Dirac cohomology of Harish-Chandra modules

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Background.

G a Lie group acting on a manifold X

$\Rightarrow G$ acts on functions on X,via

$$(g \cdot f)(x) = f(g^{-1} \cdot x).$$

E.g.:

$C^{\infty}(X)$ - a smooth representation of G

$L^2(X)$ wrt G-invt dx - a unitary rep. of G

Δ a G-invariant diff. op. on X

$\Rightarrow$ any eigenspace of Δ is G-invariant.

Conversely, Δ typically acts by scalars on irreducible G-subspaces.

So decomposing the representation is related to finding Δ-eigenspaces.

G: connected real reductive Lie group
(e.g. $SL(n,\mathbb{R})$, $U(p,q)$, $Sp(2n,\mathbb{R})$,...)

Θ: Cartan involution ($\Theta(g) =^t \bar{g}^{-1}$)

$K = G^{\Theta}$: maximal compact subgroup
($SO(n)$, $U(p) \times U(q)$, $U(n)$,...)

Representation of G:

a complex topological vector space V

with a continuous G-action

by linear operators

V_K: the space of K-finite vectors in V

V_K has an action of the Lie algebra $\mathfrak{g}_0$

(K-finite $\Rightarrow$ smooth)

$\mathfrak{g} = (\mathfrak{g}_0)_{\mathbb{C}}$ also acts

Get a $(\mathfrak{g}, K)$-module; an algebraic version of V.

A $(\mathfrak{g}, K)$-module is a vector space M

with a Lie algebra action of $\mathfrak{g}$

and a locally finite action of K,

which are compatible.

(I.e., induce the same action of
$\mathfrak{k}_0$ = the Lie algebra of K.)

Such M can be decomposed under K as

$$M = \bigoplus_{\delta \in \widehat{K}} m_\delta V_\delta.$$

M is a Harish-Chandra module if it is finitely generated and all $m_\delta < \infty$.

Example: $G = SU(1,1) \quad (\cong SL(2,\mathbb{R}))$.

The Lie algebra is

$\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ = 2x2 matrices of trace 0

$\mathfrak{g}$ has a basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The commutation relations are

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Since h spans $\mathfrak{k}$, it diagonalizes on $(\mathfrak{g}, K)$-modules and has integer eigenvalues. The possible modules are

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \cdots \\ k & k+2 & k+4 & \ldots \end{array} \tag{1}$$

$$\begin{array}{cccc} \cdots & \bullet & \bullet & \bullet \\ \ldots & -k-4 & -k-2 & -k \end{array} \tag{2}$$

$$\begin{array}{cccc} \bullet & \bullet & \cdots & \bullet \\ -n & -n+2 & \ldots & n \end{array} \tag{3}$$

$$\begin{array}{ccccc} \cdots & \bullet & \bullet & \bullet & \cdots \\ \ldots & i-2 & i & i+2 & \ldots \end{array} \tag{4}$$

where $k > 0$ and $n \geq 0$ are integers.

Each dot represents a 1-dim eigenspace for h. Numbers are the corresponding eigenvalues.

In each picture, e raises the eigenvalue by 2, and f lowers the eigenvalue by 2.

Since ef-fe=h and since we know ef at each lowest weight space and fe at each highest weight space, pictures (1), (2) and (3) define unique modules.

For picture (4), it is enough to determine $ef + fe$. We use

$$\mathrm{Cas}_{\mathfrak{g}} = \frac{1}{2}h^2 + ef + fe,$$

which commutes with $\mathfrak{g}$ and so acts by a scalar on any irreducible module.

Fixing this scalar determines the module.

(Not all values are allowed, the module may break up.)

In general, can define $\mathsf{Cas}_{\mathfrak{g}} \in Z(\mathfrak{g}) \subset U(\mathfrak{g})$:

Fix a nondegenerate invariant bilinear form B on $\mathfrak{g}$ (e.g. $\mathrm{tr}\, XY$)

Take dual bases b_i, d_i of $\mathfrak{g}$ w.r.t. B

Write $\mathsf{Cas}_{\mathfrak{g}} = \sum b_i d_i$.

In general, $Z(\mathfrak{g})$ has generators other than $\text{Cas}_{\mathfrak{g}}$;

all of them act as scalars on irred. modules;

get infinitesimal character $\chi_M : Z(\mathfrak{g}) \to \mathbb{C}$.

Harish-Chandra proved that $Z(\mathfrak{g}) \cong P(\mathfrak{h}^*)^W$, so inf. chars correspond to $\mathfrak{h}^*/W$.

