Lectures on nonlinear sigma-models in projective superspace

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Brief outline of the lectures:

- Matrix realization of the $\mathcal{N}$-extended super-Poincaré group.
- $\mathcal{N}$-extended superspace and superfields.
- General $\mathcal{N} = 1$ rigid supersymmetric nonlinear σ-models.
- General $\mathcal{N} = 2$ rigid supersymmetric nonlinear σ-models in $\mathcal{N} = 1$ superspace (formulation in terms of chiral superfields).
- Adding auxiliary bosonic directions to $\mathcal{N} = 2$ superspace.
- Projective superspace approach.
- Projective multiplets.
- Generalized Legendre transform construction.
- $\mathcal{N} = 2$ σ-models on cotangent bundles of Kähler manifolds.

The concept of supersymmetry was introduced in theoretical physics in the early 1970s. It is a symmetry between bosons and fermions in relativistic theories (field theory, string theory, etc.). The discovery of supersymmetry immediately led to the appearance of new research directions in high-energy physics, due to quite remarkable properties of supersymmetry, including the following:

- Supersymmetry has nontrivial manifestations at the quantum level.
- Local supersymmetry implies gravity (supergravity).
- Special version of local supersymmetry ($\mathcal{N} = 2$ supergravity) fulfills Einstein's dream of unifying gravity and electromagnetism.

The year 1979 was very special both for physics and geometry:

100 years since the birth of Einstein

- Intimate connection between supersymmetry and complex geometry

B. Zumino

- Concept of hyperkähler geometry

E. Calabi

Historically, it was just coincidence. However, what followed in the next 30 years was remarkably fruitful collaboration between supersymmetry and hyperkäher geometry, in particular presented in:

N. Hitchin, A. Karlhede, U. Lindström & M. Roček, *Hyperkähler Metrics and Supersymmetry,* (1987).

These lectures will give an overview of some of these developments.

Kähler manifolds are target spaces for rigid supersymmetric sigma-models with four supercharges (D $\leq$ 4).

B. Zumino (1979)

Hyperkähler manifolds are target spaces for rigid supersymmetric sigma-models with eight supercharges (D $\leq$ 6).

L. Alvarez-Gaumé & D. Z. Freedman (1981)

Quaternionic Kähler manifolds are target spaces for locally supersymmetric sigma-models with eight supercharges (D $\leq$ 6).

J. Bagger & E. Witten (1983)

Bosonic nonlinear sigma-model is a field theory over a space-time $\mathbb{S}$ in which the fields take values in a Riemannian manifold $(\mathcal{M}, g)$ (target space). If $\mathbb{S}$ is 4D Minkowski space, the σ-model action is

$$S = -\frac{1}{2}\int \mathrm{d}^4x\, g_{\mu\nu}(\phi)\partial^a\varphi^\mu\partial_a\varphi^\nu \ ,$$

where $\varphi^\mu(x)$ are scalar fields on $\mathbb{S}$ and local coordinates on $\mathcal{M}$.

Unlike general Kähler metrics, the hyperkähler and quaternionic Kähler metrics are difficult to construct explicitly.

Off-shell supersymmetry, *provided its power is properly elaborated*, is a device to generate hyperkähler and quaternionic Kähler structures.

A. Karlhede, U. Lindström & M. Roček (1984)

A. Galperin, E. Ivanov, V. Ogievetsky & E. Sokatchev (1986)

N. Hitchin, A. Karlhede, U. Lindström & M. Roček (1987)

There exists one-to-one correspondence between $4n$-dimensional quaternion Kähler manifolds and $4(n+1)$-dimensional hyperkähler spaces possessing a homothetic conformal Killing vector, and hence an isometric action of SU(2) rotating the complex structures.

A. Swann (1991)

Such hyperkähler spaces are called *Swann bundles* in the mathematics literature, and *hyperkähler cones* in the physics literature.

Hyperkähler cones are target spaces for $\mathcal{N}=2$ rigid superconformal σ-models.

B. de Wit, B. Kleijn & S. Vandoren (2000)
B. de Wit, M. Roček & S. Vandoren (2001)

It is sufficient to develop techniques to generate arbitrary $\mathcal{N}=2$ rigid supersymmetric nonlinear σ-models, and hence hyperkähler metrics.

Matrix realization of the Poincaré group

Denote by $\mathfrak{P}(4)$ the universal covering group of the restricted Poincaré group $\mathrm{ISO}_0(3,1)$. The principle of Poincaré invariance states that $\mathfrak{P}(4)$ must be a subgroup of the symmetry group of any quantum field theory.

Traditional realization of $\mathfrak{P}(4)$:

The group of linear inhomogeneous transformations on the space of 2×2 Hermitian matrices (with $\vec{\sigma}$ the Pauli matrices)

$$\boldsymbol{x} := x^m \sigma_m = \boldsymbol{x}^\dagger = (x_{\alpha\dot{\beta}}) \ , \qquad \sigma_m = (\mathbb{1}_2, \vec{\sigma}) \ , \qquad x^m \in \mathbb{R}^4$$

defined to act as follows:

$$\boldsymbol{x} \ \to \ \boldsymbol{x}' = x'^m \sigma_m = N \boldsymbol{x} N^\dagger + \boldsymbol{b} \ , \qquad \boldsymbol{b} = b^m \sigma_m \ ,$$

where

$$N = (N_\alpha{}^\beta) \in \mathrm{SL}(2,\mathbb{C}) \ , \qquad\qquad b^m \in \mathbb{R}^4.$$

$N^\dagger := \bar{N}^{\mathrm{T}}$ the Hermitian conjugate of N,
$\bar{N} = (\bar{N}_{\dot{\alpha}}{}^{\dot{\beta}})$ the complex conjugate of N.

Its equivalent form:

The group of linear inhomogeneous transformations on the space of 2×2 Hermitian matrices

$$\tilde{\boldsymbol{x}} := x^m \tilde{\sigma}_m = \tilde{\boldsymbol{x}}^\dagger = (x^{\dot{\alpha}\beta}) \ , \qquad \tilde{\sigma}_m = (\mathbb{1}_2, -\vec{\sigma}) \ , \qquad x^m \in \mathbb{R}^4$$

defined to act as follows:

$$\tilde{\boldsymbol{x}} \ \to \ \tilde{\boldsymbol{x}}' = x'^m \tilde{\sigma}_m = (N^{-1})^\dagger \tilde{\boldsymbol{x}} N^{-1} + \tilde{\boldsymbol{b}} \ , \qquad \tilde{\boldsymbol{b}} = b^m \tilde{\sigma}_m \ .$$

$(\sigma_m)_{\alpha\dot{\beta}}$ and $(\tilde{\sigma}_m)^{\dot{\alpha}\beta}$ are invariant tensors of the Lorenz group.

It is advantageous to realize $\mathfrak{P}(4)$ as a subgroup of $\mathrm{SU}(2,2)$ consisting of all block triangular matrices of the form:

$$
(N,b) := \left(\begin{array}{c|c} N & 0 \\ \hline -\mathrm{i}\,\tilde{\boldsymbol{b}}\,N & (N^{-1})^{\dagger} \end{array} \right) = (\mathbb{1}_2, b)\,(N,0)\ ,
$$

where

$$
N \in \mathrm{SL}(2,\mathbb{C})\ , \qquad \tilde{\boldsymbol{b}} := b^m \tilde{\sigma}_m = \tilde{\boldsymbol{b}}^{\dagger}\ , \qquad b^m \in \mathbb{R}^4\ .
$$

Minkowski space $\mathbb{M}^4 \equiv \mathbb{R}^{3,1} = \mathrm{ISO}_0(3,1)/\mathrm{SO}_0(3,1)$ can be realized as the coset space

$$
\mathbb{M}^4 = \mathfrak{P}(4)/\mathrm{SL}(2,\mathbb{C})\ .
$$

Its points are naturally parametrized by the Cartesian coordinates $x^m \in \mathbb{R}^4$ corresponding to the coset representative:

$$
(\mathbb{1}_2, x) = \left(\begin{array}{c|c} \mathbb{1}_2 & 0 \\ \hline -\mathrm{i}\,\tilde{\boldsymbol{x}} & \mathbb{1}_2 \end{array} \right) = \exp \left(\begin{array}{c|c} 0 & 0 \\ \hline -\mathrm{i}\,\tilde{\boldsymbol{x}} & 0 \end{array} \right)\ .
$$

From here one can read off the action of $\mathfrak{P}(4)$ on $\mathbb{M}^4$:

$$
(N,b)\,(\mathbb{1}_2, x) = (\mathbb{1}_2, x')\,(N,0)\ , \qquad x'^m = (\Lambda(N))^m{}_n\, x^n + b^m\ ,
$$

which is the standard action of $\mathrm{ISO}_0(3,1)$ on Minkowski space.
Here $\Lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}_0(3,1)$ is the doubly covering homomorphism defined by

$$
(\Lambda(N))^m{}_n = -\frac{1}{2}\mathrm{tr}(\tilde{\sigma}^m N \sigma_n N^{\dagger})\ .
$$

Matrix realization of the super-Poincaré group

Supersymmetry is the only consistent & nontrivial extension of the Poincaré symmetry that is compatible with the principles of QFT.

R. Haag, J. Lopuszański & M. Sohnius (1975)

Denote by $\mathfrak{P}(4|\mathcal{N})$ the $\mathcal{N}$-extended super-Poincaré group. It can be realized as a subgroup of $\mathrm{SU}(2,2|\mathcal{N})$. Any element $g \in \mathfrak{P}(4|\mathcal{N})$ is a $(4+\mathcal{N}) \times (4+\mathcal{N})$ supermatrix of the form:

$$g = s(b, \boldsymbol{\varepsilon})\, h(N)\ , \qquad \boldsymbol{\varepsilon} := (\epsilon, \bar{\epsilon})$$

$$s(b, \boldsymbol{\varepsilon}) := \left(\begin{array}{c|c||c} \mathbb{1}_2 & 0 & 0 \\ \hline -\mathrm{i}\, \tilde{\boldsymbol{b}}_{(+)} & \mathbb{1}_2 & 2\bar{\epsilon} \\ \hline\hline 2\epsilon & 0 & \mathbb{1}_{\mathcal{N}} \end{array} \right) = \left(\begin{array}{c|c||c} \delta_\alpha{}^\beta & 0 & 0 \\ \hline -\mathrm{i}\, b_{(+)}^{\dot{\alpha}\beta} & \delta^{\dot{\alpha}}{}_{\dot{\beta}} & 2\bar{\epsilon}^{\dot{\alpha}j} \\ \hline\hline 2\epsilon_i{}^\beta & 0 & \delta_i{}^j \end{array} \right) ,$$

$$h(N) := \left(\begin{array}{c|c||c} N & 0 & 0 \\ \hline 0 & (N^{-1})^\dagger & 0 \\ \hline\hline 0 & 0 & \mathbb{1}_{\mathcal{N}} \end{array} \right) = \left(\begin{array}{c|c||c} N_\alpha{}^\beta & 0 & 0 \\ \hline 0 & (\bar{N}^{-1})_{\dot{\beta}}{}^{\dot{\alpha}} & 0 \\ \hline\hline 0 & 0 & \delta_i{}^j \end{array} \right) ,$$

where $N \in \mathrm{SL}(2, \mathbb{C})$ and $i, j = 1, \ldots, \mathcal{N}$.

$$b^m_{(\pm)} := b^m \pm \mathrm{i}\, \epsilon_i \sigma^m \bar{\epsilon}^i = b^m \pm \mathrm{i}\, \epsilon_i^\alpha (\sigma^m)_{\alpha\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha} i}\ , \qquad \overline{b^m} = b^m\ .$$

$h(N)$ Lorentz transformation

$s(b, 0)$ space-time translation

$s(0, \boldsymbol{\epsilon})$ supersymmetry transformation described by $2\mathcal{N}$ anti-commuting complex parameters ϵ_i^α and their complex conjugates

$$\bar{\epsilon}^{\dot{\alpha} i} := \overline{\epsilon_i^\alpha}\ .$$

V. Akulov & D. Volkov (1973)

Group structure:

$$s(b,\boldsymbol{\varepsilon})s(c,\boldsymbol{\eta}) = s(d,\boldsymbol{\varepsilon}+\boldsymbol{\eta}) ~,$$
$$h(N)s(b,\epsilon)h(N^{-1}) = s(\Lambda(N)b,\epsilon N^{-1}) ~,$$

where $\qquad d^m := b^m + c^m + \mathrm{i}(\eta_i\sigma^m\bar{\epsilon}^i - \epsilon_i\sigma^m\bar{\eta}^i) ~.$

$\mathcal{N}$-extended Minkowski superspace is the homogeneous space

$$\mathbb{M}^{4|4\mathcal{N}} = \mathfrak{P}(4|\mathcal{N})\Big/\mathrm{SL}(2,\mathbb{C}) ~,$$

where $\mathrm{SL}(2,\mathbb{C})$ is identified with the set of all matrices $h(N)$.
The points of $\mathbb{M}^{4|4\mathcal{N}}$ can be parametrized by the variables

$$z^A = (x^a, \theta^\alpha_i, \bar{\theta}^i_{\dot{\alpha}}) ~, \qquad \Theta := (\theta, \bar{\theta})$$

which correspond to the following coset representative:

$$s(z) := s(x,\Theta) = \left(\begin{array}{c|c||c} \mathbb{1}_2 & 0 & 0 \\ \hline -\mathrm{i}\,\tilde{\boldsymbol{x}}_{(+)} & \mathbb{1}_2 & 2\bar{\theta} \\ \hline\hline 2\theta & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right) = \exp\left(\begin{array}{c|c||c} 0 & 0 & 0 \\ \hline -\mathrm{i}\,\tilde{\boldsymbol{x}} & 0 & 2\bar{\theta} \\ \hline\hline 2\theta & 0 & 0 \end{array}\right) ~.$$

The action of $\mathfrak{P}(4|\mathcal{N})$ on $\mathbb{M}^{4|4\mathcal{N}}$ is naturally defined by

$$g = s(b,\boldsymbol{\varepsilon})\,h(N) : \quad s(z) ~\to~ s(z') := s(b,\boldsymbol{\varepsilon})h(N)s(z)h(N^{-1}) ~.$$

Poincaré transformation is generated by $g = s(b,0)\,h(N)$

$$x'^a = (\Lambda(N))^a{}_b\, x^b + b^a ~, \qquad \theta'^\alpha_i = \theta^\beta_i(N^{-1})_\beta{}^\alpha ~.$$

Supersymmetry transformation is generated by $g = s(0,\boldsymbol{\varepsilon})$

$$x'^a = x^a + \mathrm{i}(\theta_i\sigma^m\bar{\epsilon}^i - \epsilon_i\sigma^m\bar{\theta}^i) ~, \qquad \theta'^\alpha_i = \theta^\alpha_i + \epsilon^\alpha_i ~.$$

$\mathcal{N}$-extended super-Poincaré algebra $\mathfrak{p}(4|\mathcal{N})$

We can represent group elements as

$$s(b, \boldsymbol{\varepsilon}) = \exp \mathrm{i} \Big\{ - b^a P_a + \epsilon_i^\alpha \, Q_\alpha^i + \bar{\epsilon}_{\dot\alpha}^i \, \bar{Q}_i^{\dot\alpha} \Big\} \ ,$$

and

$$h(\mathrm{e}^{\frac{1}{2}\omega^{ab}\sigma_{ab}}) = \left(\begin{array}{c|c||c} \mathrm{e}^{\frac{1}{2}\omega^{ab}\sigma_{ab}} & 0 & 0 \\ \hline 0 & \mathrm{e}^{\frac{1}{2}\omega^{ab}\tilde{\sigma}_{ab}} & 0 \\ \hline\hline 0 & 0 & \mathbb{1}_{\mathcal{N}} \end{array} \right) = \exp \frac{\mathrm{i}}{2} \, \omega^{ab} J_{ab} \ ,$$

with

$$\sigma_{ab} := -\frac{1}{4}\Big(\sigma_a \tilde{\sigma}_b - \sigma_b \tilde{\sigma}_a\Big) \ , \qquad \tilde{\sigma}_{ab} := -\frac{1}{4}\Big(\tilde{\sigma}_a \sigma_b - \tilde{\sigma}_b \tilde{\sigma}_a\Big) \ ,$$

and $\omega^{ab} = -\omega^{ba}$ real parameters.

