {\rm End}^-_\gamma(\mathcal{E}))\,. \tag{2}$$

Note that this turns Dirac's first order differential operator: $i\partial\!\!\!/_{\rm A} - m$, into a Dirac operator of Yang-Mills-Higgs type: $\partial\!\!\!/_{\rm YMH} = i\partial\!\!\!/_{\rm A} - \Phi_{\rm H}$.

More explicitly, let $\mathcal{E} := \mathcal{S} \otimes_M E \twoheadrightarrow M$ be a twisted spinor bundle. Accordingly, fix the grading by the involution:

$$\tau_{\mathcal{E}} := \tau_{\rm M} \otimes \tau_{\rm E}\,, \tag{3}$$

i.e.

$$\mathcal{E}^+ \;:=\; \mathcal{S}_{\rm R} \otimes_M E_{\rm R} \oplus \mathcal{S}_{\rm L} \otimes_M E_{\rm L}\,, \tag{4}$$

$$\mathcal{E}^- \;:=\; \mathcal{S}_{\rm R} \otimes_M E_{\rm L} \oplus \mathcal{S}_{\rm L} \otimes_M E_{\rm R}\,. \tag{5}$$

Whence, the Clifford action and the action of the Higgs field reads, respectively:

$$\gamma_{\mathcal{E}} \;:=\; \gamma_{\rm W} \otimes {\rm id}_{\rm E}\,, \tag{6}$$

$$\Phi_{\rm H} \;:=\; {\rm id}_{\mathcal{S}} \otimes \phi_{\rm H}\,. \tag{7}$$

It is common to re-write this as follows:

$$\mathcal{E} \equiv \begin{array}{c} \mathcal{E}_{\rm R} \\ \oplus \\ \mathcal{E}_{\rm L} \end{array} := \begin{array}{c} \mathcal{S} \otimes_M E_{\rm R} \\ \oplus \\ \mathcal{S} \otimes_M E_{\rm L} \end{array}\,. \tag{8}$$

Then, respectively, the grading involution and the Dirac operator reads:

$$\tau_{\mathcal{E}} = \begin{pmatrix} \tau_{\rm M} & 0 \\ 0 & -\tau_{\rm M} \end{pmatrix}, \qquad i\partial\!\!\!/_{\rm A} - \Phi_{\rm H} = \begin{pmatrix} i\partial\!\!\!/_{\rm A} & -\Phi_{\rm H,RL} \\ -\Phi_{\rm H,LR} & i\partial\!\!\!/_{\rm A} \end{pmatrix}. \tag{9}$$

How does one get back Dirac's original equation from
Yang-Mills-Higgs type Dirac operators?

2) *Einstein's "biggest blunder"*:

$$\mathcal{I}_{\mathrm{EH},\Lambda} := \int_M *(scal(g_{\mathrm{M}}) + \Lambda)\,, \tag{10}$$

with $\Lambda \in \mathbb{R}$ being the "cosmological constant".

According to *"Lovelock's Theorem"*, the functional $\mathcal{I}_{\mathrm{EH},\Lambda}$ is the most general functional whose Euler-Lagrange equations fulfil:

– The field equations with respect to g_{M} are tensorial,
– The field equations are linear to highest order,
– The field equations are div-free.

Indeed,

$$\mathcal{EL}_{\mathrm{EH},\Lambda} = 0 \quad \Leftrightarrow \quad \mathcal{R}ic(g_{\mathrm{M}}) = \Lambda\, g_{\mathrm{M}}\,. \tag{11}$$

Hence, the stationary points of $\mathcal{I}_{\mathrm{EH},\Lambda}$ are Einstein manifolds with scalar curvature satisfying

$$scal(g_{\mathrm{M}}) \sim \Lambda\,. \tag{12}$$

How to describe this in terms of Dirac operators?

Lemma 1.1 *Let $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^- \twoheadrightarrow M$ be an Hermitian vector bundle. The restriction of the universal Dirac-Lagrangian to the subset of symmetric Dirac operators of Yang-Mills-Higgs type gives rise to the density:*

$$\mathcal{L}_{\mathrm{D}}(\not{\partial}_{\mathrm{YMH}}) = -\tfrac{\epsilon\,\mathrm{rank}(\mathcal{E})}{4}\ *(scal(g_{\mathrm{M}}) + \Lambda_{\mathrm{H}})\ , \tag{13}$$

with the "cosmological term" being given by

$$\Lambda_{\mathrm{H}} := \lambda\, \|\Phi_{\mathrm{H}}\|^2\,. \tag{14}$$

Here, $\lambda \in \mathbb{R}$ is a purely numerical constant that is basically determined by $\dim(M)$ *and*

$$\|\Phi_{\mathrm{H}}\|^2 := \mathrm{tr}_{\mathcal{E}}(\Phi_{\mathrm{H}}^\dagger \circ \Phi_{\mathrm{H}})\,. \tag{15}$$

The proof is a straightforward calculation of the Dirac potential V_{D}. Furthermore, one notices the vanishing of the Dirac field ξ_{D} of both simple type and Yang-Mills-Higgs type Dirac operators.