Here $\mathfrak{h}$ is a CSA of $\mathfrak{g}$ and W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$.

Dirac operator on $\mathbb{R}^n$:

Look for D such that $D^2 = -\sum \partial_i^2$.

If $D = \sum e_i \partial_i$, get

$$e_i^2 = -1; \qquad e_i e_j + e_j e_i = 0, \quad i \neq j$$

So the coefficients should belong to the Clifford algebra $C(\mathbb{R}^n)$.

Identifying $\partial_i \leftrightarrow e_i$, get

$$D = \sum e_i \otimes e_i \in D_{cc}(\mathbb{R}^n) \otimes C(\mathbb{R}^n),$$

where $D_{cc}(\mathbb{R}^n)$ denotes the algebra of constant coefficient diff. ops on $\mathbb{R}^n$.

Back to G:

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition ($\mathfrak{p} = \mathfrak{k}^{\perp}$)

$C(\mathfrak{p})$: Clifford algebra of $\mathfrak{p}$ wrt B

Dirac operator:

$$D = \sum_i b_i \otimes d_i \qquad \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$$

(b_i a basis of $\mathfrak{p}$, d_i dual basis)

D is independent of choice of basis and K-invariant.

$D^2 = -\,\mathrm{Cas}_{\mathfrak{g}} \otimes 1 + diag(\mathrm{Cas}_{\mathfrak{k}}) + const.$

Parthasarathy - used D to construct the discrete series.

Let M be a $(\mathfrak{g}, K)$-module

Let S be a spin module for $C(\mathfrak{p})$
($S = \bigwedge(U)$, $U \subset \mathfrak{p}$ max. isotropic.)

$M \otimes S$ is a $(U(\mathfrak{g}) \otimes C(\mathfrak{p}), \tilde{K})$-module
($\tilde{K}$ = spin double cover of K)

In particular, D acts on $M \otimes S$

Dirac cohomology:

$$H_D(M) = \operatorname{Ker} D \Big/ \operatorname{Im} D \cap \operatorname{Ker} D$$

(introduced by Vogan)

Properties:

1. $H_D(M)$ is a $\tilde{K}$-module

2. For M unitary, D is symmetric, so

$$H_D(M) = \text{Ker}\, D = \text{Ker}\, D^2$$

and $D^2 \geq 0$ (Parthasarathy's Dirac inequality)

For $SL(2)$: exactly (1)-(3) have $H_D \neq 0$.

$H_D(M)$ is equal to h.wt.+1 and/or l.wt.-1

More notation:

$\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$: a fundamental CSA of $\mathfrak{g}$

$\Lambda \in \mathfrak{h}^*$: infinitesimal character of M (assume $\mathfrak{g}$-dominant)

$\mu \in \mathfrak{t}^* \subset \mathfrak{h}^*$: the h.wt. of $E_\mu \subset H_D(M)$

Main fact:

$$\Lambda = \mu + \rho_{\mathfrak{k}} \qquad \text{up to} \quad W$$

(conjectured by Vogan; proved by Huang-P.; generalized to other settings:
Kostant: replace $\mathfrak{k}$ by quadratic $\mathfrak{r} \subset \mathfrak{g}$
Kumar, Alexeev-Meinrenken: noncommutative equivariant cohomology
Huang-P.: $\mathfrak{g}$ superalgebra of Riemannian type
Kac-Moseneder-Frajria-Papi: $\mathfrak{g}$ affine)

Partial converse:

If M is unitary and if $\mu = w\Lambda - \rho_{\mathfrak{k}}$ is the h.wt. of some $E_\mu \subset M \otimes S$, then $E_\mu \subset H_D(M)$.

Problems:

1) Classify irreducible unitary M with $H_D \neq 0$

2) Calculate $H_D(M)$ for given M

Motivation:

1. unitarity - sharpening the Dirac inequality

2. irred. unitary M with $H_D \neq 0$ are interesting:

   - discrete series - Parthasarathy;

   - $A_q(\lambda)$ modules - Huang-Kang-P.;

   - unitary h.wt. modules - Enright, Huang-P.-Renard; more directly by Huang-P.-Protsak in special cases;

- some unipotent reps - Barbasch-P.
- also fd mod. - Kostant, Huang-Kang-P.

3. irred. unitary M with $H_D \neq 0$ should form a nice part of the unitary dual

4. H_D is related to $\mathfrak{n}$-cohomology (Huang-P.-Renard)

5. H_D is related to $(\mathfrak{g}, K)$-cohomology (Huang-Kang-P)

6. can construct reps with $H_D \neq 0$ via Dirac induction (P.-Renard)

Example: Wallach modules of $\mathfrak{sp}(2n,\mathbb{R})$

(with Huang and Protsak)

$\mathfrak{t}$ = CSA in $\mathfrak{g} = \mathfrak{sp}(2n,\mathbb{C})$ and $\mathfrak{k} = \mathfrak{gl}(n,\mathbb{C})$.