Here P_a, J_{ab}, Q_α^i and $\bar{Q}_i^{\dot\alpha}$ are the generators of the Lie superalgebra $\mathfrak{p}(4|\mathcal{N})$ of $\mathfrak{P}(4|\mathcal{N})$.

Making use of

$$s(b, \boldsymbol{\varepsilon}) s(c, \boldsymbol{\eta}) = s(d, \boldsymbol{\varepsilon} + \boldsymbol{\eta})$$

leads to the (anti-)commutation relations:

$$\begin{aligned} [P_a, P_b] &= 0 \ , \\ [P_a, Q_\alpha^i] &= [P_a, \bar{Q}_{\dot\alpha i}\,] = 0 \ , \\ \{Q_\alpha^i \, , \, Q_\beta^j\} &= \{\bar{Q}_{\dot\alpha i} \, , \, \bar{Q}_{\dot\beta j}\} = 0 \ , \qquad i, j = 1, \dots, \mathcal{N} \\ \{Q_\alpha^i \, , \, \bar{Q}_{\dot\beta j}\} &= 2\delta_j^i \, (\sigma_c)_{\alpha\dot\beta} \, P^c \ . \end{aligned}$$

P^a the energy-momentum 4-vector

Q_i^α & $\bar{Q}_i^{\dot\alpha}$ the supersymmetry generators

Routine: Raising and lowering of (iso) spinor indices

$$\psi_\alpha = \varepsilon_{\alpha\beta}\,\psi^\beta\ , \qquad \psi^\alpha = \varepsilon^{\alpha\beta}\,\psi_\beta$$

where $\varepsilon_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are 2×2 antisymmetric matrices normalized as

$$\varepsilon^{12} = \varepsilon_{21} = 1\ .$$

The same conventions for dotted spinor indices ($\psi_{\dot\alpha}$ and $\psi^{\dot\alpha}$), and for SU(2) isospinor ones (ψ_i and ψ^i).

Routine: Minkowski metric

$$\eta_{mn} = \mathrm{diag}\,(-1, +1, +1, +1)\ .$$

Generalization: $\mathfrak{P}_{\mathrm{A}}(4|\mathcal{N})$

The $\mathcal{N}$-extended super-Poincaré group $\mathfrak{P}(4|\mathcal{N})$ can be generalized to include $\mathrm{U}(\mathcal{N})$ automorphisms of the super-Poincaré algebra $\mathfrak{p}(4|\mathcal{N})$. The resulting supergroup is denoted $\mathfrak{P}_{\mathrm{A}}(4|\mathcal{N})$. Any element $g \in \mathfrak{P}_{\mathrm{A}}(4|\mathcal{N})$ is a $(4+\mathcal{N}) \times (4+\mathcal{N})$ supermatrix of the form:

$$g = s(b, \boldsymbol{\varepsilon})\, h(N, U)\;, \qquad \boldsymbol{\varepsilon} := (\epsilon, \bar{\epsilon})$$

$$s(b, \boldsymbol{\varepsilon}) := \left(\begin{array}{c|c||c} \mathbb{1}_2 & 0 & 0 \\ \hline -\mathrm{i}\,\tilde{\boldsymbol{b}}_{(+)} & \mathbb{1}_2 & 2\bar{\epsilon} \\ \hline\hline 2\epsilon & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right)\;,$$

$$h(N, U) := \left(\begin{array}{c|c||c} N & 0 & 0 \\ \hline 0 & (N^{-1})^{\dagger} & 0 \\ \hline\hline 0 & 0 & U \end{array}\right)\;, \qquad U = (U_i{}^j) \in \mathrm{U}(\mathcal{N})\;.$$

$\mathcal{N}$-extended Minkowski superspace is the homogeneous space

$$\mathbb{M}^{4|4\mathcal{N}} = \mathfrak{P}_{\mathrm{A}}(4|\mathcal{N}) \Big/ \mathrm{SL}(2, \mathbb{C}) \times \mathrm{U}(\mathcal{N})\;.$$

V. Akulov & D. Volkov (1973)

Generalization: The possibility of central charges for $\mathcal{N} > 1$

R. Haag, J. Lopuszański & M. Sohnius (1975)

A brief review of induced representations

Consider a homogeneous space

$$X = G/H = \{x\} \ ,$$

for some Lie group G and its subgroup H. For simplicity, assume that there exists a global cross-section (or coset representative) $s(x)$

$$s(x)\colon X \to G \quad \text{such that} \quad \pi \circ s = \mathrm{id} \quad \Longleftrightarrow \quad \pi\Big(s(x)\Big) = x \ ,$$

with $\pi\colon G \to G/H$ the natural projection. We then have the following unique decomposition in the group:

$$\forall g \in G \ , \quad \exists h \in H \quad \text{such that} \quad g = s(x)\, h \ .$$

Now, one can express the fact that G acts on $X = G/H$ as follows:

$$g\, s(x) = s(g \cdot x)\, \boldsymbol{h}(g, x) \equiv s(x')\, \boldsymbol{h}(g, x) \ , \quad \text{for some} \quad \boldsymbol{h}(g, x) \in H \ .$$

Here $\boldsymbol{h}(g, x)$ is called the cocycle.

$$\boldsymbol{h}(g_1 g_2, x) = \boldsymbol{h}(g_1, g_2 x)\, \boldsymbol{h}(g_2, x) \ .$$

Let R be a finite-dimensional representation of H on a vector space $\mathcal{V}$. We then can define a representation T of G in a linear space of fields $\varphi(x)$ over X with their values in $\mathcal{V}$, $\varphi : X \to \mathcal{V}$, by the rule:

$$\Big[T(g)\varphi\Big](g \cdot x) \equiv \varphi'(x') = R\Big(\boldsymbol{h}(g, x)\Big)\varphi(x) \ .$$

T is known as the induced representation.

The Maurer-Cartan forms

Denote by $\mathcal{G}$ and $\mathcal{H}$ the Lie algebras of G and H, respectively. Suppose there exists a complement $\mathcal{K}$ of $\mathcal{H}$ in $\mathcal{G}$ such that

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{H} \ , \qquad [\mathcal{H}, \mathcal{H}] \in \mathcal{H} \ , \qquad [\mathcal{H}, \mathcal{K}] \in \mathcal{K} \ .$$

Let $\{\mathcal{T}_a\}$ be a basis of $\mathcal{K}$, and $\{\mathcal{T}_i\}$ a basis for $\mathcal{H}$.

$$s^{-1}\mathrm{d}s = E + \Omega \ ,$$

$$E = \mathrm{d}x^\mu E_\mu{}^a(x)\mathcal{T}_a \equiv E^a\mathcal{T}_a \qquad \text{vielbein or frame}$$

$$\Omega = \mathrm{d}x^\mu\,\Omega_\mu{}^i(x)\mathcal{T}_i \equiv E^a\,\Omega_a{}^i(x)\mathcal{T}_i \qquad \text{connection}$$

Group transformation

$$x \ \rightarrow \ x' = g \cdot x \ , \qquad s(x) \ \rightarrow \ s(x') = gs(x)\boldsymbol{h}^{-1}(g,x)$$

leads to: $s^{-1}\mathrm{d}s \ \rightarrow \ \boldsymbol{h}\Big(s^{-1}\mathrm{d}s\Big)\boldsymbol{h}^{-1} - \mathrm{d}\boldsymbol{h}\,\boldsymbol{h}^{-1}$, and hence

$$E \ \rightarrow \ \boldsymbol{h}E\boldsymbol{h}^{-1} \ , \qquad \Omega \ \rightarrow \ \boldsymbol{h}\Omega\boldsymbol{h}^{-1} - \mathrm{d}\boldsymbol{h}\,\boldsymbol{h}^{-1} \ .$$

Covariant derivative

Let $\varphi(x)$ be a field with the group transformation law:

$$\varphi(x) \ \rightarrow \ \varphi'(x') = \boldsymbol{h}(g,x)\varphi(x) \ ,$$

where, for simplicity of notation, $\boldsymbol{h}(g,x)$ stands for $R\Big(\boldsymbol{h}(g,x)\Big)$. The covariant derivative of φ is defined as

$$\mathcal{D}\varphi := (\mathrm{d} + \Omega)\varphi = E^a\,\mathcal{D}_a\varphi \ , \qquad \mathcal{D}_a\varphi := (E_a + \Omega_a)\varphi \ ,$$

with $\{E_a = E_a{}^\mu(x)\partial_\mu\}$ the dual basis of $\{E^a = \mathrm{d}x^\mu E_\mu{}^a(x)\}$.

In the case of $\mathcal{N}$-extended superspace $\mathbb{M}^{4|4\mathcal{N}}$, we have

$$
\begin{aligned}
G &\rightarrow \mathfrak{P}(4|\mathcal{N}) , \qquad & g &\rightarrow s(b,\boldsymbol{\varepsilon})h(N) \\
H &\rightarrow \mathrm{SL}(2,\mathbb{C}) , \qquad & h &\rightarrow h(N) \\
x^\mu &\rightarrow z^A = (x^a, \theta^\alpha_i, \bar\theta^i_{\dot\alpha}) & & \\
s(x) &\rightarrow s(x,\Theta) & & \\
\boldsymbol{h}(g,x) &\rightarrow h(N) & & \text{no dependence on } z^A
\end{aligned}
$$

The Maurer-Cartan forms:

$$
s^{-1}\mathrm{d}s = \left(\begin{array}{c|c||c} 0 & 0 & 0 \\ \hline -\mathrm{i}\,\tilde{\boldsymbol{e}} & 0 & 2\mathrm{d}\bar\theta \\ \hline\hline 2\mathrm{d}\theta & 0 & 0 \end{array}\right) , \qquad e^a := \mathrm{d}x^a + \mathrm{i}(\theta_i\sigma^a\mathrm{d}\bar\theta^i - \mathrm{d}\theta_i\sigma^m\bar\theta^i) .
$$

The vielbein:

$$
e^A = \mathrm{d}z^M e_M{}^A(z) = \left(e^a,\, \mathrm{d}\theta^a_i,\, \mathrm{d}\bar\theta^i_{\dot\alpha}\right) .
$$

comprises the supersymmetric one-forms, i.e. those one-forms which are invariant under the supersymmetry transformations.

The connection:

$$
\Omega = 0 .
$$

The covariant derivatives:

$$
\mathrm{d} \equiv \mathrm{d}z^M \frac{\partial}{\partial z_M} = e^A D_A , \qquad D_A = \left(\partial_a, D^i_\alpha, \bar D^{\dot\alpha}_i\right)
$$

$$
D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + \mathrm{i}\,(\sigma^b)_{\alpha\dot\beta}\,\bar\theta^{\dot\beta i}\,\partial_b , \qquad \bar D_{\dot\alpha i} = -\frac{\partial}{\partial\bar\theta^{\dot\alpha i}} - \mathrm{i}\,\theta^\beta_i(\sigma^b)_{\beta\dot\alpha}\,\partial_b .
$$

A tensor superfield $\mathcal{U}(z)$, with all indices suppressed, is defined to possess the following transformation law under the super-Poincaré group:

$$g = s(b, \boldsymbol{\varepsilon})h(N) : \quad U(z) \quad \longrightarrow \quad U'(z') = R(N)U(z) \ ,$$

with R a finite-dimensional representation of $\mathrm{SL}(2, \mathbb{C})$.

A. Salam & J. Strathdee (1974)

Infinitesimal supersymmetry transformation ($g = s(0, \boldsymbol{\varepsilon})$):

$$\boxed{\delta\mathcal{U} := \mathcal{U}'(z) - \mathcal{U}(z) = \mathrm{i}\,\{\epsilon^\alpha_i\, Q^i_\alpha + \bar{\epsilon}^i_{\dot{\alpha}}\, \bar{Q}^{\dot{\alpha}}_i\}\,\mathcal{U}}$$

where the supersymmetry generators have the form:

$$Q^i_\alpha = \mathrm{i}\,\frac{\partial}{\partial\theta^\alpha_i} + (\sigma^b)_{\alpha\dot{\beta}}\,\bar{\theta}^{\dot{\beta}i}\,\partial_b \ , \qquad \bar{Q}_{\dot{\alpha}i} = -\mathrm{i}\,\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}i}} - \theta^\beta_i(\sigma^b)_{\beta\dot{\alpha}}\,\partial_b \ .$$

If $\mathcal{U}(z)$ is a tensor superfield, then

$$D_A\mathcal{U}(z)$$

is also a tensor superfield.

The covariant derivatives commute with the supersymmetry transformations

$$[D_A, \epsilon^\beta_j\, Q^j_\beta] = [D_A, \bar{\epsilon}^j_{\dot{\beta}}\, \bar{Q}^{\dot{\beta}}_j] = 0 \ .$$

The algebra of spinor covariant derivatives:

$$\{D^i_\alpha, D^j_\beta\} = \{\bar{D}_{\dot{\alpha}i}, \bar{D}_{\dot{\beta}j}\} = 0 \ ,$$
$$\{D^i_\alpha, \bar{D}_{\dot{\beta}j}\} = -2\mathrm{i}\delta^i_j\,(\sigma^c)_{\alpha\dot{\beta}}\,\partial_c \ .$$

Chiral superfields

Let us return to the coset representative

$$s(z) = \left(\begin{array}{c|c||c} \mathbb{1}_2 & 0 & 0 \\ \hline -\mathrm{i}\,\tilde{\boldsymbol{x}}_{(+)} & \mathbb{1}_2 & 2\bar{\theta} \\ \hline\hline 2\theta & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right) , \qquad z^A = (x^a, \theta_i^\alpha, \bar{\theta}^i_{\dot{\alpha}})$$

and consider its first $(4+\mathcal{N}) \times 2$ block-column

$$\mathcal{C}(x_{(+)}, \theta) = \begin{pmatrix} \mathbb{1}_2 \\ -\mathrm{i}\,\tilde{\boldsymbol{x}}_{(+)} \\ 2\theta \end{pmatrix} .$$

The super-Poincaré transformation law of $\mathcal{C}(x_{(+)}, \theta)$ is

$$\mathcal{C}(x_{(+)}, \theta) \;\to\; \mathcal{C}(x'_{(+)}, \theta') := s(b, \boldsymbol{\varepsilon})\, h(N)\, \mathcal{C}(x_{(+)}, \theta)\, N^{-1} .$$

It follows that the variables $x^a_{(+)}$ and θ_i^α transform via themselves (that is, they do not mix with $\bar{\theta}^{\dot{\alpha} i}$) under $\mathfrak{P}(4|\mathcal{N})$. This means that all superfields, which depend on $x^a_{(+)}$ and θ_i^α only, preserve this property under the super-Poincaré group:

$$\Phi(z) := \varphi(x_{(+)}, \theta) \quad \Longrightarrow \quad \Phi'(z) = \Phi(g^{-1} \cdot z) = \varphi'(x_{(+)}, \theta) .$$

Such superfields are singled out by the following first-order differential constraints:

$$\bar{D}_{\dot{\alpha} i} \Phi = 0 \quad \Longleftrightarrow \quad \Phi(x, \theta, \bar{\theta}) = \mathrm{e}^{\mathrm{i}\theta_i \sigma^m \bar{\theta}^i \partial_m}\, \varphi(x, \theta)$$

and are called chiral superfields.

Supersymmetric action principle

In order to construct supersymmetric field theories, we have to learn how to generate supersymmetric invariants.

Berezin or Grassmann integral (one Grassmann variable)

$$\int \mathrm{d}\theta \, f(\theta) = \frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta) \equiv \frac{\mathrm{d}}{\mathrm{d}\theta} f(\theta)\Big|_{\theta=0} \, .$$

$\mathcal{N} = 1$ supersymmetric action

Let $L(z)$ be a real scalar superfield. Then

$$S := \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, L = \frac{1}{16} \int \mathrm{d}^4x \, D^\alpha \bar{D}^2 D_\alpha L \Big|_{\theta=0} = \frac{1}{16} \int \mathrm{d}^4x \, \bar{D}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}} L \Big|_{\theta=0}$$

is invariant under the $\mathcal{N} = 1$ super-Poincaré group.

