ϑ-functions, dual pairs, $H^*(\Gamma\backslash G/K)$

$$\vartheta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} \qquad \tau \in \mathcal{H} = \{\tau\in\mathbb{C} \mid \mathrm{Im}(\tau) > 0\} \simeq SL_2(\mathbb{R})/SO(2)$$

matrix version: on $G/K = \widetilde{G}/\widetilde{K}$, $\quad G = Sp(2n, \mathbb{R})$, $K = U(n)$

$$1 \longrightarrow \{\pm 1\} \longrightarrow \widetilde{G} \xrightarrow{\pi} G \longrightarrow 1$$

<u>dual pairs in G and $\widetilde{G}$:</u>

(1) $V, (\,,\,)$ orthogonal, $W, \langle\,,\,\rangle$ symplectic $\Big\} \Rightarrow V\otimes W$ symplectic

$O(V) \times Sp(W) \Rightarrow Sp(V\otimes W)$ mutual centralisers

splits $\rightarrow$ $\pi^{-1}(O(V)) \times \pi^{-1}(Sp(W)) \Rightarrow \widetilde{Sp}(V\otimes W)$

(2) $V, (\,,\,)$ hermitian, $W, \langle\,,\,\rangle$ skew-hermitian $\Big\} \Rightarrow V\otimes W$ symplectic

$U(V) \times U(W) \Rightarrow Sp(V\otimes W)$

$\pi^{-1}(U(V)) \times \pi^{-1}(U(W)) \Rightarrow \widetilde{Sp}(V\otimes W)$

<u>symmetric spaces</u>: $G = O(V) = O(p,q) \supset K = O(p)\times O(q)$

$G' = Sp(2n, \mathbb{R}) \supset K' = U(n)$

$$G/K \times \widetilde{G'}/\widetilde{K'} \longrightarrow \widetilde{Sp}(V\otimes W)/(\text{max. cpt.}) = \underline{\underline{\widetilde{G}}} / \underline{\underline{\widetilde{K}}}$$

<u>ϑ-correspondence</u>: general ϑ-fns on $\uparrow$ give rise to <u>integral operators</u>

$$\Gamma(G/K, \mathcal{L}) \longrightarrow \Gamma(\widetilde{G'}/\widetilde{K'}, \mathcal{L}')$$

bundles

$\Gamma \subset G$ $\quad\cup\quad$ Γ-invariant sections $\qquad \cup \quad$ $\widetilde{\Gamma'}$-invariant sections

$\Gamma' \subset G'$

arithmetic subgroups $\qquad$ automorphic forms

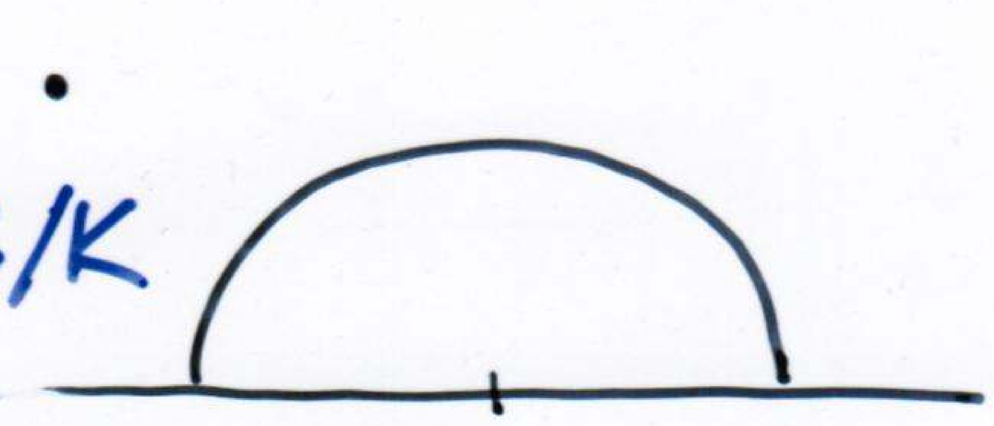
<u>Kudla–Millson</u>: constructed natural closed differential forms on $G/K \xrightarrow{\vartheta\text{-corr.}}$ automorphic forms on $\widetilde{G'}/\widetilde{K'}$ with values in $H^*(\Gamma\backslash G/K)$ whose Fourier coefficients are cohomology classes of (comb. of) subvarieties $\Gamma_H\backslash H/K_H \hookrightarrow \Gamma\backslash G/K$, $H = O(p',q) \subset \underbrace{O(p,q)}_{G}$