The Einstein equation implies that $\Lambda_{\rm H}$ must be constant. Accordingly, an admissible $(M, g_{\rm M})$ must be an Einstein manifold with the scalar curvature fulfilling:

$$scal(g_{\rm M}) \sim \Lambda_{\rm H}\,. \tag{16}$$

However,

$$\Lambda_{\rm H} = const. \qquad \Rightarrow \quad \Phi_{\rm H} \in \mathfrak{S}ec(M, S_{\Lambda_{\rm H}/\lambda})\,, \tag{17}$$

with

$$S_{\Lambda_{\rm H}/\lambda} \hookrightarrow {\rm End}_\gamma(\mathcal{E}) \twoheadrightarrow M \tag{18}$$

being the sphere sub-bundle of radius $|\Lambda_{\rm H}/\lambda|$.

Note that $\Lambda_{\rm H} = 0 \;\Leftrightarrow\; \Phi_{\rm H} = 0$ corresponds to the case where $g_{\rm M}$ and $\Phi_{\rm H}$ are treated as independent variables.

Lemma 1.2 *Assume that the Yang-Mills gauge group:*

$$\mathcal{G}_{\rm YM} := {\rm Aut}_\gamma(\mathcal{E}) \subset \mathcal{G}_{\rm D,tot}\,, \tag{19}$$

acts transitively on $S_{\Lambda_{\rm H}/\lambda} \twoheadrightarrow M$. *In this case, any choice of a "fermionic mass matrix":*

$$m_{\rm D} \in S_{\Lambda_{\rm H}/\lambda} \subset {\rm End}_\gamma(\mathcal{E})\,, \tag{20}$$

yields a reduction of $\mathcal{G}_{\rm YM}$ *to the isotropy group:*

$$\mathcal{I}(m_{\rm D}) := \{h \in \mathcal{G}_{\rm YM} \,|\, [h, m_{\rm D}] = 0\}\,. \tag{21}$$

□

Note that for $\mathcal{E} = \mathcal{S} \otimes_M E \twoheadrightarrow M$:

$${\rm Aut}_\gamma(\mathcal{E}) = {\rm Aut}(E)\,, \tag{22}$$

which is the Yang-Mills gauge group of $E \twoheadrightarrow M$.

Remark:
The mechanism of *"spontaneous symmetry breaking"* in terms of Yang-Mills-Higgs type Dirac operators is analogous to the mechanism of spontaneous symmetry breaking provided by the usual Higgs potential.

In particular, since

$$\mathrm{spec}(m_{\mathrm{D}}) = const. \tag{23}$$

throughout M, the Clifford module bundle decomposes:

$$\mathcal{E} = \ker(m_{\mathrm{D}}) \oplus \left(\bigoplus_{m \in \mathrm{spec}(m_{\mathrm{D}})} \mathcal{E}_m = \mathcal{E}_m^+ \oplus \mathcal{E}_m^- \right) . \tag{24}$$

When restricted to an eigen bundle of m_{D}, the quantized Yang-Mills-Higgs connection formally looks like Dirac's (gauge covariant) first order differential operator:

$$\not{\partial}_{\mathrm{YMH}}|_{\mathcal{E}_m} = \not{\partial}_{\mathrm{A}} + \mu \,, \tag{25}$$

where,

$$\mu \in \mathrm{End}^-_\gamma(\mathcal{E}_m) \,. \tag{26}$$

In the case of $\mathcal{E} = \mathcal{S} \otimes_M E \twoheadrightarrow M$, it follows that

$$\mathcal{E} = \mathcal{S} \otimes_M \left(\ker(m_{\mathrm{D}}) \oplus \left(\bigoplus_{m \in \mathrm{spec}(m_{\mathrm{D}})} E_m \right) \right) . \tag{27}$$

Whence, for non-degenerated mass spectra all eigen bundles $E_m \twoheadrightarrow M$ are complex line bundles:

$$\not{\partial}_{\mathrm{A}} + \mu = \begin{pmatrix} \not{\partial}_{\mathrm{A}} & m \\ m & \not{\partial}_{\mathrm{A}} \end{pmatrix} : \begin{array}{c} \mathfrak{Sec}(M, \mathcal{S}_{\mathrm{R}} \otimes_M E_m) \\ \oplus \\ \mathfrak{Sec}(M, \mathcal{S}_{\mathrm{L}} \otimes_M E_m) \end{array} \longrightarrow \begin{array}{c} \mathfrak{Sec}(M, \mathcal{S}_{\mathrm{L}} \otimes_M E_m) \\ \oplus \\ \mathfrak{Sec}(M, \mathcal{S}_{\mathrm{R}} \otimes_M E_m) \end{array} . \tag{28}$$

This may happen only if $\mathcal{H}_{\mathrm{YM}} \simeq U(1)$. Also, the complex vector bundle $\ker(m_{\mathrm{D}}) \twoheadrightarrow M$ carries the trivial representation of the reduced gauge group $\mathcal{H}_{\mathrm{YM}}$.