Proof:

$$\begin{aligned}
\delta_{\mathrm{SUSY}} S &= \frac{\mathrm{i}}{16} \int \mathrm{d}^4x \, D^\alpha \bar{D}^2 D_\alpha \Big(\epsilon Q + \bar{\epsilon}\bar{Q}\Big) L \Big|_{\theta=0} \\
&= \frac{\mathrm{i}}{16} \int \mathrm{d}^4x \, \Big(\epsilon Q + \bar{\epsilon}\bar{Q}\Big) D^\alpha \bar{D}^2 D_\alpha L \Big|_{\theta=0} \\
&= -\frac{1}{16} \int \mathrm{d}^4x \, \Big(\epsilon D + \bar{\epsilon}\bar{D}\Big) D^\alpha \bar{D}^2 D_\alpha L \Big|_{\theta=0} \\
&= \int \mathrm{d}^4x \, \text{total space-time derivative} \; = 0 \, .
\end{aligned}$$

Made use of:

$$Q_\alpha = \mathrm{i} \frac{\partial}{\partial \theta^\alpha} + (\sigma^b)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_b \, , \qquad D_\alpha = \frac{\partial}{\partial \theta^\alpha} + \mathrm{i} \, (\sigma^b)_{\alpha\dot{\beta}} \bar{\theta}^{\dot{\beta}} \partial_b$$

General $\mathcal{N}=1$ supersymmetric nonlinear sigma-model

B. Zumino (1979)

$$S = \int \mathrm{d}^4x \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, K(\Phi^a, \bar\Phi^{\bar b}) \;, \qquad \bar D_{\dot\alpha}\Phi^a = 0$$

Here the dynamical variables $\Phi^a(z)$ are chiral scalar superfields. The Lagrangian $K(\Phi^a, \bar\Phi^{\bar b})$ can be interpreted as a Kähler potential of some Kähler manifold $\mathcal{M}$ (target space) with the Kähler metric

$$g_{a\bar b}(\Phi,\bar\Phi) := \frac{\partial^2 K}{\partial\Phi^a\partial\bar\Phi^{\bar b}} \equiv K_{a\bar b} \;, \qquad g_{ab} = g_{\bar a\bar b} = 0 \;,$$

$$K_{a_1\dots a_n\,\bar b_1\dots\bar b_m} := \frac{\partial^{n+m}K}{\partial\Phi^{a_1}\dots\partial\Phi^{a_n}\,\bar\Phi^{\bar b_1}\dots\bar\Phi^{\bar b_m}} \;.$$

Kähler invariance

$$K(\Phi,\bar\Phi) \quad\longrightarrow\quad K(\Phi,\bar\Phi) + \Lambda(\Phi) + \bar\Lambda(\bar\Phi) \;,$$

for arbitrary holomorphic function $\Lambda(\Phi)$, follows from the identities:

$$D^\alpha \bar D^2 D_\alpha = \bar D_{\dot\alpha} D^2 \bar D^{\dot\alpha}$$

and

$$S = \int \mathrm{d}^4x\, \mathcal{L} \;, \qquad \mathcal{L} := \frac{1}{16} D^\alpha \bar D^2 D_\alpha K(\Phi,\bar\Phi)\Big|_{\theta=0}$$

and

$$\bar D_{\dot\alpha}\Phi^a = 0 \quad\longrightarrow\quad \bar D_{\dot\alpha}\Lambda(\Phi) = 0 \;.$$

If $\{U_{(i)}\}$ is an atlas on $\mathcal{M}$, and $K_{(i)}$ is the local Kähler potential corresponding to the chart $U_{(i)}$, then one and the same point $p\in\mathcal{M}$ can belong to several charts. The above consideration shows that the Lagrangian $\mathcal{L}$ is independent of the choice of $K_{(i)}$ made.

The component Lagrangian

Introduce the component fields of $\Phi^a(z)$:

$$\Phi^a(x,\theta,\bar\theta) = \mathrm{e}^{\mathrm{i}\theta\sigma^m\bar\theta\partial_m}\left\{\varphi^a(x) + \theta\,\psi^a(x) + \theta^2 F^a(x)\right\} .$$

Here φ^a and F^a are complex scalar fields, while ψ^a_α a spinor field. Direct calculations lead to

$$\begin{aligned}\mathcal{L} = &-g_{a\bar b}(\varphi,\bar\varphi)\Big(\partial^m\varphi^a\,\partial_m\bar\varphi^{\bar b} + \frac{\mathrm{i}}{4}\psi^a\sigma^m\overleftrightarrow{\nabla}_m\,\bar\psi^{\bar b}\Big) + g_{a\bar b}(\varphi,\bar\varphi)\,\mathcal{F}^a\bar{\mathcal{F}}^{\bar b}\\ &+\frac{1}{16}R_{a\bar b c\bar d}(\varphi,\bar\varphi)\,\psi^a\psi^c\,\bar\psi^{\bar b}\bar\psi^{\bar d} ,\end{aligned}$$

where $\nabla_m\psi^a$ denotes the covariant derivative of ψ^a,

$$\nabla_m\psi^a := \partial_m\psi^a + (\partial_m\varphi^b)\,\Gamma^a_{bc}(\varphi,\bar\varphi)\,\psi^c ,$$

and

$$\mathcal{F}^a := F^a - \frac{1}{4}\Gamma^a_{bc}(\varphi,\bar\varphi)\,\psi^b\psi^c .$$

Finally, $\Gamma^a_{bc}(\varphi,\bar\varphi)$ and $R_{a\bar b c\bar d}(\varphi,\bar\varphi)$ denote the Christoffel symbols and the Riemann tensor associated with the Kähler metric $g_{a\bar b}(\varphi,\bar\varphi)$.

$$\Gamma^a_{bc} = g^{a\bar d}K_{bc\bar d} , \qquad R_{a\bar b c\bar d} = K_{ac\bar b\bar d} - g_{e\bar f}\,\Gamma^e_{ac}\,\Gamma^{\bar f}_{\bar b\bar d} .$$

The equations of motion for $\bar F$s:

$$\mathcal{F}^a = 0 \quad\longleftrightarrow\quad F^a = \frac{1}{4}\Gamma^a_{bc}(\varphi,\bar\varphi)\,\psi^b\psi^c .$$

The field F^a and its conjugate $\bar F^{\bar a}$ appear in the action without derivatives. On the equations of motion, they become functions of other fields. Their sole role is to have supersymmetry linearly realized. Such fields are called auxiliary.

How to construct $\mathcal{N} = 2$ supersymmetric nonlinear sigma-models?

A possible approach is to work in terms of $\mathcal{N} = 1$ superfields. Start from the general $\mathcal{N} = 1$ supersymmetric nonlinear sigma-model

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar{\theta}\, \mathfrak{K}(\Phi^a, \bar{\Phi}^{\bar{b}}) , \qquad \bar{D}_{\dot{\alpha}}\Phi^a = 0 ,$$

associated with some Kähler manifold $\mathcal{M}$, and look for those those target space geometries which are compatible with an additional hidden supersymmetry.

Ansatz for the second supersymmetry:

U. Lindström & M. Roček (1983)

C. Hull, A. Karlhede, U. Lindström & M. Roček (1986)

$$\delta\Phi^a = \frac{1}{2}\bar{D}^2\Big(\bar{\epsilon}(\bar{\theta})\,\bar{\Omega}^a\Big) , \qquad \delta\bar{\Phi}^{\bar{a}} = \frac{1}{2}D^2\Big(\epsilon(\theta)\,\Omega^{\bar{a}}\Big) ,$$

for some functions

$$\Omega^{\bar{a}} = \Omega^{\bar{a}}(\Phi, \bar{\Phi})$$

associated with the Kähler manifold $\mathcal{M}$. The explicit form of the transformation parameter ϵ is

$$\epsilon(\theta) = \tau + \epsilon^\alpha\theta_\alpha , \qquad \tau = \text{const} , \quad \epsilon^\alpha = \text{const} .$$

ϵ^α second SUSY transformation,

τ central charge transformation.

Results of the analysis:

C. Hull, A. Karlhede, U. Lindström & M. Roček (1986)

(1) The action is invariant under the transformations introduced if

$$\boxed{\omega_{bc} := g_{b\bar{a}}\,\Omega^{\bar{a}}{}_{,c} = -\omega_{cb}}$$

and

$$\boxed{\omega_{bc\,,\bar{a}} := \partial_{\bar{a}}\omega_{bc} = \nabla_{\bar{a}}\omega_{bc} = 0}$$

and

$$\boxed{\nabla_a\omega_{bc} = 0}$$

It can be shown that $\omega_{bc}(\Phi)$ is a globally defined holomorphic two-form on $\mathcal{M}$. The above equations mean that this two-form is covariantly constant.

(2) The first and the second supersymmetries form the $\mathcal{N} = 2$ super-Poincaré algebra

$$\begin{aligned} \{Q^i_\alpha\,,\,Q^j_\beta\} &= \{\bar{Q}_{\dot{\alpha}i}\,,\,\bar{Q}_{\dot{\beta}j}\} = 0\;, \qquad i,j = 1,2 \\ \{Q^i_\alpha\,,\,\bar{Q}_{\dot{\beta}j}\} &= 2\delta^i_j\,(\sigma_c)_{\alpha\dot{\beta}}\,P^c\;. \end{aligned}$$

on the equations of motion if

$$\boxed{\bar{\Omega}^a{}_{,\bar{c}}\,\Omega^{\bar{c}}{}_{,b} = -\delta^a{}_b}$$

Denote $J \equiv J_3$ the complex structure chosen on the target space $\mathcal{M}$,

$$J_3 = \begin{pmatrix} \mathrm{i}\,\delta^a{}_b & 0 \\ 0 & -\mathrm{i}\,\delta^{\bar a}{}_{\bar b} \end{pmatrix} \ .$$

There exist two more covariantly constant complex structures

$$J_1 := \begin{pmatrix} 0 & \bar\Omega^a{}_{,\bar b} \\ \Omega^{\bar a}{}_{,b} & 0 \end{pmatrix} \ , \qquad J_2 := \begin{pmatrix} 0 & \mathrm{i}\,\bar\Omega^a{}_{,\bar b} \\ -\mathrm{i}\,\Omega^{\bar a}{}_{,b} & 0 \end{pmatrix}$$

such that (i) $\mathcal{M}$ is Kähler with respect to each of them; and
(ii) the operators $J_A = (J_1, J_2, J_3)$ form the quaternionic algebra:

$$J_A\, J_B = -\delta_{AB}\, \mathbb{1} + \varepsilon_{ABC} J_C \ .$$

Therefore the target space $\mathcal{M}$ is a hyperkähler manifold.

C. Hull, A. Karlhede, U. Lindström & M. Roček (1986)

These authors also presented the following explicit expression for $\bar\Omega^a$:

$$\bar\Omega^a = \omega^{ab}(\Phi)\mathfrak{K}_b(\Phi, \bar\Phi) \ .$$

Although $\bar\Omega^a$ changes under the Kähler transformations as

$$\begin{aligned} \mathfrak{K}(\Phi, \bar\Phi) \ &\rightarrow \ \mathfrak{K}(\Phi, \bar\Phi) + \Lambda(\Phi) + \bar\Lambda(\bar\Phi) \ , \\ \omega^{ab}\mathfrak{K}_b \ &\rightarrow \ \omega^{ab}\mathfrak{K}_b + \omega^{ab}\Lambda_b \ , \end{aligned}$$

the SUSY transformation $\delta\Phi^a = \frac{1}{2}\bar D^2(\bar\epsilon\,\bar\Omega^a)$ remains invariant.

The Lagrangian of the $\mathcal{N} = 2$ supersymmetric σ-model

$$S = \int \mathrm{d}^4x\mathrm{d}^2\theta\mathrm{d}^2\bar\theta\, \mathfrak{K}(\Phi^a, \bar\Phi^{\bar b}) \ , \qquad \bar D_{\dot\alpha}\Phi^a = 0$$

is the hyperkähler potential of $\mathcal{M}$.

Similarly to the component (i.e. $\mathcal{N}=0$) formulation of general $\mathcal{N}=2$ supersymmetric nonlinear sigma-models

Alvarez-Gaumé & Freedman (1981)

Bagger & Witten (1983)

their formulation in terms of $\mathcal{N}=1$ superfields described above, is just an *existence theorem*.

- The $\mathcal{N}=1$ formulation is not suitable from the point of view of generating $\mathcal{N}=2$ supersymmetric nonlinear sigma-models.
- The $\mathcal{N}=1$ formulation provides `NO` insight from the point of view of constructing $\mathcal{N}=2$ superconformal nonlinear sigma-models.

What is necessary: $\mathcal{N}=2$ superspace techniques.

Conceptual problem: Multiplets in the standard $\mathcal{N}=2$ superspace $\mathbb{M}^{4|8}$ are not suitable (say, too long) for sigma-model constructions.

Idea to circumvent the problem: Look for an extension of $\mathbb{M}^{4|8}$ by bosonic (twistor-like) variables.

The correct superspace setting was found in 1983–1984 independently by three groups who pursued somewhat different goals.

$$\mathbb{M}^{4|8}\times\mathbb{C}P^1=\mathbb{M}^{4|8}\times S^2$$

Rosly (83)

Galperin, Ivanov, Kalitsin, Ogievetsky & Sokatchev (1984)

Karlhede, Lindström & Roček (1984)

The algebra of $\mathcal{N}=2$ spinor covariant derivatives:

$$\{D^i_\alpha\,,\,D^j_\beta\}=0\;,\qquad\{\bar D^i_{\dot\alpha}\,,\,\bar D^j_{\dot\beta}\}=0\;,$$
$$\{D^i_\alpha\,,\,\bar D^j_{\dot\beta}\}=2{\rm i}\,\varepsilon^{ij}\,(\sigma^m)_{\alpha\dot\beta}\,\partial_m\;.$$

Following Rosly, introduce an isotwistor $v^i\in\mathbb{C}^2\setminus\{0\}$ and define

$$\mathfrak{D}_\alpha:=v_i\,D^i_\alpha\;,\qquad\bar{\mathfrak{D}}_{\dot\alpha}:=v_i\,\bar D^i_{\dot\alpha}\;.$$

Then

$$\{\mathfrak{D}_\alpha,\mathfrak{D}_\beta\}=\{\mathfrak{D}_\alpha,\bar{\mathfrak{D}}_{\dot\beta}\}=\{\bar{\mathfrak{D}}_{\dot\alpha},\bar{\mathfrak{D}}_{\dot\beta}\}=0\;.$$

(Grassmann) Analyticity constraints

$$\mathfrak{D}_\alpha\phi=\bar{\mathfrak{D}}_{\dot\alpha}\phi=0\;,\qquad\phi=\phi(z,v,\bar v)\;,\qquad\bar v_i:=(v^i)^*\;.$$

The constraints $\mathfrak{D}_\alpha\phi=\bar{\mathfrak{D}}_{\dot\alpha}\phi=0$ do not change if we replace $v^i\to c\,v^i$, with $c\in\mathbb{C}^*$, in the definition of $\mathfrak{D}_\alpha$ and $\bar{\mathfrak{D}}_{\dot\alpha}$.
It is natural to restrict our attention to so-called *isotwistor* superfields which (i) obey the constrains $\mathfrak{D}_\alpha\phi=\bar{\mathfrak{D}}_{\dot\alpha}\phi=0$ and (ii) only scale under arbitrary re-scalings of v:

$$\phi(z,c\,v,\bar c\,\overline{v})=c^{n_+}\,\bar c^{n_-}\,\phi(z,v,\overline{v})\;,\qquad c\in\mathbb{C}^*$$

for some parameters $n_\pm$ such that n_+-n_- is an integer. By redefining $\phi(z,v,\bar v)\to\phi(z,v,\bar v)/(v^\dagger v)^{n_-}$, we can always choose $n_-=0$.

$$\phi^{(n)}(z,c\,v,\bar c\,\overline{v})=c^n\,\phi^{(n)}(z,v,\overline{v})\;,\qquad c\in\mathbb{C}^*$$

$\phi^{(n)}(z,v,\overline{v})$ is said to have weight n.