$\Gamma_H = H\cap\Gamma$, $K_H = H \cap K$.

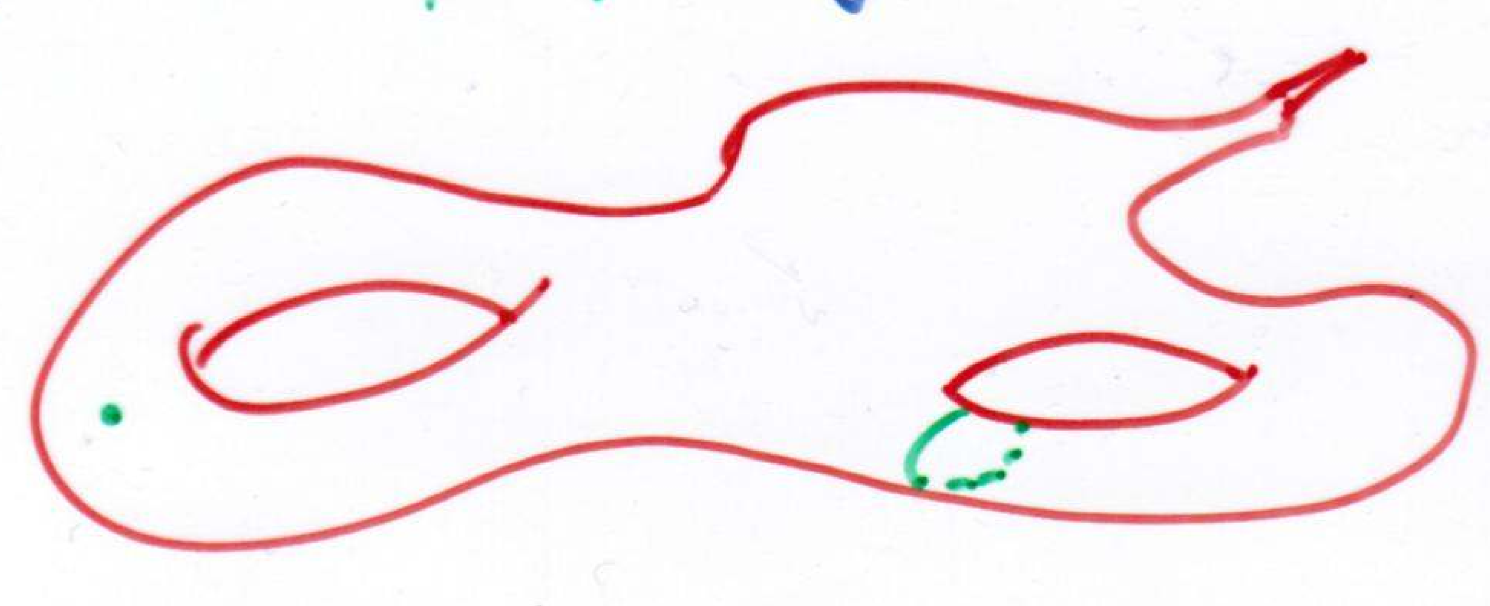
<u>Ex</u>: $\mathcal{H} = SL_2(\mathbb{R})/SO(2) \simeq SO_0(1,2)/SO(2) \simeq O(1,2)/O(1)\times O(2)$