Furthermore, since a connection on $\mathcal{E} \twoheadrightarrow M$ is compatible with the gauge reduction:

$$\mathcal{G}_{\mathrm{YM}} \rightsquigarrow \mathcal{H}_{\mathrm{YM}} \simeq \mathcal{I}(m_{\mathrm{D}}), \tag{29}$$

if and only if

$$\nabla_X^{\mathrm{End}(\mathcal{E})} m_{\mathrm{D}} = 0 \,, \tag{30}$$

for all $X \in \mathfrak{Sec}(M, TM)$, the mapping:

$$\begin{aligned} m^2_{\mathrm{YM}} : \mathcal{A}_{\mathrm{Cl}}(\mathcal{E}) &\longrightarrow \mathcal{C}^\infty(M) \\ \partial_{\mathrm{A}} &\mapsto 2^{-n} \, \|\partial_{\mathrm{A}}^{\mathrm{End}(\mathcal{E})} m_{\mathrm{D}}\|^2 \,, \end{aligned} \tag{31}$$

yields the usual *"Yang-Mills mass matrix"*. Its rank equals the co-dimension of the sphere sub-bundle $\mathcal{S}_{\Lambda_{\mathrm{D}}/\lambda} \subset \mathrm{End}_\gamma(\mathcal{E})$. The defect of m^2_{YM} equals the rank of the reduced gauge group $\mathcal{H}_{\mathrm{YM}} \subset \mathrm{Aut}_\gamma(\mathcal{E})$.

Note that

$$\partial_{\rm A} = \partial'_{\rm A} + \alpha_{\rm G}\,, \tag{32}$$

with $\partial'_{\rm A}$ being $\mathcal{H}_{\rm YM}-$reducible. Then,

$$m^2_{\rm YM}(\partial_{\rm A})\ \sim\ \|m_{\rm D}\|^2\,\langle{\rm ad}(T_a),{\rm ad}(T_b)\rangle\,g_{\rm M}(\alpha^a_{\rm G},\alpha^b_{\rm G})\,, \tag{33}$$

where $\alpha_{\rm G} \stackrel{\rm loc.}{=} \alpha^a_{\rm G}\otimes T_a \in \Omega^1(U,{\rm End}_\gamma(\mathcal{E}))$, such that $[T_a, m_{\rm D}] \neq 0$.

2 Real Dirac operators, the STM and the mass of the Higgs

Definition 2.1 *A Hermitian Clifford module bundle* $(\mathcal{E}, \gamma_{\mathcal{E}}) \twoheadrightarrow (M, g_{\rm M})$ *is called a "real Clifford module bundle", if there exists a* $\mathbb{C}-$*anti-linear involution* $J_{\mathcal{E}}$ *making the complex vector bundle* $\mathcal{E} = \mathcal{M}_{\mathcal{E}} \otimes \mathbb{C} \twoheadrightarrow M$ *real and*

$$\begin{aligned} J_{\mathcal{E}} \circ \gamma_{\mathcal{E}}(\alpha) &= \pm\gamma_{\mathcal{E}}(\alpha) \circ J_{\mathcal{E}}\,, \\ J_{\mathcal{E}} \circ \tau_{\mathcal{E}} &= \pm\tau_{\mathcal{E}} \circ J_{\mathcal{E}}\,, \\ \langle J_{\mathcal{E}}(z), J_{\mathcal{E}}(w)\rangle_{\mathcal{E}} &= \pm\langle w, z\rangle_{\mathcal{E}}\,, \end{aligned} \tag{34}$$

for all $\alpha \in T^*M$ *and* $z, w \in \mathcal{E}$.