The isotwistor $v^i\in\mathbb{C}^2\setminus\{0\}$ is seen to be defined modulo the equivalence relation $v^i\sim c\,v^i$, with $c\in\mathbb{C}^*$, hence it parametrizes $\mathbb{C}P^1$.

Supersymmetric field theory in $\mathbb{R}^{4|8} \times S^2$

Harmonic superspace approach

Galperin, Ivanov, Kalitsin, Ogievetsky & Sokatchev (1984)

`Conceptual setup:`

Use the equivalence relation $v^i \sim c\, v^i$, with $c \in \mathbb{C}^*$, to switch to a description in terms of the following normalized isotwistors:

$$u^{+i} := \frac{v^i}{\sqrt{v^\dagger v}}\ , \qquad u_i^- := \frac{\bar{v}_i}{\sqrt{v^\dagger v}} = \overline{u^{+i}} \quad \Longrightarrow \quad ({u_i}^-, {u_i}^+) \in \mathrm{SU}(2)\ .$$

The $u_i^\pm$ are called *harmonics*, and are defined modulo the equivalence relation $u_i^\pm \sim \exp(\pm \mathrm{i}\alpha)\, u_i^\pm$, with $\alpha \in \mathbb{R}$. Clearly, the harmonics parametrize $S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$.

Associated with an isotwistor superfield $\phi^{(n)}(z, v, \overline{v})$,

$$\mathfrak{D}_\alpha \phi^{(n)} = \bar{\mathfrak{D}}_{\dot{\alpha}} \phi^{(n)} = 0\ , \quad \phi^{(n)}(z, c\, v, \bar{c}\, \overline{v}) = c^n\, \phi^{(n)}(z, v, \overline{v})\ , \quad c \in \mathbb{C}^*$$

is the following superfield

$$\varphi^{(n)}(z, u^+, u^-) := \phi^{(n)} \left(z, \frac{v}{\sqrt{v^\dagger v}}, \frac{\bar{v}}{\sqrt{v^\dagger v}} \right) = \frac{1}{(\sqrt{v^\dagger v})^n} \phi^{(n)}(z, v, \overline{v})$$

obeying the homogeneity condition

$$\varphi^{(n)}(z, \mathrm{e}^{\mathrm{i}\alpha}\, u^+, \mathrm{e}^{-\mathrm{i}\alpha}\, u^-) = \mathrm{e}^{\mathrm{i}n\alpha}\, \varphi^{(n)}(z, u^+, u^-)\ .$$

They say $\varphi^{(n)}(z, u^\pm)$ has $\mathrm{U}(1)$ charge n.

Within the harmonic superspace approach, $\varphi^{(n)}(z, u^\pm)$ is required to be a *smooth* charge-n function over $\mathrm{SU}(2)$ or, equivalently, a *smooth* tensor field over the two-sphere $S^2 \simeq \mathrm{SU}(2)/\mathrm{U}(1)$: harmonic superfield

Supersymmetric action principle includes integration over S^2, in addition to integration over the space-time and (half of) Grassmann variables.

Let $L^{(4)}(z, u^\pm)$ be a real harmonic superfield of U(1) charge +4, and

$$\mathcal{L}^{(4)}(z, v, \bar{v}) := (v^\dagger v)^2 \, L^{(4)}(z, u^+, u^-)$$

the corresponding weight-n isotwistor superfield. Associated with $\mathcal{L}^{(4)}$ is the following $\mathcal{N} = 2$ supersymmetric invariant:

$$S := \int \mathrm{d}^4 x \int \mathrm{d}^2\mu \, \Delta^{(-4)} \mathcal{L}^{(4)} \ .$$

Here

$$\mathrm{d}^2\mu := \frac{v_i \mathrm{d}v^i \bar{v}^j \mathrm{d}\bar{v}_j}{(v^\dagger v)^2} = \frac{v_i \mathrm{d}v^i \bar{v}^j \mathrm{d}\bar{v}_j}{(\bar{v}_k v^k)^2}$$

can be recognized to be the usual measure on S^2. Indeed, introducing a complex (inhomogeneous) coordinate ζ in the north chart of $\mathbb{C}P^1$ as

$$v^i = v^{\underline{1}} \, (1, \zeta) \ , \qquad \zeta := \frac{v^{\underline{2}}}{v^{\underline{1}}} \qquad i = \underline{1}, \underline{2}$$

one obtains

$$\mathrm{d}^2\mu = \frac{\mathrm{d}\zeta \mathrm{d}\bar{\zeta}}{(1 + \zeta\bar{\zeta})^2} \ .$$

The operator $\Delta^{(-4)}$ in the expression for S is

$$\Delta^{(-4)} := \frac{1}{16} \nabla^\alpha \nabla_\alpha \bar{\nabla}_{\dot{\beta}} \bar{\nabla}^{\dot{\beta}} \ , \qquad \nabla_\alpha := \frac{1}{v^\dagger v} \bar{v}_i D^i_\alpha \ , \quad \bar{\nabla}_{\dot{\beta}} := \frac{1}{v^\dagger v} \bar{v}_i \bar{D}^i_{\dot{\beta}}$$

Supersymmetric field theory in $\mathbb{R}^{4|8} \times \mathbb{C}P^1$

Projective superspace approach

Karlhede, Lindström & Roček (1984)

Lindström & Roček (1988, 90)

Gonzalez-Rey, Lindström, Roček, von Unge & Wiles (1998)

`Conceptual setup:`

Off-shell supermultiplets are described in terms of isotwistor weight-n superfields $Q^{(n)}(z, v)$,

$$\mathfrak{D}_\alpha Q^{(n)} = \bar{\mathfrak{D}}_{\dot\alpha} Q^{(n)} = 0 \ , \quad Q^{(n)}(z, c\, v) = c^n\, Q^{(n)}(z, v) \ , \quad c \in \mathbb{C}^*$$

which are *holomorphic* over an open domain of $\mathbb{C}P^1$

$$\frac{\partial}{\partial \bar{v}_i} Q^{(n)} = 0 \ .$$

Such superfields are called projective. *There is no need to require $Q^{(n)}(z, v)$ to be smooth over $\mathbb{C}P^1$, for:*

Supersymmetric action principle includes a contour integral in $\mathbb{C}P^1$, and integral over the space-time and (half of) Grassmann variables.

$$S := \frac{1}{2\pi} \oint v_i \mathrm{d}v^i \int \mathrm{d}^4x\, \Delta^{(-4)} \mathcal{L}^{(2)} \ , \qquad \frac{\partial}{\partial u_k} S = 0$$

where $\mathcal{L}^{(2)}(z, v)$ be a (smile-conjug.) real weight-2 projective superfield, and

$$\Delta^{(-4)} := \frac{1}{16} \nabla^\alpha \nabla_\alpha \bar\nabla_{\dot\beta} \bar\nabla^{\dot\beta} \ , \qquad \nabla_\alpha := \frac{1}{u_k v^k} u_i D^i_\alpha \ , \quad \bar\nabla_{\dot\beta} := \frac{1}{u_k v^k} u_i \bar{D}^i_{\dot\beta}$$

Here u_k is a fixed isotwistor chosen to be arbitrary modulo the condition $u_k v^k \neq 0$ along the integration contour.

Projective superfields in the north chart of $\mathbb{C}P^1$

Introduce the inhomogeneous complex coordinate ζ on $\mathbb{C}P^1 \setminus \{\infty\}$:

$$v^i = v^{\underline{1}}\,(1,\zeta)\ , \qquad \zeta \in \mathbb{C}\ .$$

Given a projective weight-n superfield $Q^{(n)}(z,v)$, we can associate with it a new object $Q^{[n]}(z,\zeta)$ defined as

$$Q^{(n)}(z,v) \;\longrightarrow\; Q^{[n]}(z,\zeta) \propto Q^{(n)}(z,v)\ , \qquad \frac{\partial}{\partial\bar{\zeta}}Q^{[n]} = 0$$

It can be represented as

$$Q^{[n]}(z,\zeta) = \sum_p^q Q_k(z)\zeta^k\ , \qquad -\infty \le p < q \le +\infty$$

with $Q_k(z)$ some ordinary $\mathcal{N}=2$ superfields. Here p and q are invariants of the supersymmetry transformations.

The analyticity conditions

$$\mathfrak{D}_\alpha Q^{(n)} = \bar{\mathfrak{D}}_{\dot\alpha} Q^{(n)} = 0\ , \qquad \mathfrak{D}_\alpha := v_i\, D^i_\alpha\ , \quad \bar{\mathfrak{D}}_{\dot\alpha} := v_i\, \bar{D}^i_{\dot\alpha}$$

take the form:

$$D^{\underline{2}}_\alpha Q^{[n]}(\zeta) = \zeta\, D^{\underline{1}}_\alpha Q^{[n]}(\zeta)\ , \qquad \bar{D}_{\dot\alpha\,\underline{2}} Q^{[n]}(\zeta) = -\frac{1}{\zeta}\,\bar{D}_{\dot\alpha\,\underline{1}} Q^{[n]}(\zeta)\ .$$

Interpretation:

The dependence of the component superfields Q_k of $Q^{[n]}(\zeta)$
on $\theta^\alpha_{\underline{2}}$ and $\bar{\theta}^{\underline{2}}_{\dot\alpha}$,
is uniquely determined in terms of their dependence
on $\theta^\alpha_{\underline{1}} \equiv \theta^\alpha$ and $\bar{\theta}^{\underline{1}}_{\dot\alpha} \equiv \bar{\theta}_{\dot\alpha}$ (Grassmann variables of $\mathcal{N}=1$ superspace).

Smile conjugation

Consider a projective superfield $Q(z,\zeta)$,

$$Q(z,\zeta) \equiv Q^{[n]}(z,\zeta) = \sum_{-\infty}^{+\infty} Q_k(z)\zeta^k \ .$$

It obeys the analyticity conditions

$$D^{\underline{2}}_\alpha Q(\zeta) = \zeta\, D^{\underline{1}}_\alpha Q(\zeta)\ , \qquad \bar{D}_{\dot\alpha\,\underline{2}} Q(\zeta) = -\frac{1}{\zeta}\,\bar{D}_{\dot\alpha\,\underline{1}} Q(\zeta)\ .$$

Let $\bar{Q}(z,\bar\zeta)$ be the complex conjugate of $Q(z,\zeta)$:

$$\bar{Q}(z,\bar\zeta) = \sum_{-\infty}^{+\infty} \bar{Q}_k(z)\bar\zeta^k\ , \qquad \bar{Q}_k(z) := \overline{Q_k(z)}\ .$$

It is not a projective superfield, for it obeys the constrains

$$D^{\underline{2}}_\alpha \bar{Q}(\bar\zeta) = -\frac{1}{\bar\zeta}\, D^{\underline{1}}_\alpha \bar{Q}(\bar\zeta)\ , \qquad \bar{D}_{\dot\alpha\,\underline{2}} \bar{Q}(\bar\zeta) = \bar\zeta\, \bar{D}_{\dot\alpha\,\underline{1}} \bar{Q}(\bar\zeta)$$

which do not coincide with the analyticity conditions. However, the following object

$$\breve{Q}(z,\zeta) := \bar{Q}\left(z, -\frac{1}{\zeta}\right) = \sum_{-\infty}^{+\infty} (-1)^k \bar{Q}_{-k}(z)\zeta^k$$

does obey the analyticity conditions, and therefore it is a projective superfield. They say $\breve{Q}(\zeta)$ is the smile-conjugate of $Q(\zeta)$.

Real projective superfields:

$$\breve{Q}(z,\zeta) = Q(z,\zeta) = \sum_{-\infty}^{+\infty} Q_k(z)\zeta^k\ , \qquad \bar{Q}_k(z) = (-1)^k Q_{-k}(z)$$

$\mathcal{N}=2$ supersymmetric action in $\mathcal{N}=1$ superspace

$$S := \frac{1}{2\pi}\oint_\gamma v_i \mathrm{d}v^i \int \mathrm{d}^4x\, \Delta^{(-4)}\mathcal{L}^{(2)}\Big|_{\theta_i=\bar\theta^i=0}~,$$

where $\mathcal{L}^{(2)}(z,v)$ be a real weight-2 projective superfield, and

$$\Delta^{(-4)} := \frac{1}{16}\nabla^2\bar\nabla^2~, \qquad \nabla_\alpha := \frac{1}{u_k v^k}u_i D^i_\alpha~, \quad \bar\nabla_{\dot\beta} := \frac{1}{u_k v^k}u_i \bar D^i_{\dot\beta}$$

Here u_k is a fixed isotwistor such that $u_k v^k \neq 0$ at each point of γ.

Without loss of generality, we can assume that the integration contour γ does not pass through the "north pole" $v^i \sim (0,1)$.

Introduce the complex variable ζ on $\mathbb{C}P^1 \setminus \{\infty\}$ $\qquad v^i = v^{\underline{1}}(1,\zeta)$

Use the fact that S is independent of u_i to fix $\qquad u_i = (1,0)$

Represent the Lagrangian in the form

$$\mathcal{L}^{(2)}(z,v) = \mathrm{i}\,v^{\underline{1}}v^{\underline{2}}\mathcal{L}(z,\zeta) = \mathrm{i}(v^{\underline{1}})^2\,\zeta\,\mathcal{L}(z,\zeta)~, \qquad \breve{\mathcal{L}} = \mathcal{L}~.$$

Then, the action reduces to

$$S = \frac{1}{16}\oint \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}}\int \mathrm{d}^4x\,\zeta\,(D^{\underline{1}})^2(\bar D_{\underline{2}})^2\mathcal{L}(z,\zeta)\Big|_{\theta_i=\bar\theta^i=0}~.$$

Finally, making use of the analyticity of $\mathcal{L}$,

$$D^{\underline{2}}_\alpha\mathcal{L}(\zeta) = \zeta\, D^{\underline{1}}_\alpha\mathcal{L}(\zeta)~, \qquad \bar D^{\dot\alpha}_{\underline{2}}\mathcal{L}(\zeta) = -\frac{1}{\zeta}\,\bar D^{\dot\alpha}_{\underline{1}}\mathcal{L}(\zeta)~,$$

the action turns into

$$S = \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}\zeta}{\zeta}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{L}(z,\zeta)\Big|_{\theta_{\underline{2}}=\bar\theta^{\underline{2}}=0}$$

where the integration is carried out over the $\mathcal{N}=1$ superspace.

Important property of projective multiplets

$$Q^{[n]}(z,\zeta) = \sum_{p}^{q} Q_k(z)\zeta^k \;, \qquad -\infty \leq p < q \leq +\infty$$

In terms of Q_k, the analyticity conditions are

$$D^{\underline{2}}_\alpha Q_k = D^{\underline{1}}_\alpha Q_{k-1} \;, \qquad \bar{D}^{\dot\alpha}_{\underline{2}} Q_{k-1} = -\bar{D}^{\dot\alpha}_{\underline{1}} Q_k \;.$$

Suppose the series terminates from below, that is $p > -\infty$.

Then Q_p and Q_{p+1} are constrained $\mathcal{N}=1$ superfields

$$\boxed{\bar{D}^{\dot\alpha} Q_p = 0 \;, \qquad \bar{D}^2 Q_{p+1} = 0} \qquad \bar{D}^{\dot\alpha} := \bar{D}^{\dot\alpha}_{\underline{1}}$$

Q_p is chiral, while Q_{p+1} is said to be linear.

Suppose the series terminates from above, that is $q < \infty$.

Then Q_q and Q_{q-1} are constrained $\mathcal{N}=1$ superfields

$$\boxed{D_\alpha Q_q = 0 \;, \qquad D^2 Q_{q-1} = 0} \qquad D_\alpha := D^{\underline{1}}_\alpha$$

Q_p is antichiral, while Q_{p+1} is said to be antilinear.