image of $O(0,2)$ = point, of $O(1,1)$ = geodesic

G/K



$\Gamma\backslash G/K$

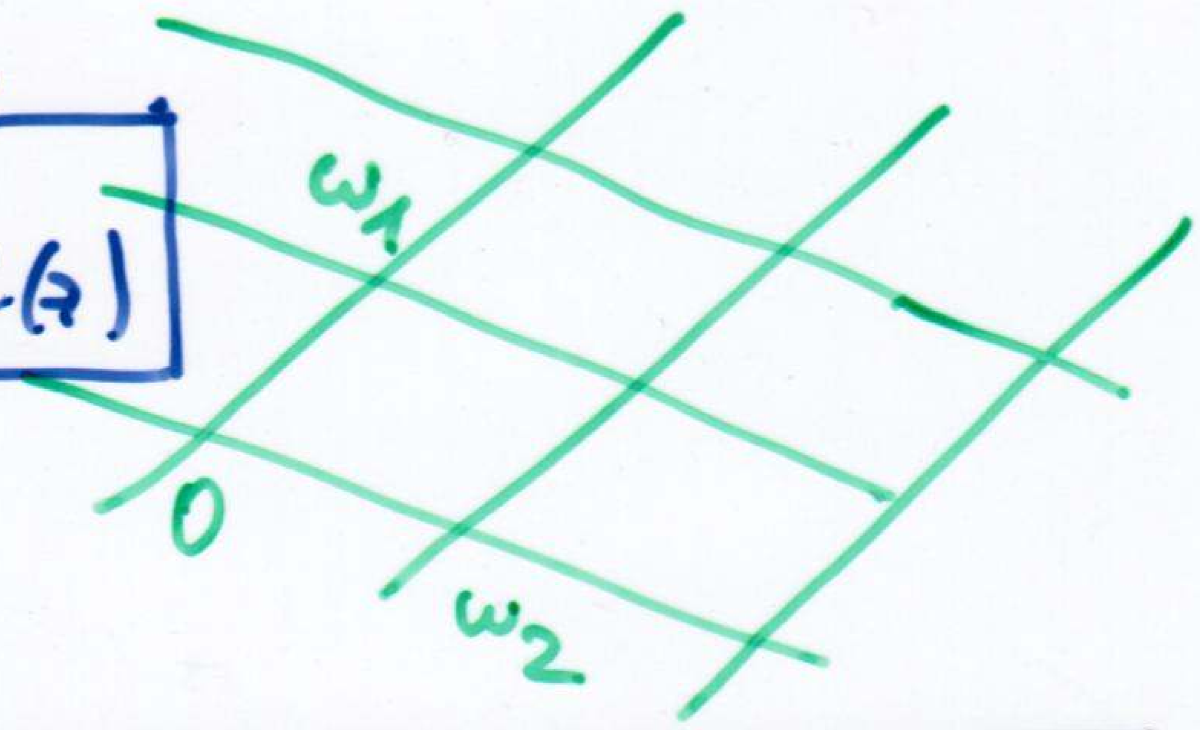


Metaplectic group, oscillator (= Weil) representation

$$\boxed{\theta(z,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z}} \qquad (\tau\in\mathcal{H},\ z\in\mathbb{C})$$

$$\theta(z+1,\tau) = \theta(z,\tau), \quad \theta(z+\tau,\tau) = e^{-2\pi i\left(z+\frac{\tau}{2}\right)}\theta(z,\tau)$$

Def. A θ-function w.r.t. lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 \subset \mathbb{C}$ is a holomorphic $f: \mathbb{C}\longrightarrow\mathbb{C}$ s.t. $\forall u\in L\ \exists \binom{a(u)}{b(u)}\in\mathbb{C}^2\ \forall z\in\mathbb{C}$ $\boxed{f(z+u) = e^{a(u)z+b(u)}f(z)}$



Note: $f\in\Gamma(\mathbb{C}/L, \mathcal{O}(\mathcal{L}))$
hol. sections line bundle

Transformation rules in z (τ and $L=\mathbb{Z}\tau+\mathbb{Z}$ fixed): action of the Heisenberg group (H. Weyl) – combines translations and multiplication by e^{az+b}.