In particular, a real Clifford module bundle is called a "Majorana module bundle", provided

$$J_{\mathcal{E}} \circ \tau_{\rm M} = -\tau_{\rm M} \circ J_{\mathcal{E}}\,. \tag{35}$$

Definition 2.2 *A real Clifford module bundle* $(\mathcal{S}, \langle\cdot,\cdot\rangle_{\mathcal{S}}, \tau_{\mathcal{S}}, \gamma_{\mathcal{S}}, J_{\mathcal{S}})$ *is called a "Dirac module bundle", provided there is a Majorana module bundle* $(\mathcal{W}, \langle\cdot,\cdot\rangle_{\mathcal{W}}, \tau_{\mathcal{W}}, \gamma_{\mathcal{W}}, J_{\mathcal{W}})$ *over* $(M, g_{\rm M})$*, such that*

$$\mathcal{S} := {}^2\mathcal{W} = \mathcal{W} \otimes \mathbb{C}^2 \tag{36}$$

and

$$\tau_{\mathcal{S}} = \begin{pmatrix} \mathrm{id}_{\mathcal{W}} & 0 \\ 0 & -\mathrm{id}_{\mathcal{W}} \end{pmatrix}, \quad \gamma_{\mathcal{S}} = \begin{pmatrix} 0 & \gamma_{\mathcal{W}} \\ \gamma_{\mathcal{W}} & 0 \end{pmatrix}, \quad J_{\mathcal{S}} = \begin{pmatrix} 0 & J_{\mathcal{W}} \\ J_{\mathcal{W}} & 0 \end{pmatrix}. \tag{37}$$

Finally,

$$\left\langle \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\mathcal{S}} = \langle u_1, v_2\rangle_{\mathcal{W}} \pm \langle v_1, u_2\rangle_{\mathcal{W}}\,, \tag{38}$$

depending on whether $\langle J_{\mathcal{W}}(u), J_{\mathcal{W}}(v)\rangle_{\mathcal{W}} = \pm\langle v, u\rangle_{\mathcal{W}}$*, for all* $u, v \in \mathcal{W}$.

Definition 2.3 *The doubling*

$$(\mathcal{P}, \langle\cdot,\cdot\rangle_{\mathcal{P}}, \tau_{\mathcal{P}}, \gamma_{\mathcal{P}}, J_{\mathcal{P}}) \tag{39}$$

of a real Clifford module bundle $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}})$ *is called a "Pauli module bundle", whereby*

$$\tau_{\mathcal{P}} := \begin{pmatrix} \tau_{\mathcal{E}} & 0 \\ 0 & -\tau_{\mathcal{E}} \end{pmatrix}, \quad \gamma_{\mathcal{P}} := \begin{pmatrix} \gamma_{\mathcal{E}} & 0 \\ 0 & \gamma_{\mathcal{E}} \end{pmatrix}, \quad J_{\mathcal{P}} := \begin{pmatrix} 0 & J_{\mathcal{E}} \\ J_{\mathcal{E}} & 0 \end{pmatrix}. \tag{40}$$

Proposition 2.1 *The most general real Dirac operator on a Pauli module bundle reads:*

$$D\!\!\!/_{\mathcal{P}} = \begin{pmatrix} D\!\!\!/_{\mathcal{E}} & \phi_{\mathcal{E}} - \mathcal{F}_{\mathcal{E}} \\ \phi_{\mathcal{E}} + \mathcal{F}_{\mathcal{E}} & D\!\!\!/_{\mathcal{E}}^{\rm cc} \end{pmatrix}. \tag{41}$$

Here, respectively, $D\!\!\!/_{\mathcal{E}}$ *is any Dirac operator on the underlying real Clifford module bundle* $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}})$ *and*

$$\begin{aligned} \phi_{\mathcal{E}}^{\rm cc} &= +\phi_{\mathcal{E}}\,, \\ \mathcal{F}_{\mathcal{E}}^{\rm cc} &= -\mathcal{F}_{\mathcal{E}} \end{aligned} \tag{42}$$

are general even endomorphisms thereof.

The proof needs some thought but is true.

Definition 2.4 *Let* $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}})$ *be a real Clifford module bundle. A real Dirac operator* $\not{P}_{\rm D} \in \mathcal{D}_\gamma(\mathcal{P})$ *is called "of Pauli type", provided*

$$\begin{aligned}\phi_{\mathcal{E}} &:= 0\,,\\ \mathcal{F}_{\mathcal{E}} &:= i\delta_\gamma(F_{\rm D})\end{aligned} \tag{43}$$

is defined by the relative curvature of $\not{D}_{\mathcal{E}} \in \mathcal{D}_\gamma(\mathcal{E})$.

Note that $\delta_\gamma(F_{\rm D}) \in \mathfrak{Sec}(M, {\rm End}^+(\mathcal{E}))$ is even and real for all real Dirac operators $\not{D}_{\mathcal{E}} \in \mathcal{D}(\mathcal{E})$. Whence, $\mathcal{F}_{\mathcal{E}}^{\rm cc} = -\mathcal{F}_{\mathcal{E}}$, as required.