Very special case: $q - p = 2$

$$\boxed{\bar{D}_{\dot\alpha} Q_p = 0 \;, \qquad \bar{D}^2 Q_{p+1} = D^2 Q_{p+1} = 0 \;, \qquad D_\alpha Q_{p+2} = 0}$$

Projective multiplets suitable for σ-model constructions

Real $\mathcal{O}(2n)$ multiplet, $n = 2, 3\dots$

$$
\begin{aligned}
H^{(2n)}(z,v) &= H_{i_1\dots i_{2n}}(z) v^{i_1}\dots v^{i_{2n}} \\
&\qquad = (\mathrm{i}\, v^{\underline{1}} v^{\underline{2}})^n H^{[2n]}(z,\zeta) = (v^{\underline{1}})^{2n} (\mathrm{i}\,\zeta)^n H^{[2n]}(z,\zeta) \ , \\
H^{[2n]}(z,\zeta) &= \sum_{k=-n}^{n} H_k(z)\zeta^k \ , \qquad \bar{H}_k = (-1)^k H_{-k} \ , \\
\bar{D}_{\dot\alpha} H_{-n} &= 0 \ , \qquad \bar{D}^2 H_{-n+1} = 0 \ .
\end{aligned}
$$

$\mathcal{N}=2$ tensor multiplet $\equiv$ Real $\mathcal{O}(2)$ multiplet

$$
\eta(\zeta) = \frac{1}{\zeta}\varphi + G - \zeta\,\bar\varphi \ , \qquad \bar{D}_{\dot\alpha}\varphi = 0 \ , \qquad \bar{D}^2 G = 0 \ , \quad \bar{G} = G \ .
$$

General $\mathcal{N}=2$ σ-model couplings of tensor multiplets

$$
S = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{L}\Big(\eta(\zeta);\zeta\Big)
$$

Karlhede, Lindström & Roček (1984)

Upon evaluation of the contour integral, the action reduces to that constructed originally in the $\mathcal{N}=1$ superspace setting:

Lindström & Roček (1983)

General $\mathcal{N}=2$ σ-model couplings of $\mathcal{O}(2n)$ multiplets

$$
S = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{L}\Big(H^{[\dots]}(\zeta);\zeta\Big)
$$

Ketov, Lokhvitsky & Tyutin (1987)

Lindström & Roček (1988)

`Polar multiplet = arctic + antarctic multiplets`

$$\Upsilon(z,\zeta) = \sum_{k=0}^{\infty} \Upsilon_k(z)\zeta^k \qquad \text{arctic multiplet}$$

$$\bar{D}_{\dot\alpha}\Upsilon_0 = 0~, \qquad \bar{D}^2\Upsilon_1 = 0$$

$$\breve{\Upsilon}(z,\zeta) = \sum_{k=0}^{\infty} (-1)^k \bar{\Upsilon}_k(z)\frac{1}{\zeta^k} \qquad \text{antarctic multiplet}$$

General $\mathcal{N}=2$ σ-model of polar multiplets

$$S = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{L}\Big(\Upsilon(\zeta),\breve{\Upsilon}(\zeta);\zeta\Big)$$

Lindström & Roček (1988)

In all cases considered, the Lagrangian may depend explicitly on ζ.
All Lagrangians

$$\mathcal{L}(\eta;\zeta)~, \qquad \mathcal{L}(H^{[\dots]};\zeta)~, \qquad \mathcal{L}(\Upsilon,\breve{\Upsilon};\zeta)$$

are analytic functions of their arguments, but otherwise arbitrary, modulo a reality condition w.r.t. smile-conjugation.

Unique properties of the arctic multiplet:

- Can be used to describe a charged hypermultiplet

  phase transformation: $\Upsilon(\zeta) \;\to \mathrm{e}^{\mathrm{i}\alpha}\Upsilon(\zeta)~, \qquad \alpha \in \mathbb{R}~.$

- Ring structure: for any arctic superfields Υ_A and Υ_B, their product

$$\Upsilon_A(\zeta)\cdot\Upsilon_B(\zeta) = \Upsilon_C(\zeta)$$

is arctic.

Generalized Legendre transform construction

Lindström & Roček (1988)

To fix the ideas, consider a $\mathcal{N}=2$ supersymmetric nonlinear sigma-model described by a single $\mathcal{O}(2n)$ multiplet ($n\geq 2$) or a polar multiplet. Upon evaluation of the contour integral, the actions becomes

$$S=\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, L_{\mathrm{off-shell}}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma},\mathcal{U}_\imath)\ ,$$

for some Lagrangian $L_{\mathrm{off-shell}}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma},\mathcal{U}_\imath)$.

The dynamical variables are:

- two *physical* superfields Φ and Σ and their conjugates $\bar{\Phi}$ and $\bar{\Sigma}$;
- some number of *auxiliary* superfields $\mathcal{U}_\imath$.

[the index $\imath$ may take a finite ($2n-3$ in the case of $\mathcal{O}(2n)$ multiplets) or infinite (in the case of polar multiplet) number of values].

The physical superfields Φ and Σ are chiral and complex linear

$$\bar{D}_{\dot{\alpha}}\Phi=0\ ,\qquad\qquad \bar{D}^2\Sigma=0\ ,$$

while the auxiliary superfields $\mathcal{U}_\imath$ are unconstrained.

The $\mathcal{U}$'s are auxiliary, for their Euler-Lagrange equations are algebraic

$$\frac{\partial}{\partial\mathcal{U}_\jmath}L_{\mathrm{off-shell}}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma},\mathcal{U}_\imath)=0\quad\Longrightarrow\quad \mathcal{U}_\imath=\mathcal{U}_\imath(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma})\ .$$

This leads to an action formulated in terms of the physical superfields

$$S=\int \mathrm{d}^4x\,\mathrm{d}^4\theta\, L_{\mathrm{on-shell}}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma})\ .$$

This action is of course $\mathcal{N}=2$ supersymmetric. However, the Lagrangian $L_{\mathrm{on-shell}}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma})$ is not a hyperkähler potential, for Σ complex linear. One needs a formulation in terms of chiral superfields only.

Duality between chiral and complex linear superfields

Zumino (1980)

Gates & Siegel (1981)

free scalar multiplet	off-shell constraint	equation of motion
minimal	$\bar{D}_{\dot\alpha}\Phi = 0$	$D^2\Phi = 0$
non-minimal	$D^2\bar\Sigma = 0$	$\bar{D}_{\dot\alpha}\bar\Sigma = 0$

Free action functionals:

Minimal scalar multiplet

$$S = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \bar\Phi\, \Phi$$

Non-minimal scalar multiplet

$$S = -\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \bar\Sigma\, \Sigma$$

The two formulations are related by the first-order action:

$$S_{\text{first-order}} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta \Big\{ -\bar\Gamma\,\Gamma + \Psi\,\Gamma + \bar\Psi\bar\Gamma \Big\} \ .$$

Here Γ is complex unconstrained, while Ψ is chiral, $\bar{D}_{\dot\alpha}\Psi = 0$.

The above sigma-model action

$$S = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, L_{\text{on-shell}}(\Phi, \bar\Phi, \Sigma, \bar\Sigma) \ .$$

is equivalent to the following first-order action:

$$S_{\text{first-order}} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta \Big\{ L_{\text{on-shell}}(\Phi, \bar\Phi, \Gamma, \bar\Gamma) \ + \Psi\,\Gamma + \bar\Psi\bar\Gamma \Big\} \ .$$

Integrating out Γ and $\bar\Gamma$ leads to an action of the form

$$S_{\text{dual}} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, H(\Phi, \bar\Phi, \Psi, \bar\Psi) \ .$$

$H(\Phi, \bar\Phi, \Psi, \bar\Psi)$ is the Legendre transform of $L_{\text{on-shell}}(\Phi, \bar\Phi, \Sigma, \bar\Sigma)$.

$H(\Phi, \bar\Phi, \Psi, \bar\Psi)$ is the Kähler potential of a hyperkähler manifold.

Ancient Legendre transform construction

Lindström & Roček (1983)

Hitchin, Karlhede, Lindström & Roček (1987)

Consider a general $\mathcal{N}=2$ σ-model described by tensor multiplets

$$S = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{L}\Big(\eta(\zeta);\zeta\Big) ,$$

$$\eta(\zeta) = \frac{1}{\zeta}\varphi + G - \zeta\,\bar\varphi , \qquad \bar D_{\dot\alpha}\varphi = 0 , \qquad \bar D^2 G = 0 , \quad \bar G = G .$$

No auxiliary superfields !

Upon evaluation of the contour integral, the action becomes

$$S = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, L(\varphi,\bar\varphi,G) .$$

Lagrangian $L(\varphi,\bar\varphi,G)$ is not a hyperkähler potential, for G is real linear. To derive the relevant hyperkähler potential, we have to dualize G into a chiral superfield Ψ and its conjugate $\bar\Psi$, by considering the following first-order action:

$$S_{\mathrm{first-order}} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta \Big\{ L(\varphi,\bar\varphi,U) + U(\Psi+\bar\Psi) \Big\} .$$

Here U is real unconstrained, and Ψ is chiral, $\bar D_{\dot\alpha}\Psi = 0$. Integrating out U leads to an action of the form

$$S_{\mathrm{dual}} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, H(\varphi,\bar\varphi,\Psi+\bar\Psi) .$$

$H(\varphi,\bar\varphi,\Psi+\bar\Psi)$ is the Legendre transform of $L(\varphi,\bar\varphi,G)$.

$H(\varphi,\bar\varphi,\Psi+\bar\Psi)$ is the Kähler potential of a hyperkähler manifold.

Most general $\mathcal{N}=2$ σ-model couplings

In general, $\mathcal{N}=$ supersymmetric σ-models can describe couplings of tensor multiplets, $\mathcal{O}(2n)$ multiplets and polar multiplets.

$$S = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, \mathcal{L}(\eta, H^{[\dots]}, \Upsilon, \breve{\Upsilon}; \zeta) \ .$$

It is always possible, in principle, to dualize any tensor multiplet into a polar multiplet, and also any $\mathcal{O}(2n)$ multiplet into a polar one.

Lindström & Roček (1988)

Gonzalez-Rey, Lindström, Roček, von Unge & Wiles (1998)

As a result, the most general $\mathcal{N}=2$ σ-model can in principle be described by polar multiplets only.

$$S = \frac{1}{2\pi \mathrm{i}} \oint \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, \mathcal{L}(\Upsilon, \breve{\Upsilon}; \zeta) \ .$$

Different choices of $\mathcal{L}\Big(\Upsilon, \breve{\Upsilon}; \zeta\Big)$ may lead to one and the same hyperkähler geometry. The point is that a polar multiplet can be dualized into a polar one, and the dual Lagrangian differs, in general, from the original one.

Gates & SMK (1999)

Lindström & Roček (2009)

Example: For any complex parameter $a \in \mathbb{C}$, the Lagrangian

$$\mathcal{L}_a(\Upsilon, \breve{\Upsilon}; \zeta) = \frac{1}{1-|a|^2} \Big\{ \breve{\Upsilon}\Upsilon + \frac{\bar{a}}{2} \frac{1}{\zeta^2} \Upsilon^2 + \frac{a}{2} \zeta^2 \breve{\Upsilon}^2 \Big\}$$

is equivalent (dual) to

$$\mathcal{L}(\Upsilon, \breve{\Upsilon}) = \breve{\Upsilon}\Upsilon \ .$$

`Sigma models on cotangent bundles of Kähler manifolds`

SMK (1998)

Gates & SMK (1999, 2000)

$$S[\Upsilon,\breve{\Upsilon}] = \frac{1}{2\pi \mathrm{i}} \oint_\gamma \frac{\mathrm{d}\zeta}{\zeta} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K\Big(\Upsilon^I(\zeta), \breve{\Upsilon}^{\bar J}(\zeta)\Big)~.$$

Here $\Upsilon^I(\zeta)$ are **`arctic`** and $\breve{\Upsilon}^{\bar J}(\zeta)$ **`antarctic`** multiplets

$$\Upsilon(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n = \Phi + \Sigma\,\zeta + O(\zeta^2)~, \qquad \breve{\Upsilon}(\zeta) = \sum_{n=0}^{\infty} \bar{\Upsilon}_n (-\zeta)^{-n}~.$$

Here Φ is chiral, Σ complex linear,

$$\bar{D}_{\dot\alpha} \Phi = 0~, \qquad \bar{D}^2 \Sigma = 0~,$$

and the remaining component superfields are complex unconstrained. This theory proves to have remarkable geometric properties.

The above theory is a minimal $\mathcal{N}=2$ extension of the general 4D $\mathcal{N}=1$ supersymmetric nonlinear sigma-model

Zumino (1979)

$$S[\Phi,\bar\Phi] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, K(\Phi^I, \bar{\Phi}^{\bar J})~,$$

with K the Kähler potential of a Kähler manifold $\mathcal{M}$.

Most general σ-model couplings of polar multilpets:

Lindström & Roček (1988)

$$K\Big(\Upsilon^I(\zeta), \widetilde{\Upsilon}^{\bar J}(\zeta)\Big) \quad \longrightarrow \quad \mathcal{L}\Big(\Upsilon^I(\zeta), \widetilde{\Upsilon}^{\bar J}(\zeta); \zeta\Big)$$

Homogeneity of time (t) in CM $\longrightarrow$ no explicit t dependence

Fundamental properties of the theory:

Rigid U(1) symmetry (holomorphic action in the fibers)

$$\Upsilon(\zeta) \quad \mapsto \quad \Upsilon(\mathrm{e}^{\mathrm{i}\alpha}\zeta) \quad \Longleftrightarrow \quad \Upsilon_n(z) \quad \mapsto \quad \mathrm{e}^{\mathrm{i}n\alpha}\Upsilon_n(z)$$

is present iff the Lagrangian has no explicit dependence on ζ.

Transformation $\zeta \to \mathrm{e}^{\mathrm{i}\alpha}\zeta$ is a time translation along γ.

The $\mathcal{N} = 2$ supersymmetric sigma-model inherits all the geometric features of its $\mathcal{N} = 1$ predecessor.

Kähler invariance

$$\begin{aligned} \mathcal{N} = 1 \text{ case}: \qquad & K(\Phi, \bar{\Phi}) \quad \longrightarrow \quad K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \\ \mathcal{N} = 2 \text{ case}: \qquad & K(\Upsilon, \breve{\Upsilon}) \quad \longrightarrow \quad K(\Upsilon, \breve{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\breve{\Upsilon})\ . \end{aligned}$$

Holomorphic reparametrizations of the Kähler manifold

$$\begin{aligned} \mathcal{N} = 1 \text{ case}: \qquad & \Phi^I \quad \longrightarrow \quad \Phi'^I = f^I(\Phi) \\ \mathcal{N} = 2 \text{ case}: \qquad & \Upsilon^I(\zeta) \quad \longrightarrow \quad \Upsilon'^I(\zeta) = f^I\Big(\Upsilon(\zeta)\Big) \end{aligned}$$

Therefore, the physical superfields of the $\mathcal{N} = 2$ theory

$$\Upsilon^I(\zeta)|_{\zeta=0} \;=\; \Phi^I\ , \qquad \frac{\mathrm{d}\Upsilon^I(\zeta)}{\mathrm{d}\zeta}\Big|_{\zeta=0} \;=\; \Sigma^I\ ,$$

should be regarded, respectively, as coordinates of a point in the Kähler manifold and a tangent vector at the same point.

The variables (Φ^I, Σ^J) parametrize the holomorphic tangent bundle $T\mathcal{M}$ of the Kähler manifold.