Positive roots:

compact: $e_i - e_j$, $1 \leq i < j \leq n$.

noncompact: $e_i + e_j$, $i < j$; $2e_i$, $i = 1,\dots,n$.

$$\rho = (n, n-1, \dots, 1)$$

$$\rho_{\mathfrak{k}} = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2})$$

$W_{\mathfrak{k}}$: permutations of coordinates

$W_{\mathfrak{g}}$: permutations and arbitrary sign changes

fund. ch. for $\mathfrak{g}$: $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$

fund. ch. for $\mathfrak{k}$: $x_1 \geq x_2 \geq \cdots \geq x_n$.

(open chambers: $>$ in place of $\geq$)

Let $W^1 = \{w \in W_{\mathfrak{g}} \Big| w\rho$ is $\mathfrak{k}-$dominant$\}$

Then $W^1 \leftrightarrow \mathbb{Z}_2^n$

($\forall$ sign change, $\exists!$ rearrangement.)

Wallach modules: V_k, $k \in \{1, 2, \ldots, n\}$.

V_k has lowest weight $(\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2})$.

Infinitesimal character ($\mathfrak{g}$-dominant representative):

$$\Lambda = (\frac{k}{2} + n - k, \frac{k}{2} + n - k - 1, \ldots, \frac{k}{2} + 1, \frac{k}{2},$$
$$\frac{k}{2} - 1, \frac{k}{2} - 1, \frac{k}{2} - 2, \frac{k}{2} - 2, \ldots),$$

ending with $1, 1, 0$ if k is even and with $\frac{1}{2}, \frac{1}{2}$ if k is odd.

K-types (all of multiplicity 1):

$$(\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2}) + (d_1, d_2, \ldots, d_k, 0, \ldots, 0),$$

$d_1 \geq d_2 \geq \cdots \geq d_k \geq 0$ integers of the same parity.

All d_i even: $V_k^+ \subset V_k$, irreducible

All d_i odd: $V_k^- \subset V_k$, irreducible

$k=1$: Weil representation

These modules are nice and small and appear in many situations (dual pairs, invariant theory, important in classification...)

To calculate H_D, we need to match

$$w\Lambda - \rho_{\mathfrak{k}} = [\text{h.wt. in } S + \text{l.wt. in } V_k]'$$

where $[]'$ denotes the $\mathfrak{k}$-dominant W_K-conjugate.

Highest weights in S are $\sigma\rho - \rho_k$, $\sigma \in W^1$.

Lowest weights in V_k are

$$(\frac{k}{2}, \frac{k}{2}, \ldots, \frac{k}{2}) + (0, \ldots, 0, d_k, \ldots, d_1).$$

$w\Lambda - \rho_{\mathfrak{k}}$ is $\mathfrak{k}$-dominant $\Rightarrow$ up to $\mathrm{Stab}_{W_{\mathfrak{g}}}(\Lambda)$,

$$w \in W_\Lambda = \{w \in W^1 \Big| \; w \leftrightarrow \epsilon_1 \ldots \epsilon_{n-k+1}(+-)\ldots(+-)(+)\},$$

where the last $+$ appears only when k is even.

For each such w, $\exists!$ matching σ and $d_1, \ldots, d_k$:

$$\sigma = - - \cdots - \epsilon_1 \epsilon_2 \ldots \epsilon_{n-k};$$

$d_1 = \cdots = d_k = \#$ pluses among $\epsilon_1, \ldots, \epsilon_{n-k+1}$.

Hence

$$H_D(V_k) = \bigoplus_{w \in W_\Lambda} w\Lambda - \rho_{\mathfrak{k}}.$$

Ongoing work with Barbasch:

1. many computations of H_D for unipotent reps of complex groups, some for real

2. e.g. for $GL(n,\mathbb{C})$, a unip. rep has $H_D \neq 0$ iff the corresponding Young diagram has only two columns, of opposite parity. H_D is a single K-type with multiplicity. We can write it down explicitly.

3. we also have some more general (but still special) results of the following type: suppose $\mathfrak{m}$ is a Levi factor of a parabolic and suppose π is induced from $\pi_{\mathfrak{m}}$. Then π has $H_D \neq 0$ iff $\pi_{\mathfrak{m}}$ has $H_D \neq 0$.