Remark:
When restricted to the complexification of the distinguished real sub-bundle:

$$\mathcal{V}_{\mathcal{P}} := \left\{ {}^2\mathfrak{z} \equiv \begin{pmatrix}\mathfrak{z}\\ \mathfrak{z}\end{pmatrix} \in \mathcal{P} \,|\, \mathfrak{z} \in \mathcal{M}_{\mathcal{E}} \right\} \hookrightarrow \mathcal{P} \twoheadrightarrow M\,, \tag{44}$$

whereby $\mathcal{M}_{\mathcal{E}}^{\mathbb{C}} = \mathcal{E}$, the two mappings:

$$\begin{aligned}\mathfrak{Sec}(M,\mathcal{E}) \times \mathcal{D}(\mathcal{E}) &\longrightarrow \Omega^n(M,\mathbb{C})\\ (\psi, \not{D}_{\mathcal{E}}) &\mapsto \begin{cases} *\langle \psi, \not{D}_{\mathcal{E}}\psi\rangle_{\mathcal{E}}\,,\\ *\langle {}^2\psi, \not{P}_{\rm D}\,{}^2\psi\rangle_{\mathcal{P}}\end{cases}\end{aligned} \tag{45}$$

provide the same information.

Consider the following sequence of real Clifford module bundles and Dirac operators, starting with a Majorana module bundle $(\mathcal{W}, \langle\cdot,\cdot\rangle_{\mathcal{W}}, \tau_{\mathcal{W}}, \gamma_{\mathcal{W}}, J_{\mathcal{W}})$ and a Yang-Mills-Higgs connection:

$$\begin{array}{ccccccc}\mathcal{W} & \longrightarrow & \mathcal{S} & \longrightarrow & \mathcal{E} \equiv {}^2\mathcal{S} & \longrightarrow & \mathcal{P}\\ \not{\partial}_{\rm YMH} & \longrightarrow & \not{\partial}_{\rm A} + i\mu_{\rm D} & \longrightarrow & \not{\partial}_{\mathcal{A}} + i\mu_{\mathcal{D}} & \longrightarrow & \not{P}_{\rm D}\,,\end{array} \tag{46}$$

where, respectively:

$$\not{\partial}_{\rm A} + i\mu_{\rm D} \;:=\; \begin{pmatrix} 0 & \not{\partial}_{\rm A} - i\varphi_{\rm D}\\ \not{\partial}_{\rm A} + i\varphi_{\rm D} & 0\end{pmatrix} \in \mathcal{D}_\gamma(\mathcal{S}) \tag{47}$$

is of *simple type* and

$$\not{\partial}_{\mathcal{A}} + i\mu_{\mathcal{D}} \;:=\; \begin{pmatrix} \not{\partial}_{\rm A} + i\mu_{\rm D} & 0\\ 0 & (\not{\partial}_{\rm A} + i\mu_{\rm D})^{\rm cc}\end{pmatrix} \in \mathcal{D}_\gamma(\mathcal{E}) \tag{48}$$

is *real and of simple type.*

Theorem 2.1 *Let $(\mathcal{W}, \langle\cdot,\cdot\rangle_{\mathcal{W}}, \tau_{\mathcal{W}}, \gamma_{\mathcal{W}}, J_{\mathcal{W}})$ be a Majorana module, such that $\gamma^{\rm cc}_{\mathcal{W}} = -\gamma_{\mathcal{W}}$. Also, let $\not\partial_{\rm YMH} = \not\partial_{\rm A} + i\varphi_{\rm D} \in \mathcal{D}_\gamma(\mathcal{W})$ be a YMH type Dirac operator.*

Consider the real Dirac operator of simple type, called "Dirac-Yukawa-Majorana operator":

$$\not\partial_{\mathcal{A}} + i\mu_{\rm YM} \;:=\; \begin{pmatrix} \not\partial_{\rm A} + i\mu_{\rm D} & i\mu_{\rm M} \\ -i\mu_{\rm M} & (\not\partial_{\rm A} + i\mu_{\rm D})^{\rm cc} \end{pmatrix} \in \mathcal{D}_\gamma(\mathcal{E})\,. \tag{49}$$

Here, the **Majorana mass operator** $\mu_{\rm M} \in \Omega^0(M, {\rm End}^+_\gamma(\mathcal{S}))$ *is real and constant.*

The Euler-Lagrange equations of the fermionic part of the density:

$$\mathcal{L}_{\rm D,tot}(\not P_{\rm D}, {}^2\psi) = *\Big(\langle {}^2\psi, \not P_{\rm D}\,{}^2\psi\rangle_{\mathcal{P}} + {\rm tr}_\gamma\Big(curv(\not P_{\rm D}) - \varepsilon\, ev_g(\omega^2_{\rm D})\Big)\Big) \tag{50}$$

read:

$$\begin{aligned} i\not\partial_{\rm A}\psi &= \mu_{\rm D}\psi + \mu_{\rm M}\psi^{\rm cc}\,, & (51)\\ (i\not\partial_{\rm A}\psi)^{\rm cc} &= \mu^{\rm cc}_{\rm D}\psi^{\rm cc} + \mu_{\rm M}\psi\,. & (52)\end{aligned}$$