SMK (1998)

Equations of motion for the auxiliary superfields

To describe the theory in terms of the physical superfields Φ and Σ only, all the auxiliary superfields have to be eliminated with the aid of the corresponding algebraic equations of motion

$$\oint \frac{\mathrm{d}\zeta}{\zeta}\,\zeta^n\,\frac{\partial K(\Upsilon,\widetilde{\Upsilon})}{\partial \Upsilon^I} = \oint \frac{\mathrm{d}\zeta}{\zeta}\,\zeta^{-n}\,\frac{\partial K(\Upsilon,\widetilde{\Upsilon})}{\partial \widetilde{\Upsilon}^{\bar{J}}} = 0\ , \qquad n \geq 2\ .$$

Let $\Upsilon_*(\zeta) \equiv \Upsilon_*(\zeta;\Phi,\bar{\Phi},\Sigma,\bar{\Sigma})$ denote a unique solution subject to the initial conditions

$$\Upsilon_*(0) = \Phi\ , \qquad \dot{\Upsilon}_*(0) = \Sigma\ .$$

Perturbative elimination of the auxiliary superfields

For a general Kähler manifold $\mathcal{M}$, the auxiliary superfields $\Upsilon_2, \Upsilon_3, \dots$, and their conjugates, can be eliminated only perturbatively. Their elimination can be carried out using the *ansatz*

$$\Upsilon^I_n = \sum_{p=0}^{\infty} G^I{}_{J_1\dots J_{n+p}\,\bar{L}_1\dots\bar{L}_p}(\Phi,\bar{\Phi})\,\Sigma^{J_1}\dots\Sigma^{J_{n+p}}\,\bar{\Sigma}^{\bar{L}_1}\dots\bar{\Sigma}^{\bar{L}_p}\ , \qquad n \geq 2\ .$$

SMK & Linch (2006)

Example: **Hermitian symmetric space**

$$\nabla_L R_{I_1\bar{J}_1 I_2\bar{J}_2} = \bar{\nabla}_{\bar{L}} R_{I_1\bar{J}_1 I_2\bar{J}_2} = 0\ .$$

The $\Upsilon_*(\zeta)$ turns out to obey the generalized geodesic equation:

$$\frac{\mathrm{d}^2\Upsilon^I_*(\zeta)}{\mathrm{d}\zeta^2} + \Gamma^I_{JK}\Big(\Upsilon_*(\zeta),\bar{\Phi}\Big)\,\frac{\mathrm{d}\Upsilon^J_*(\zeta)}{\mathrm{d}\zeta}\,\frac{\mathrm{d}\Upsilon^K_*(\zeta)}{\mathrm{d}\zeta} = 0\ .$$

Gates & SMK (1999)

Two nontrivial technical issues to address:

- Elimination of the auxiliary fields in order to end up with $\Upsilon_*(\zeta)$;
- Evaluation of the contour integral

$$S_{\rm tb}[\Phi,\Sigma] = \frac{1}{2\pi {\rm i}} \oint \frac{{\rm d}\zeta}{\zeta} \int {\rm d}^4x\, {\rm d}^4\theta\, K(\Upsilon_*(\zeta), \breve{\Upsilon}_*(\zeta))$$

Outcome:

$$S_{\rm tb}[\Phi,\Sigma] = \int {\rm d}^4x\, {\rm d}^4\theta \Big\{ K(\Phi,\bar\Phi) + \mathcal{L}(\Phi,\bar\Phi,\Sigma,\bar\Sigma) \Big\} ,$$

$$\mathcal{L} = \sum_{n=1}^{\infty} \mathcal{L}_{I_1\cdots I_n \bar J_1 \cdots \bar J_n}(\Phi,\bar\Phi)\Sigma^{I_1}\dots\Sigma^{I_n}\bar\Sigma^{\bar J_1}\dots\bar\Sigma^{\bar J_n} := \sum_{n=1}^{\infty} \mathcal{L}^{(n)} .$$

Here $\mathcal{L}_{I\bar J} = -g_{I\bar J}(\Phi,\bar\Phi)$ and the coefficients $\mathcal{L}_{I_1\cdots I_n \bar J_1\cdots \bar J_n}$, for $n>1$, are tensor functions of the Kähler metric $g_{I\bar J}(\Phi,\bar\Phi) = \partial_I \partial_{\bar J} K(\Phi,\bar\Phi)$, the Riemann curvature $R_{I\bar J K \bar L}(\Phi,\bar\Phi)$ and its covariant derivatives.

Each term in the action contains equal powers of Σ and $\bar\Sigma$, since the original action is invariant under the rigid $U(1)$ transformations

$$\Upsilon(\zeta) \mapsto \Upsilon({\rm e}^{{\rm i}\alpha}\zeta) \quad \Longleftrightarrow \quad \Upsilon_n(z) \mapsto {\rm e}^{{\rm i}n\alpha}\Upsilon_n(z) .$$

`Explicit expressions for several scalar functions` $\mathcal{L}^{(n)}$

$$\begin{aligned}
\mathcal{L}^{(1)} &= -g_{I\bar J}\Sigma^I\bar\Sigma^{\bar J} , \\
\mathcal{L}^{(2)} &= \frac{1}{4} R_{I_1\bar J_1 I_2 \bar J_2}\Sigma^{I_1}\Sigma^{I_2}\bar\Sigma^{\bar J_1}\bar\Sigma^{\bar J_2} , \\
\mathcal{L}^{(3)} &= -\frac{1}{12}\Big\{\frac{1}{6}\{\nabla_{I_3},\bar\nabla_{\bar J_3}\}R_{I_1\bar J_1 I_2\bar J_2} + R_{I_1\bar J_1 I_2}{}^L R_{L\bar J_2 I_3\bar J_3}\Big\} \\
&\qquad \times \Sigma^{I_1}\dots\Sigma^{I_3}\bar\Sigma^{\bar J_1}\dots\bar\Sigma^{\bar J_3}
\end{aligned}$$

To construct a dual formulation, consider the first-order action

$$S_{\rm f.-o.} = \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, \Big\{ K(\Phi, \bar{\Phi}) + \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I \, \Sigma^I + \bar{\Psi}_{\bar{I}} \bar{\Sigma}^{\bar{I}} \Big\} ~.$$

Here the tangent vector Σ^I is now complex unconstrained, while the one-form Ψ_I is chiral, $\bar{D}_{\dot{\alpha}} \Psi_I = 0$.

Varying Ψ_I gives $\bar{D}^2 \Sigma^I = 0$, and $S_{\rm f.-o.}$ reduces to the original action. On the other hand, varying Σ^I gives

$$\frac{\partial}{\partial \Sigma^I} \mathcal{L}(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) + \Psi_I = 0 ~.$$

Eliminating Σ's and their conjugates leads to the dual action

$$S_{\rm ctb}[\Phi, \Psi] = \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, \Big\{ K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) \Big\} ~,$$

where

$$\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \cdots I_n \bar{J}_1 \cdots \bar{J}_n}(\Phi, \bar{\Phi}) \Psi_{I_1} \dots \Psi_{I_n} \bar{\Psi}_{\bar{J}_1} \dots \bar{\Psi}_{\bar{J}_n} ~,$$

$$\mathcal{H}^{I \bar{J}}(\Phi, \bar{\Phi}) = g^{I \bar{J}}(\Phi, \bar{\Phi}) ~.$$

In the dual formulation of the $\mathcal{N} = 2$ supersymmetric sigma-model, the target space is (an open neighborhood of the zero section of) the cotangent bundle $T^*\mathcal{M}$ of the Kähler manifold $\mathcal{M}$.
It is therefore a hyperkähler space, and

$$\mathbb{K}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) := K(\Phi, \bar{\Phi}) + \mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$$

the corresponding hyperkähler potential.

Gates & SMK (1999)

Parallel mathematical results:

D. Kaledin, "Hyperkähler structures on total spaces of holomorphic cotangent bundles," in D. Kaledin and M. Verbitsky, *Hyperkähler Manifolds*, International Press, Cambridge MA, 1999 [alg-geom/9710026];

"A canonical hyperkähler metric on the total space of a cotangent bundle," in *Quaternionic Structures in Mathematics and Physics*, S. Marchiafava, P. Piccinni and M. Pontecorvo (Eds.), World Scientific, 2001 [alg-geom/0011256].

B. Feix, "Hyperkähler metrics on cotangent bundles," Cambridge PhD thesis, 1999; J. reine angew. Math. **532**, 33 (2001).

Both constructions are existence theorems.

`N = 2 supersymmetric nonlinear sigma-models on cotangent bundles of` *Hermitian symmetric spaces*

If the Kähler manifold $\mathcal{M}$ is Hermitian symmetric,
in particular its Riemann curvature is covariantly constant,

$$\nabla_L R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = \bar{\nabla}_{\bar{L}} R_{I_1 \bar{J}_1 I_2 \bar{J}_2} = 0 ,$$

then the $\mathcal{N} = 2$ supersymmetric sigma-model on $T^*\mathcal{M}$ can be derived in closed form. To carry out such a construction, there have been developed two alternative methods that are based on the use of conceptually different ideas and tools:

- **Method 1** makes use of the properties that

(i) $\mathcal{M}$ is a homogeneous space, $\mathcal{M} = G/H$;
(ii) the group G acts on $\mathcal{M}$ by holomorphic isometries.

Gates & SMK (1999, 2000)
Arai & Nitta (2006)
Arai, SMK & Lindström (2007a)

- **Method 2** makes use of

(i) the covariant constancy of the curvature;
(ii) extended supersymmetry.

Arai, SMK & Lindström (2007b)
SMK & Novak (2008)

Both methods will be reviewed below.

Canonical (or Kähler normal) coordinates

S. Bochner (1947)
E. Calabi (1953)

Given a Kähler manifold $\mathcal{M}$, $\forall p_0 \in \mathcal{M}$ $\exists$ a neighborhood where holomorphic reparametrizations and Kähler transformations allow one to choose coordinates with origin at p_0 in which the Kähler potential is

$$K(\Phi,\bar{\Phi}) = g_{I\bar{J}}|\, \Phi^I \bar{\Phi}^{\bar{J}} + \sum_{m,n\geq 2}^{\infty} K^{(m,n)}(\Phi,\bar{\Phi}) \ ,$$

$$K^{(m,n)}(\Phi,\bar{\Phi}) := \frac{1}{m!n!} K_{I_1\cdots I_m \bar{J}_1\cdots\bar{J}_n}|\, \Phi^{I_1}\dots\Phi^{I_m}\bar{\Phi}^{\bar{J}_1}\dots\bar{\Phi}^{\bar{J}_n} \ .$$

There still remains freedom to perform linear reparametrizations which can be used to set the metric at the origin, $p \in \mathcal{M}$, to be $g_{I\bar{J}}| = \delta_{I\bar{J}}$. It turns out that the coefficients $K_{I_1\cdots I_m \bar{J}_1\cdots\bar{J}_n}|$ are tensor functions of the Kähler metric $g_{I\bar{J}}|$, the Riemann curvature $R_{I\bar{J}K\bar{L}}|$ and its covariant derivatives, evaluated at the origin. In particular, one finds

$$\begin{aligned}
K^{(2,2)} &= \frac{1}{4} R_{I_1\bar{J}_1 I_2\bar{J}_2}|\, \Phi^{I_1}\Phi^{I_2}\bar{\Phi}^{\bar{J}_1}\bar{\Phi}^{\bar{J}_2} \ ,\\
K^{(3,2)} &= \frac{1}{12}\nabla_{I_3} R_{I_1\bar{J}_1 I_2\bar{J}_2}|\, \Phi^{I_1}\dots\Phi^{I_3}\bar{\Phi}^{\bar{J}_1}\bar{\Phi}^{\bar{J}_2} \ ,\\
K^{(4,2)} &= \frac{1}{48}\nabla_{I_3}\nabla_{I_4} R_{I_1\bar{J}_1 I_2\bar{J}_2}|\, \Phi^{I_1}\dots\Phi^{I_4}\bar{\Phi}^{\bar{J}_1}\bar{\Phi}^{\bar{J}_2} \ ,\\
K^{(3,3)} &= \frac{1}{12}\Big\{\frac{1}{6}\{\nabla_{I_3},\bar{\nabla}_{\bar{J}_3}\} R_{I_1\bar{J}_1 I_2\bar{J}_2}| + R_{I_1\bar{J}_1 I_2}{}^{L}| R_{L\bar{J}_2 I_3\bar{J}_3}|\Big\}\\
&\qquad \times\Phi^{I_1}\dots\Phi^{I_3}\bar{\Phi}^{\bar{J}_1}\dots\bar{\Phi}^{\bar{J}_3}
\end{aligned}$$

If $\mathcal{M}$ is Hermitian symmetric, then

$$\nabla_L R_{I_1\bar{J}_1 I_2\bar{J}_2} = \bar{\nabla}_{\bar{L}} R_{I_1\bar{J}_1 I_2\bar{J}_2} = 0 \qquad \Longrightarrow \qquad K^{(m,n)} = 0 \ , \quad m \neq n \ .$$

For Hermitian symmetric spaces, there exists a closed form expression for the Kähler potential in canonical coordinates:

SMK & Novak (2008)

$$K(\Phi,\bar\Phi) = -\frac{1}{2}\boldsymbol{\Phi}^{\rm T}\boldsymbol{g}\,\frac{\ln\left(\mathbb{1}-\boldsymbol{R}_{\Phi,\bar\Phi}\right)}{\boldsymbol{R}_{\Phi,\bar\Phi}}\,\boldsymbol{\Phi}\ , \qquad \boldsymbol{\Phi} := \begin{pmatrix}\Phi^I\\ \bar\Phi^{\bar I}\end{pmatrix}\ .$$

Here

$$\boldsymbol{R}_{\Phi,\bar\Phi} := \begin{pmatrix} 0 & (R_\Phi)^I{}_{\bar J}\\ (R_{\bar\Phi})^{\bar I}{}_J & 0\end{pmatrix}\ , \qquad (R_\Phi)^I{}_{\bar J} := \frac{1}{2}R_K{}^I{}_{L\bar J}|\,\Phi^K\Phi^L\ ,$$

$$\boldsymbol{g} := \begin{pmatrix} 0 & g_{I\bar J}|\\ g_{\bar I J}| & 0\end{pmatrix}\ , \qquad (R_{\bar\Phi})^{\bar I}{}_J := \overline{(R_\Phi)^I{}_{\bar J}}\ .$$

Method 1

As before, denote by $\Upsilon_*(\zeta)\equiv\Upsilon_*(\zeta;\Phi,\bar\Phi,\Sigma,\bar\Sigma)$ the unique solution of the auxiliary field equations

$$\oint\frac{{\rm d}\zeta}{\zeta}\,\zeta^n\,\frac{\partial K(\Upsilon,\breve\Upsilon)}{\partial\Upsilon^I} = \oint\frac{{\rm d}\zeta}{\zeta}\,\zeta^{-n}\,\frac{\partial K(\Upsilon,\breve\Upsilon)}{\partial\breve\Upsilon^{\bar J}} = 0\ , \qquad n\geq 2\ .$$

under the initial conditions

$$\Upsilon_*(0)=\Phi\ , \qquad \dot\Upsilon_*(0)=\Sigma\ .$$

Using the canonical coordinates allows us to find a part of the solution:

$$\Upsilon_0(\zeta)\equiv\Upsilon_{p_0}(\zeta) := \Upsilon_*(\zeta;\Phi=0,\bar\Phi=0,\Sigma_0,\bar\Sigma_0)\ , \qquad \Upsilon_{p_0}(0)=p_0$$

with Σ_0 a tangent vector at $p_0\in\mathcal{M}$, the origin of the canonical coordinate system. It is

$$\Upsilon_0(\zeta)=\Sigma_0\,\zeta\ , \qquad \breve\Upsilon_0(\zeta) = -\frac{\bar\Sigma_0}{\zeta}$$

As a next step, we can construct a curve

$$\Upsilon_p(\zeta)\ , \qquad \Upsilon_p(0) = p \in \mathcal{M}$$

obtained from $\Upsilon_{p_0}(\zeta)$ by applying an isometry transformation $g \in G$ such that $g \cdot p_0 = p$. The holomorphic isometry transformations leave invariant the auxiliary field equations.

Let $U \subset \mathcal{M}$ be the neighborhood on which the canonical coordinate system is defined. We can construct a coset representative, $\mathcal{S}$: $U \to G$, with the following property: associated with $p \in U$ is the holomorphic isometry $\mathcal{S}[p] \in G$ of $\mathcal{M}$, $\quad q \to \mathcal{S}[p] \cdot q\ ,\ \ \forall q \in \mathcal{M}$, such that

$$\mathcal{S}[p] \cdot p_0 = p\ .$$

In local coordinates, $\mathcal{S}[p] = \mathcal{S}[\Phi, \bar{\Phi}]$, and it acts on a generic point $q \in U$ parametrized by complex variables $(\Psi^I, \bar{\Psi}^{\bar{J}})$ as follows:

$$\Psi \to \Psi' = f(\Psi; \Phi, \bar{\Phi})\ , \qquad f(0; \Phi, \bar{\Phi}) = \Phi\ .$$

Now, applying the group transformation $\mathcal{S}(\Phi, \bar{\Phi})$ to $\Upsilon_0(\zeta)$ gives

$$\Upsilon_0(\zeta) \ \to\ \Upsilon_*(\zeta) = f(\Upsilon_0(\zeta); \Phi, \bar{\Phi}) = f(\Sigma_0\, \zeta; \Phi, \bar{\Phi})\ , \qquad \Upsilon_*(0) = \Phi\ .$$

Imposing the second initial condition, $\dot{\Upsilon}_*(0) = \Sigma$, gives

$$\Sigma^I = \Sigma_0^J\, \frac{\partial}{\partial \Psi^J} f^I(\Psi; \Phi, \bar{\Phi})|_{\Psi=0}\ ,$$

and thus Σ_0 can be uniquely expressed in terms of Σ and Φ, $\bar{\Phi}$.

Geodesic equation with complex evolution parameter

In the canonical coordinate system, the curve

$$\Upsilon_0(\zeta) = \Sigma_0\,\zeta\;, \qquad \breve{\Upsilon}_0(\zeta) = -\frac{\bar{\Sigma}_0}{\zeta}$$

satisfies the equation

$$\frac{\mathrm{d}^2\Upsilon_0^I(\zeta)}{\mathrm{d}\zeta^2} = \frac{\mathrm{d}^2\Upsilon_0^I(\zeta)}{\mathrm{d}\zeta^2} + \Gamma^I_{JK}\Big(\Upsilon_0(\zeta), \bar{\Phi} = 0\Big)\,\frac{\mathrm{d}\Upsilon_0^J(\zeta)}{\mathrm{d}\zeta}\,\frac{\mathrm{d}\Upsilon_0^K(\zeta)}{\mathrm{d}\zeta} = 0\;.$$

Since the equation

$$\frac{\mathrm{d}^2\Upsilon^I(\zeta)}{\mathrm{d}\zeta^2} + \Gamma^I_{JK}\Big(\Upsilon(\zeta), \bar{\Phi}\Big)\,\frac{\mathrm{d}\Upsilon^J(\zeta)}{\mathrm{d}\zeta}\,\frac{\mathrm{d}\Upsilon^K(\zeta)}{\mathrm{d}\zeta} = 0$$

is invariant under holomorphic isometries, we conclude that $\Upsilon_*(\zeta)$ obeys the generalized geodesic equation:

$$\frac{\mathrm{d}^2\Upsilon_*^I(\zeta)}{\mathrm{d}\zeta^2} + \Gamma^I_{JK}\Big(\Upsilon_*(\zeta), \bar{\Phi}\Big)\,\frac{\mathrm{d}\Upsilon_*^J(\zeta)}{\mathrm{d}\zeta}\,\frac{\mathrm{d}\Upsilon_*^K(\zeta)}{\mathrm{d}\zeta} = 0\;.$$

Corollary:

$$\Upsilon_*(\zeta) = \sum_{n=0}^{\infty} \Upsilon_n \zeta^n = \Phi + \Sigma\,\zeta + \Upsilon_2\,\zeta^2 + O(\zeta^3)\;,$$

where

$$\Upsilon_2^I = -\frac{1}{2}\Gamma^I{}_{JK}(\Phi, \bar{\Phi})\,\Sigma^J\Sigma^K\;.$$

`Example:` Two-sphere $\mathcal{M} = \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$

In the north chart, the Kähler potential and metric are

$$K(z,\bar z) \;=\; r^2 \ln\left(1 + \frac{z\bar z}{r^2}\right)~, \qquad g_{z\bar z}(z,\bar z) \;=\; \left(1 + \frac{z\bar z}{r^2}\right)^{-2}~,$$

with $1/r^2$ being proportional to the curvature.

Fractional linear (isometry) transformation

$$z \;\to\; \mathcal{S}_{[\Phi,\bar\Phi]}(z) \;=\; \frac{z+\Phi}{-\bar\Phi z/r^2 + 1}~, \qquad \mathcal{S}_{[\Phi,\bar\Phi]}(0) \;=\; \Phi$$

induces

$$\Upsilon_*(\zeta) \;=\; \frac{\Phi(1+\Phi\bar\Phi/r^2) + \zeta\Sigma}{1 + \Phi\bar\Phi/r^2 - \zeta\bar\Phi\Sigma/r^2}~,$$

and then

$$K\Big(\Upsilon_*(\zeta), \breve\Upsilon_*(\zeta)\Big) \;=\; r^2 \ln\left\{\Big(1+\Phi\bar\Phi/r^2\Big)\Big(1 - \frac{1}{r^2}\frac{\Sigma\bar\Sigma}{(1+\Phi\bar\Phi/r^2)^2}\Big)\right\}$$
$$+\; \Lambda(\Upsilon_*(\zeta)) + \bar\Lambda(\breve\Upsilon_*(\zeta))~,$$

with $\Lambda(\Phi)$ some holomorphic function.

The action becomes

$$S[\Upsilon_*, \breve\Upsilon_*] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \left\{K(\Phi,\bar\Phi) + r^2 \ln\left(1 - \frac{1}{r^2}\, g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Sigma\bar\Sigma\right)\right\}~,$$

and is well-defined under the global restriction

$$g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Sigma\,\bar\Sigma \;<\; r^2~.$$

Relativistic mechanics: $\qquad v^2 \;<\; c^2~.$

The tangent bundle Lagrangian is

$$\mathcal{L}_{\rm tb}(\Phi,\bar\Phi,\Sigma,\bar\Sigma) = K(\Phi,\bar\Phi) + r^2 \ln\Big(1 - \frac{1}{r^2}\, g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Sigma\,\bar\Sigma\Big)$$

Dual formulation:

(a) Replace

$$S[\Sigma,\bar\Sigma] = r^2 \int \mathrm{d}^8 z\ \ln\Big(1 - \frac{1}{r^2}\, g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Sigma\bar\Sigma\Big)\ , \qquad \bar D^2 \Sigma = 0$$

with the first-order action

$$S[\Sigma,\bar\Sigma,\Psi,\bar\Psi] = \int \mathrm{d}^8 z\ \Big\{ r^2 \ln\Big(1 - \frac{1}{r^2}\, g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Sigma\bar\Sigma\Big) + \Sigma\Psi + \bar\Sigma\bar\Psi \Big\}\ ,$$

$$\bar D_{\dot\alpha}\Psi = 0\ ,$$

where Σ is a complex unconstrained superfield.

(b) Eliminate Σ using its equation of motion

$$\frac{g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\bar\Sigma}{1 - g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Sigma\bar\Sigma/r^2} = \Psi\ , \qquad g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Sigma\,\bar\Sigma < r^2\ .$$

No restriction on Ψ (this is similar to relativistic mechanics).

Complex dynamical variables (Φ,Ψ) parametrize $T^*\mathbb{C}P^1$.

One ends up with the hyperkähler potential

$$\mathbb{K}(\Phi,\bar\Phi,\Psi,\bar\Psi) = K(\Phi,\bar\Phi) + \mathcal{H}(\Phi,\bar\Phi,\Psi,\bar\Psi)\ ,$$

$$\mathcal{H}(\Phi,\bar\Phi,\Psi,\bar\Psi) = r^2 \mathcal{F}(\kappa)\ , \qquad \kappa = \frac{1}{r^2}\, g^{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Psi\bar\Psi$$

$$\mathcal{F}(x) := \frac{1}{x}\Big\{\sqrt{1+4x} - 1 - \ln\frac{1+\sqrt{1+4x}}{2}\Big\}\ , \qquad \mathcal{F}(0) = 1$$

corresponding to the Eguchi-Hanson metric.

`Example:` Projective plane $\mathcal{M} = \mathrm{SU}(1,1)/\mathrm{U}(1) \equiv \mathbf{H}$

Kähler potential and metric

$$K(z,\bar z) \;=\; -\,r^2 \ln\Big(1 - \frac{z\bar z}{r^2}\Big) ~, \qquad g_{z\bar z}(z,\bar z) \;=\; \Big(1 - \frac{z\bar z}{r^2}\Big)^{-2} ~,$$

with $1/r^2$ being proportional to the curvature.

Elimination of the auxiliary superfields gives

$$S[\Upsilon_*, \breve{\Upsilon}_*] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \Big\{ K(\Phi,\bar\Phi) - r^2 \ln\Big(1 + \frac{1}{r^2}\, g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\, \Sigma\,\bar\Sigma\Big)\Big\}$$

The action is defined on $T\mathbf{H}$, *no restriction* on Σ.

Dual formulation

$$\frac{g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\bar\Sigma}{1 + g_{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Sigma\bar\Sigma/r^2} \;=\; \Psi \quad\longrightarrow\quad g^{\Phi\bar\Phi}(\Phi,\bar\Phi)\,\Psi\,\bar\Psi \;<\; r^2 ~.$$

Hyperkähler structure on the open disc bundle in the cotangent bundle $T^*\mathbf{H}$.

For a Riemann surface Γ of genus > 1, there exists no complete hyperkähler metric on $T^*\Gamma$. However, a hyperkähler metric can be constructed in a neighbourhood of the zero section in the cotangent bundle $T^*\Gamma$.

Method 1 was successfully applied to the four series of compact Hermitian symmetric spaces

$$\frac{\mathrm{U}(m+n)}{\mathrm{U}(m)\times\mathrm{U}(n)}~, \qquad \frac{\mathrm{Sp}(n)}{\mathrm{U}(n)}~,$$

$$\frac{\mathrm{SO}(2n)}{\mathrm{U}(n)}~, \qquad \frac{\mathrm{SO}(n+2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}~, \quad n>2$$

as well as to their non-compact versions

$$\frac{\mathrm{U}(m,n)}{\mathrm{U}(m)\times\mathrm{U}(n)}~, \qquad \frac{\mathrm{Sp}(n,\mathbb{R})}{\mathrm{U}(n)}~,$$

$$\frac{\mathrm{SO}^*(2n)}{\mathrm{U}(n)}~, \qquad \frac{\mathrm{SO}_0(n,2)}{\mathrm{SO}(n)\times\mathrm{SO}(2)}~, \quad n>2$$

on case by case basis. This construction was finalized in:

Arai, SMK & Lindström (2007a)

General results:

• If the Hermitian symmetric space $\mathcal{M}$ is compact, then the hyperkähler structure is defined on the whole $T^*\mathcal{M}$.

• If $\mathcal{M}$ is non-compact, then the hyperkähler structure is defined on a neighbourhood of the zero section of $T^*\mathcal{M}$.

Method 1 turned out, however, to be impractical in the case of the exceptional Hermitian symmetric spaces

$$\frac{\mathrm{E}_6}{\mathrm{SO}(10)\times\mathrm{U}(1)}~, \qquad \frac{\mathrm{E}_7}{\mathrm{E}_6\times\mathrm{U}(1)}$$

Method 2 is based on considerations of extended SUSY

$\mathcal{N}=2$ supersymmetry transformation:

$$\delta\Upsilon(\zeta) = \mathrm{i}\left(\varepsilon^\alpha_i Q^i_\alpha + \bar\varepsilon^i_{\dot\alpha}\bar Q^{\dot\alpha}_i\right)\Upsilon(\zeta)$$

where $\Upsilon(\zeta)$ is viewed as a $\mathcal{N}=2$ superfield.

Reduce to $\mathcal{N}=1$ superspace. Then, the second hidden supersymmetry proves to act on the physical superfields Φ and Σ as

$$\delta\Phi^i = \bar\varepsilon_{\dot\alpha}\bar D^{\dot\alpha}\Sigma^I~, \qquad \delta\Sigma^I = -\varepsilon^\alpha D_\alpha\Phi^I + \bar\varepsilon_{\dot\alpha}\bar D^{\dot\alpha}\Upsilon^I_2~.$$

Upon elimination of the auxiliary superfields,

$$\Upsilon^I_2 = -\frac{1}{2}\Gamma^I{}_{JK}(\Phi,\bar\Phi)\,\Sigma^J\Sigma^K~.$$

Require the tangent-bundle action

$$S_{\rm tb}[\Phi,\Sigma] = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\Big\{K(\Phi,\bar\Phi) + \mathcal{L}(\Phi,\bar\Phi,\Sigma,\bar\Sigma)\Big\}~,$$

$$\mathcal{L} = \sum_{n=1}^{\infty}\mathcal{L}_{I_1\cdots I_n\bar J_1\cdots\bar J_n}(\Phi,\bar\Phi)\Sigma^{I_1}\dots\Sigma^{I_n}\bar\Sigma^{\bar J_1}\dots\bar\Sigma^{\bar J_n}$$

to be invariant under the supersymmetry transformation;
make use of the fact the the Riemann curvature is covariantly constant

$$\nabla_L R_{I_1\bar J_1 I_2\bar J_2} = \bar\nabla_{\bar L} R_{I_1\bar J_1 I_2\bar J_2} = 0~,$$

and hence

$$\nabla_L\mathcal{L}_{I_1\cdots I_n\bar J_1\cdots\bar J_n} = \bar\nabla_{\bar L}\mathcal{L}_{I_1\cdots I_n\bar J_1\cdots\bar J_n} = 0~.$$

Outcome:

The Lagrangian $\mathcal{L}$ obeys the linear differential equation:

$$\boxed{\tfrac{1}{2}\Sigma^K\Sigma^L\, R_{K\bar{J}L}{}^I\,\mathcal{L}_I + \mathcal{L}_{\bar{J}} + g_{I\bar{J}}\,\Sigma^I = 0\;, \qquad \mathcal{L}_I := \tfrac{\partial\mathcal{L}}{\partial\Sigma^I}}$$

and its conjugate.

Solution:

$$\mathcal{L}(\Phi,\bar{\Phi},\Sigma,\bar{\Sigma}) = -g_{I\bar{J}}\bar{\Sigma}^{\bar{J}}\,\frac{\mathrm{e}^{\mathcal{R}_{\Sigma,\bar{\Sigma}}}-1}{\mathcal{R}_{\Sigma,\bar{\Sigma}}}\,\Sigma^I\;,$$

$$\mathcal{R}_{\Sigma,\bar{\Sigma}} := -\frac{1}{2}\Sigma^K\bar{\Sigma}^{\bar{L}}\,R_{K\bar{L}I}{}^J\,\Sigma^I\frac{\partial}{\partial\Sigma^J}\;.$$

In the dual, cotangent bundle formulation

$$S_{\rm ctb}[\Phi,\Psi] = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\Big\{\,K(\Phi,\bar{\Phi}) + \mathcal{H}(\Phi,\bar{\Phi},\Psi,\bar{\Psi})\Big\}\;,$$

$$\mathcal{H} = \sum_{n=1}^{\infty}\mathcal{H}^{I_1\cdots I_n\bar{J}_1\cdots\bar{J}_n}(\Phi,\bar{\Phi})\Psi_{I_1}\dots\Psi_{I_n}\bar{\Psi}_{\bar{J}_1}\dots\bar{\Psi}_{\bar{J}_n}\;,$$

“Hamiltonian” $\mathcal{H}$ obeys the nonlinear differential equation:

$$\boxed{\mathcal{H}^I\,g_{I\bar{J}} - \tfrac{1}{2}\,\mathcal{H}^K\mathcal{H}^L\,R_{K\bar{J}L}{}^I\,\Psi_I = \bar{\Psi}_{\bar{J}}\;, \qquad \mathcal{H}^I = \tfrac{\partial\mathcal{H}}{\partial\Psi_I}}$$

To derive it, make use of the properties of Legendre transformation.

Using the above results, the case of $E_6/SO(10)\times U(1)$ was worked out for the first time.