When restricted to $\tau_{\mathcal{S}}\psi = \psi$, these equations become equivalent to:

$$i\not\partial_{\rm A}\chi = \varphi_{\rm D}\chi + m_{\rm M}\chi^{\rm cc} \qquad \Leftrightarrow \qquad \left\{\begin{aligned} i\not\partial\nu &= m_{{\rm D},\nu}\nu + m_{{\rm M},\nu}\nu^{\rm cc}\,,\\ i\not\partial_{\rm A} e &= \varphi_{{\rm D},e}e\,,\end{aligned}\right. \tag{53}$$

$$(i\not\partial_{\rm A}\chi)^{\rm cc} = \varphi^{\rm cc}_{\rm D}\chi^{\rm cc} + m_{\rm M}\chi\,, \tag{54}$$

whereby $\chi = (\nu, e) \in \mathfrak{Sec}(M, \mathcal{W} = \mathcal{W}_\nu \oplus \mathcal{W}_e)$. The splitting of the Majorana module bundle into an "uncharged" and "charged" sector is defined in terms of the kernel of the Majorana mass operator.

Furthermore, when restricted to the thus defined class of Pauli type Dirac operators, the universal Dirac-Lagrangian explicitly reads:

$$\mathcal{L}_{\rm D}(\not P_{\rm D}) =$$
$$\int_M *\Big({\rm tr}_\gamma(curv(\not\partial_{\mathcal{A}})) + a_1\,{\rm tr}_g F^2_{\mathcal{A}} + a_2\,\epsilon\,{\rm tr}_g(\partial^{{\rm End}(\mathcal{E})}_{\mathcal{A}}\mu_{\rm YM})^2 - a_2\,{\rm tr}_{\mathcal{E}}\mu^4_{\rm YM} - a_4\,{\rm tr}_{\mathcal{E}}\mu^2_{\rm YM}\Big) \tag{55}$$

with $a_1 = (n-3),\ a_2 = 2(n-2)(\frac{n-1}{n})^2,\ a_3 = 2\frac{(n-1)^3}{n^2},\ a_4 = 2\,.$

Furthermore,

$$\begin{aligned} {\rm tr}_g(\partial^{{\rm End}(\mathcal{E})}_{\mathcal{A}}\mu_{\rm YM})^2 &= -4{\rm Re}\,{\rm tr}_g(\partial^{{\rm End}(\mathcal{W}_e)}_{\rm A}\varphi_{{\rm D},e})^2\,, & (56)\\ a\,{\rm tr}_{\mathcal{E}}\mu^4_{\rm YM} + {\rm tr}_{\mathcal{E}}\mu^2_{\rm YM} &= 4{\rm Re}\Big(a\,{\rm tr}_{\mathcal{W}_e}\varphi^4_{{\rm D},e} - {\rm tr}_{\mathcal{W}_e}\varphi^2_{{\rm D},e} + \Lambda_{{\rm DM},\nu}\Big)\,, & (57)\end{aligned}$$

whereby $a \equiv 2\,\frac{(n-1)^3}{n^2}$ and

$$\begin{aligned} \Lambda_{{\rm DM},\nu} \equiv\ & a\,{\rm tr}_{\mathcal{W}_\nu}m^4_{{\rm D},\nu} - {\rm tr}_{\mathcal{W}_\nu}m^2_{{\rm D},\nu} + a\,{\rm tr}_{\mathcal{W}_\nu}m^4_{{\rm M},\nu} - {\rm tr}_{\mathcal{W}_\nu}m^2_{{\rm M},\nu}\\ & - 2a\,{\rm tr}_{\mathcal{W}_\nu}(m_{{\rm D},\nu}\circ m_{{\rm M},\nu})^2 \end{aligned} \tag{58}$$

is a "true cosmological constant".

The proof basically consists of a (rather tedious) calculation of the Dirac potential of the Pauli type Dirac operator associated with the Dirac-Yukawa-Majorana operator.

Remarks:

– To obtain a non-trivial "kinetic term" $\partial_{\mathrm{A}}^{\mathrm{End}(\mathcal{W}_{\mathrm{e}})}\varphi_{\mathrm{D},e}$ for the Higgs field, it is essential that the Dirac-Yukawa-Majorana operator is of simple type. The same holds true with respect to the specific form of the "cosmological constant" $\Lambda_{\mathrm{DM},\nu}$.

– To obtain the correct Dirac/Majorana equation, it is crucial that the Dirac-Yukawa-Majorana operator is induced by a Dirac operator of Yang-Mills-Higgs type.