Arai, SMK & Lindström (2007b)

Closed-form results: Tangent-bundle formulation

$$\mathcal{L}(\Phi,\bar\Phi,\Sigma,\bar\Sigma) = -\frac{1}{2}\boldsymbol{\Sigma}^{\rm T}\boldsymbol{g}\,\frac{\ln\left(\mathbb{1}+\boldsymbol{R}_{\Sigma,\bar\Sigma}\right)}{\boldsymbol{R}_{\Sigma,\bar\Sigma}}\,\boldsymbol{\Sigma}\,, \qquad \boldsymbol{\Sigma} := \begin{pmatrix}\Sigma^I\\ \bar\Sigma^{\bar I}\end{pmatrix}\,.$$

Here

$$\boldsymbol{R}_{\Sigma,\bar\Sigma} := \begin{pmatrix} 0 & (R_\Sigma)^I{}_{\bar J} \\ (R_{\bar\Sigma})^{\bar I}{}_J & 0\end{pmatrix}\,, \qquad \boldsymbol{g} := \begin{pmatrix} 0 & g_{I\bar J}\\ g_{\bar I J} & 0\end{pmatrix}$$

$$(R_\Sigma)^I{}_{\bar J} := \frac{1}{2}R_K{}^I{}_{L\bar J}\,\Sigma^K\Sigma^L\,, \qquad (R_{\bar\Sigma})^{\bar I}{}_J := \overline{(R_\Sigma)^I{}_{\bar J}}\,.$$

Closed-form results: Cotangent-bundle formulation

$$\mathcal{H}(\Phi,\bar\Phi,\Psi,\bar\Psi) = \frac{1}{2}\boldsymbol{\Psi}^{\rm T}\boldsymbol{g}^{-1}\mathcal{F}\left(-\boldsymbol{R}_{\Psi,\bar\Psi}\right)\boldsymbol{\Psi}\,, \qquad \boldsymbol{\Psi} := \begin{pmatrix}\Psi_I\\ \bar\Psi_{\bar I}\end{pmatrix}\,,$$

where

$$\mathcal{F}(x) = \frac{1}{x}\left\{\sqrt{1+4x} - 1 - \ln\frac{1+\sqrt{1+4x}}{2}\right\}\,, \qquad \mathcal{F}(0) = 1\,.$$

The operator $\boldsymbol{R}_{\Psi,\bar\Psi}$ is defined as

$$\boldsymbol{R}_{\Psi,\bar\Psi} := \begin{pmatrix} 0 & (R_\Psi)_I{}^{\bar J} \\ (R_{\bar\Psi})_{\bar I}{}^J & 0\end{pmatrix}\,,$$

$$(R_\Psi)_I{}^{\bar J} = (R_\Psi)_{IK}\,g^{K\bar J}\,, \qquad (R_\Psi)_{KL} := \frac{1}{2}R_K{}^I{}_L{}^J\,\Psi_I\Psi_J\,.$$

The hyperkähler potential for $T^*\mathcal{M}$:

$$\mathbb{K}(\Phi,\bar\Phi,\Psi,\bar\Psi) = K(\Phi,\bar\Phi) + \frac{1}{2}\boldsymbol{\Psi}^{\rm T}\boldsymbol{g}^{-1}\mathcal{F}\left(-\boldsymbol{R}_{\Psi,\bar\Psi}\right)\boldsymbol{\Psi}\,.$$

SMK & Novak (2008)

In the mathematical literature, there exists a different representation for $\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ derived in:

O. Biquard and P. Gauduchon, "Hyperkähler metrics on cotangent bundles of Hermitian symmetric spaces," in: *Geometry and Physics*, J. Andersen, J. Dupont, H. Petersen and A. Swann (Eds.) (Lect. Notes Pure Appl. Math. **184**), Marcel Dekker, 1997, p. 287.

The Biquard-Gauduchon representation is

$$\mathcal{H}(\Phi, \bar{\Phi}, \Psi, \bar{\Psi}) = \Psi^{\dagger} \check{g}^{-1} \mathcal{F}(-\mathbb{R}_{\Psi,\bar{\Psi}})\, \Psi \ ,$$

where

$$(\mathbb{R}_{\Psi,\bar{\Psi}})_I{}^J := \frac{1}{2} R_I{}^{J\bar{K}L} \Psi_L \bar{\Psi}_{\bar{K}}$$

and $\check{g}$ denotes an off-diagonal block of the Kähler metric

$$\boldsymbol{g} := \begin{pmatrix} 0 & g_{I\bar{J}} \\ g_{\bar{I}J} & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & \hat{g} \\ \check{g} & 0 \end{pmatrix} \ .$$

The above unified formula was derived by Biquard and Gauduchon with the aid of purely algebraic means involving the root theory for Hermitian symmetric spaces (in conjunction with some guesswork based on the use of the Calabi metrics for $T^*\mathbb{C}P^n$).

In the supersymmetric setting described above, the results were derived from a regular procedure. No guesswork was needed.

The case of arbitrary Kähler manifold $\mathcal{M}$

SMK (2009)

The second hidden supersymmetry becomes

$$\delta \Phi^i = \bar{\varepsilon}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} \Sigma^I \ , \qquad \delta \Sigma^I = -\varepsilon^\alpha D_\alpha \Phi^I + \bar{\varepsilon}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}\, \Upsilon_2^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \ .$$

Upon elimination of the auxiliary superfields,

$$\Upsilon_2^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = -\frac{1}{2}\Gamma^I_{JK}(\Phi, \bar{\Phi})\, \Sigma^J \Sigma^K + G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) \ ,$$

$$G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) := \sum_{p=1}^{\infty} G^I{}_{J_1 \dots J_{p+2}\, \bar{L}_1 \dots \bar{L}_p}(\Phi, \bar{\Phi})\, \Sigma^{J_1} \dots \Sigma^{J_{p+2}}\, \bar{\Sigma}^{\bar{L}_1} \dots \bar{\Sigma}^{\bar{L}_p} \ ,$$

where $G^I{}_{J_1 \dots J_{p+2}\, \bar{L}_1 \dots \bar{L}_p}(\Phi, \bar{\Phi})$ are tensor functions of the Kähler metric, the Riemann curvature $R_{I\bar{J}K\bar{L}}(\Phi, \bar{\Phi})$ and its covariant derivatives.

$$G^I(\Phi, \bar{\Phi}, \Sigma, \bar{\Sigma}) = \frac{1}{6} \nabla_{J_1} R_{J_2 \bar{L} J_3}{}^I(\Phi, \bar{\Phi})\, \Sigma^{J_1} \Sigma^{J_2} \Sigma^{J_3} \bar{\Sigma}^{\bar{L}} + \mathcal{O}(\Sigma^4 \bar{\Sigma}^2) \ .$$

Holomorphic reparametrizations of the Kähler manifold:

$$\begin{aligned}
\Phi^I \quad &\longrightarrow \quad \Phi'^I = f^I(\Phi) \ , \\
\Sigma^I \quad &\longrightarrow \quad \Sigma'^I = \frac{\partial f^I(\Phi)}{\partial \Phi^J} \Sigma^J \ , \\
\Upsilon_2^I \quad &\longrightarrow \quad \Upsilon'^I_2 = \frac{1}{2} \frac{\partial^2 f^I(\Phi)}{\partial \Phi^J \partial \Phi^K}\, \Sigma^J \Sigma^K + \frac{\partial f^I(\Phi)}{\partial \Phi^J}\, \Upsilon_2^J \ .
\end{aligned}$$

Require the tangent-bundle action

$$S_{\rm tb}[\Phi,\Sigma] = \int {\rm d}^4x\,{\rm d}^4\theta\,\Big\{\,K(\Phi,\bar\Phi) + \mathcal{L}(\Phi,\bar\Phi,\Sigma,\bar\Sigma)\Big\}\,,$$
$$\mathcal{L} = \sum_{n=1}^{\infty}\mathcal{L}_{I_1\cdots I_n\bar J_1\cdots\bar J_n}(\Phi,\bar\Phi)\Sigma^{I_1}\dots\Sigma^{I_n}\bar\Sigma^{\bar J_1}\dots\bar\Sigma^{\bar J_n}$$

to be supersymmetric. This proves to imply that $\mathcal{L}(\Phi,\bar\Phi,\Sigma,\bar\Sigma)$ and $G^I(\Phi,\bar\Phi,\Sigma,\bar\Sigma)$ obey the following equations:

$$\frac{\partial\mathcal{L}}{\partial\Sigma^J}\,\frac{\partial G^J}{\partial\bar\Sigma^{\bar I}} = \frac{\partial\Xi}{\partial\bar\Sigma^{\bar I}}\,,$$
$$\nabla_I\mathcal{L} + \frac{\partial\mathcal{L}}{\partial\Sigma^J}\,\frac{\partial G^J}{\partial\Sigma^I} = \frac{\partial\Xi}{\partial\Sigma^I}\,,$$
$$\frac{1}{2}R_{K\bar IL}{}^J\,\frac{\partial\mathcal{L}}{\partial\Sigma^J}\,\Sigma^K\Sigma^L + \frac{\partial\mathcal{L}}{\partial\bar\Sigma^{\bar I}} + g_{J\bar I}\,\Sigma^J - \frac{\partial\mathcal{L}}{\partial\Sigma^J}\,\nabla_{\bar I}G^J = -\nabla_{\bar I}\Xi\,,$$

where

$$\Xi = \Sigma^I\nabla_I\mathcal{L} + 2G^I\,\frac{\partial\mathcal{L}}{\partial\Sigma^I}\,,$$

and we have defined

$$\nabla_I\mathcal{L} := \sum_{n=1}^{\infty}(\nabla_I\mathcal{L}_{J_1\cdots J_n\bar L_1\cdots\bar L_n}(\Phi,\bar\Phi))\Sigma^{J_1}\dots\Sigma^{J_n}\bar\Sigma^{\bar L_1}\dots\bar\Sigma^{\bar L_n}$$
$$= \frac{\partial\mathcal{L}}{\partial\Phi^I} - \frac{\partial\mathcal{L}}{\partial\Sigma^K}\,\Gamma^K_{IJ}\,\Sigma^J\,,$$

and similarly for $\nabla_{\bar I}G^J$.

Special case:

$$\nabla_L R_{I_1\bar J_1I_2\bar J_2} = \nabla_{\bar L}R_{I_1\bar J_1I_2\bar J_2} = 0 \quad\Longrightarrow\quad \nabla_I\mathcal{L} = G^I = \Xi = 0\,.$$

Cotangent-bundle formulation

$$S_{\rm ctb}[\Phi, \Psi] = \int \mathrm{d}^4x\, \mathrm{d}^4\theta \,\Big\{ K(\Phi, \bar\Phi) + \mathcal{H}(\Phi, \bar\Phi, \Psi, \bar\Psi) \Big\} ,$$

$$\mathcal{H}(\Phi, \bar\Phi, \Psi, \bar\Psi) = \sum_{n=1}^{\infty} \mathcal{H}^{I_1 \cdots I_n \bar J_1 \cdots \bar J_n}(\Phi, \bar\Phi) \Psi_{I_1} \dots \Psi_{I_n} \bar\Psi_{\bar J_1} \dots \bar\Psi_{\bar J_n} .$$

Here the chiral variables (Φ^I, Ψ_J) parametrize the cotangent bundle $T^*\mathcal{M}$, and the hyperkähler potential of $T^*\mathcal{M}$ is

$$\mathbb{K}(\Phi, \bar\Phi, \Psi, \bar\Psi) := K(\Phi, \bar\Phi) + \mathcal{H}(\Phi, \bar\Phi, \Psi, \bar\Psi) .$$

The second hidden supersymmetry becomes

$$\delta \Phi^I = \frac{1}{2} \bar D^2 \Big\{ \overline{\epsilon \theta} \, \frac{\partial \mathbb{K}}{\partial \Psi_I} \Big\} , \qquad \delta \Psi_I = -\frac{1}{2} \bar D^2 \Big\{ \overline{\epsilon \theta} \, \frac{\partial \mathbb{K}}{\partial \Phi^I} \Big\} .$$

Introduce the condensed notation

$$\phi^a := (\Phi^I , \Psi_I) , \qquad \bar\phi^{\bar a} = (\bar\Phi^{\bar I} , \bar\Psi_{\bar I}),$$

as well as the symplectic matrix $\mathbb{J} = (\mathbb{J}^{ab})$, its inverse $\mathbb{J}^{-1} = (-\mathbb{J}_{ab})$ and their complex conjugates,

$$\mathbb{J}^{ab} = \mathbb{J}^{\bar a \bar b} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} , \qquad \mathbb{J}_{ab} = \mathbb{J}_{\bar a \bar b} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} .$$

The supersymmetry transformation:

$$\delta \phi^a = \frac{1}{2} \mathbb{J}^{ab} \, \bar D^2 \Big\{ \overline{\epsilon \theta} \, \frac{\partial \mathbb{K}}{\partial \phi^b} \Big\} = \frac{1}{2} \bar D^2 \Big\{ \overline{\epsilon \theta} \, \boldsymbol{\Omega}^a \Big\} , \qquad \boldsymbol{\Omega}^a := \mathbb{J}^{ab} \frac{\partial \mathbb{K}}{\partial \phi^b} .$$

These results can now be linked to the general 1986-analysis of $\mathcal{N} = 2$ sigma-models in $\mathcal{N} = 1$ superspace (Hull *et al.*) reviewed earlier.

Holomorphic two-form

By definition, the anti-holomorphic two-form is

$$\bar{\boldsymbol{\omega}}_{\bar{b}\bar{c}} = \boldsymbol{g}_{a\bar{b}}\,\bar{\boldsymbol{\Omega}}^a{}_{,\bar{c}}\ ,$$

with $\boldsymbol{g}_{a\bar{b}}$ the Kähler metric

$$\boldsymbol{g}_{a\bar{b}} = \frac{\partial^2\mathbb{K}}{\partial\phi^a\partial\bar{\phi}^{\bar{b}}} = \begin{pmatrix} \frac{\partial^2\mathbb{K}}{\partial\Phi^I\partial\bar{\Phi}^{\bar{J}}} & \frac{\partial^2\mathbb{K}}{\partial\Phi^I\partial\bar{\Psi}_{\bar{J}}} \\ \frac{\partial^2\mathbb{K}}{\partial\Psi_I\partial\bar{\Phi}^{\bar{J}}} & \frac{\partial^2\mathbb{K}}{\partial\Psi_I\partial\bar{\Psi}_{\bar{J}}} \end{pmatrix}\ .$$

Recalling the explicit form of $\bar{\boldsymbol{\Omega}}^a$,

$$\bar{\boldsymbol{\Omega}}^a := \mathbb{J}^{ab}\,\frac{\partial\mathbb{K}}{\partial\phi^b}\ ,$$

$\bar{\boldsymbol{\omega}}_{\bar{b}\bar{c}}$ is indeed seen to be antisymmetric,

$$\bar{\boldsymbol{\omega}}_{\bar{a}\bar{b}} = \boldsymbol{g}_{\bar{a}c}\,\mathbb{J}^{cd}\,\boldsymbol{g}_{d\bar{b}}\ , \qquad \boldsymbol{\omega}_{ab} = \boldsymbol{g}_{a\bar{c}}\,\mathbb{J}^{\bar{c}\bar{d}}\,\boldsymbol{g}_{\bar{d}b}\ .$$

Direct calculations show that

$$\boldsymbol{\omega}_{ab} = \mathbb{J}_{ab} + \mathcal{O}(\Psi\bar{\Psi})\ .$$

Since $\boldsymbol{\omega}_{ab}$ must be holomorphic, we immediately conclude that

$$\boldsymbol{\omega}_{ab} = \mathbb{J}_{ab}\ , \qquad \bar{\boldsymbol{\omega}}_{\bar{a}\bar{b}} = \mathbb{J}_{\bar{a}\bar{b}} \quad \Longrightarrow \quad \boldsymbol{\omega}^{ab} = \boldsymbol{g}^{a\bar{c}}\boldsymbol{g}^{b\bar{d}}\bar{\boldsymbol{\omega}}_{\bar{c}\bar{d}} = \mathbb{J}^{ab}\ .$$

As a result, the holomorphic symplectic two-form $\boldsymbol{\omega}^{(2,0)}$ of $T^*\mathcal{M}$ coincides with the canonical holomorphic symplectic two-form,

$$\boldsymbol{\omega}^{(2,0)} := \frac{1}{2}\boldsymbol{\omega}_{ab}\,\mathrm{d}\phi^a \wedge \mathrm{d}\phi^b = \mathrm{d}\Phi^I \wedge \mathrm{d}\Psi_I\ .$$