– The specific form of Pauli type Dirac operators is essential to obtain the correct relative signs within the various terms of $\mathcal{L}_{\mathrm{D}}(\not{P}_{\mathrm{D}})$ and to cancel out the curvature term in the fermionic part of the total Lagrangian.

The connection enters directly the fermionic part of the total action. In contrast, connections enter the bosonic part of the total action only via their curvatures. Curvature terms never enter the fermionic action for reasons of renormalization and connections never enter the bosonic action for reasons of gauge invariance.

– The Standard Model Action is recovered by disregarding Majorana masses:

$$\mu_{\mathrm{M}} \equiv 0\,. \tag{59}$$

Whence, Yang-Mills-Higgs type Dirac operators

$$\not{\partial}_{\mathrm{YMH}} = \delta_\gamma \circ (\partial_{\mathrm{A}} + H) \tag{60}$$

maybe regarded to provide the appropriate "square root" of the STM action.

The geometrical description of the Standard Model in terms of Yang-Mills-Higgs type Dirac operators also allow to make a prediction with regard to the *mass of the Higgs boson*:

$$m_{\mathrm{H}} = 184 \pm 20 \text{ GeV}\,. \tag{61}$$

This holds true in "top-mass approximation" and on "one-loop level".

According to the usual (non-geometrical) description of the Standard Model the mass of the Higgs boson is expected to be within the range:

$$114 \text{ GeV} < m_{\mathrm{H}} \leq 193 \text{ GeV}\,. \tag{62}$$

– The Yang-Mills Action is recovered by also disregarding Dirac masses:

$$\mu_{\mathrm{D}} \equiv 0\,. \tag{63}$$

Whence, Clifford type Dirac operators

$$\not{\partial}_{\mathrm{A}} = \delta_\gamma \circ \partial_{\mathrm{A}} \tag{64}$$

maybe regarded to provide the appropriate "square root" of the YM action.

3 A bit more geometry concerning Dirac operators

$$\begin{array}{rcl} \mathcal{L}_{\mathrm{D}} : \mathcal{D}(\mathcal{E}) & \longrightarrow & \Omega^n(M,\mathbb{C}) \\ \not{D} & \mapsto & *\mathrm{tr}_{\mathcal{E}} V_{\mathrm{D}} \end{array} \tag{65}$$

is invariant with respect to the action of the translation subgroup $\mathcal{T}_{\mathrm{D}} \subset \mathcal{G}_{\mathrm{D}}$.

Whence it is constant along the *"Dirac principal fibering"*:

$$\begin{array}{rcl} \mathcal{D}(\mathcal{E}) & \twoheadrightarrow & \mathcal{D}(\mathcal{E})/\mathcal{T}_{\mathrm{D}} \\ \not{D} & \mapsto & [\not{D}]\,. \end{array} \tag{66}$$

Consider the case of a twisted Grassmann bundle:

$$\mathcal{E} := \Lambda_{\mathrm{M}} \otimes_M E \twoheadrightarrow M\,. \tag{67}$$

In this case:

$$\mathcal{D}(\mathcal{E})/\mathcal{T}_{\mathrm{D}} \;\simeq\; \mathcal{M}_{\mathrm{D}} := \mathfrak{Sec}(M, \mathcal{E}_{\mathrm{EH}} \times_M \mathrm{End}(\mathcal{E}))/_{\sim}\,, \tag{68}$$

where

$$(g_{\mathrm{M}}, \Phi) \;\sim\; (g'_{\mathrm{M}}, \Phi') \quad :\Leftrightarrow \quad \left\{ \begin{array}{rcl} g'_{\mathrm{M}} & = & g_{\mathrm{M}}\,, \\ \Phi' & = & \Phi + \gamma(\alpha)\,, \quad \alpha \in \Omega^1(M, \mathrm{End}_\gamma(\mathcal{E}))\,. \end{array} \right. \tag{69}$$

The $\mathcal{T}_{\mathrm{D}}-$principal fibering $\mathcal{D}(\mathcal{E}) \twoheadrightarrow \mathcal{M}_{\mathrm{D}}$ is trivial but in a non-canonical way, in general. A natural class of global sections are provided by connections on $E \twoheadrightarrow M$:

$$\begin{array}{rcl} \sigma_{\mathrm{A}} : \mathcal{M}_{\mathrm{D}} & \longrightarrow & \mathcal{D}(\mathcal{E}) \\ [(g_{\mathrm{M}}, \Phi)] & \mapsto & \mathrm{d}_{\mathrm{A}} + \delta_{\mathrm{A}} + \Phi\,. \end{array} \tag{70}$$

In particular, because of the canonical inclusion:

$$\begin{array}{rcl} \mathfrak{Sec}(M, \mathcal{E}_{\mathrm{EH}}) & \hookrightarrow & \mathcal{M}_{\mathrm{D}} \\ g_{\mathrm{M}} & \mapsto & [(g_{\mathrm{M}}, 0)]\,, \end{array} \tag{71}$$

every connection on $E \twoheadrightarrow M$ yields the inclusion:

$$\begin{array}{rcl} \sigma_{\mathrm{A}} : \mathfrak{Sec}(M, \mathcal{E}_{\mathrm{EH}}) & \hookrightarrow & \mathcal{D}(\mathcal{E}) \\ g_{\mathrm{M}} & \mapsto & \mathrm{d}_{\mathrm{A}} + \delta_{\mathrm{A}}\,, \end{array} \tag{72}$$

such that

$$\sigma^*_{\mathrm{A}} \mathcal{L}_{\mathrm{D}} \;\sim\; \mathcal{L}_{\mathrm{EH}}\,. \tag{73}$$

This is independent of the choice of the connection on $E \twoheadrightarrow M$ (i.e. independent of the section $\sigma_{\rm A}$), because of the translational invariance of the universal Dirac-Lagrangian. Thus, the image of the inclusion

$$\sigma_{\rm A} : \mathfrak{S}ec(M, \mathcal{E}_{\rm EH}) \hookrightarrow \mathcal{D}(\mathcal{E}) \tag{74}$$

maybe geometrically interpreted as "making the metric gauge covariant".

Note that for trivial $E = M \times \mathbb{C}^N \twoheadrightarrow M$, there is a canonical section

$$\begin{aligned} \sigma_0 : \mathfrak{S}ec(M, \mathcal{E}_{\rm EH}) &\hookrightarrow \mathcal{D}(\mathcal{E}) \\ g_{\rm M} &\mapsto d + \delta \,, \end{aligned} \tag{75}$$

such that all other sections $\sigma_{\rm A}$ read:

$$\sigma_{\rm A}(g_{\rm A}) = d + \delta + A \,, \quad A \in \Omega^1(M, {\rm End}_\gamma(\mathcal{E})) \,. \tag{76}$$

Whence, the Yang-Mills gauge potential maybe considered as a "perturbation" of the metric.

Theorem 3.1 *Let $(\mathcal{E}, \langle\cdot,\cdot\rangle_{\mathcal{E}}, \tau_{\mathcal{E}}, \gamma_{\mathcal{E}}, J_{\mathcal{E}})$ be a real Clifford bi-module.*

On the induced Pauli module there exists a class of real Dirac operators of simple type:

$$\not{D} = \not{\partial}_{\mathcal{A}} + \tau_{\mathcal{P}} \circ \phi \,, \tag{77}$$

such that the restriction of the universal Dirac-Lagrangian to this class decomposes as:

$$\begin{aligned} \mathcal{L}_{\rm D}(\not{D}) &= *{\rm tr}_\gamma\Big(curv(\not{D}) - \epsilon ev_g(\omega_{\rm D}^2)\Big) \\ &= *\Big({\rm tr}_\gamma(curv(\not{\partial}_{\mathcal{A}})) - {\rm tr}_{\mathcal{P}}\phi^2\Big) \\ &= \mathcal{L}_{\rm EH} + \mathcal{L}_{\rm YM} + \mathcal{L}_{\rm H} \in \Omega^n(M) \,. \end{aligned} \tag{78}$$

Here,

$$\begin{aligned} \mathcal{L}_{\rm EH} &\equiv \lambda_{\rm EH} * scal(g_{\rm M}) \,, \\ \mathcal{L}_{\rm YM} &\equiv \lambda_{\rm YM}\, {\rm tr}_{\mathcal{E}} F_{\rm A} \wedge * F_{\rm A} \,, \\ \mathcal{L}_{\rm H} &\equiv \lambda_{\rm H}\, {\rm tr}_{\mathcal{E}}(\, {\rm d}_{\rm A}\varphi \wedge * \,{\rm d}_{\rm A}\varphi) - V_{\rm H} \end{aligned} \tag{79}$$

and the *Higgs potential*:

$$V_{\rm H} = \lambda\, {\rm tr}_{\mathcal{E}}\varphi^4 - \mu^2\, {\rm tr}_{\mathcal{E}}\varphi^2 \,. \tag{80}$$

$\lambda_{\rm EH}$, $\lambda_{\rm YM}$, $\lambda_{\rm H}$, λ and μ^2 are appropriate real constants, basically fixed by $\dim(M)$ and ${\rm rank}(\mathcal{E})$.

"...It often happens that the requirements of simplicity and beauty are the same, but where they clash the latter must take precedence"

("Paul Dirac – The man and his work", Cambridge Univ. Press 1998)

End of lectures

Thank you!