Transformation rules in τ ($\mathbb{Z}\tau+\mathbb{Z}$ "fixed", basis changed)

$$\theta(\tau) := \theta(0,\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau}$$

$$\boxed{\begin{array}{l}\theta(\tau+2) = \theta(\tau), \quad \theta\left(-\frac{1}{\tau}\right) = \underbrace{\sqrt{\frac{\tau}{i}}}_{=1 \text{ at } \tau=i}\,\theta(\tau) \\ \forall\tau\in\mathcal{H}\end{array}}$$

$\forall \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\mathbb{Z})$, $2\,|\,ab, cd \Longleftarrow$

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \zeta\underbrace{\sqrt{c\tau+d}}_{\text{branch of }\sqrt{\ }}\theta(\tau) \quad \forall\tau\in\mathcal{H}, \quad \zeta^8 = 1.$$

Action of the metaplectic group $\widetilde{SL_2(\mathbb{R})}$ (A. Weil) – two-fold cover of $SL_2(\mathbb{R})$.

Action of the semi-direct product (Heisenberg) $\rtimes \widetilde{SL_2(\mathbb{R})}$.

Key point: rigidity of unitary representations of (Heis) $\Longrightarrow$ existence of $\widetilde{SL_2(\mathbb{R})}$, of θ, properties of θ...

Lie algebra representation (on $C^\infty(\mathbb{R})$): x coordinate on $\mathbb{R}$

$$\mathrm{Lie}(\mathrm{Heis}) \xrightarrow{\sim} \mathrm{span}_{\mathbb{R}}\left(i,\ ix,\ \frac{d}{dx}\right)$$

$$\mathrm{Lie}(\widetilde{SL_2}) = \mathrm{Lie}(SL_2) \xrightarrow{\sim} \mathrm{span}_{\mathbb{R}}\left(\frac{ix^2}{2},\ \frac{i}{2}\left(\frac{d}{dx}\right)^2,\ \frac{1}{2}\left(x\frac{d}{dx}+\frac{d}{dx}x\right)\right)$$

$$X = \begin{pmatrix}0&1\\0&0\end{pmatrix}, \quad Y = \begin{pmatrix}0&0\\1&0\end{pmatrix}, \quad H = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$[X,Y] = H$
$[H,X] = 2X$
$[H,Y] = -2Y$

Fourier, Poisson and θ: fix $\psi : (\mathbb{R},+) \twoheadrightarrow U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$

$$\boxed{\psi(x) = e^{2\pi i x}}$$

(1) $\mathbb{Z} \subset \mathbb{R}$ is ψ-selfdual: $\mathbb{Z} = \{x \in \mathbb{R} \mid \psi(x\mathbb{Z}) = 1\}$

(2) Fourier transform $(\mathcal{F}f)(x) = \int \psi(xy) f(y)\, dy$ is self-dual:

$\boxed{\mathcal{F} \circ \mathcal{F} = [-1]}$, $([r]f)(x) = f(rx)$, $r \in \mathbb{R}$ $\boxed{\mathcal{F} \circ [r] = |r|^{-1} [r^{-1}] \circ \mathcal{F} \quad 0 \neq r \in \mathbb{R}}$

(3) The gaussian $\varphi_0(x) = e^{-\pi x^2}$, $\boxed{\mathcal{F}\varphi_0 = \varphi_0}$

(4) Poisson formula: $\boxed{\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} (\mathcal{F}f)(n)}$ (f rapidly decreasing)

Back to $\theta(-\frac{1}{\tau})$: for $y \in \mathbb{R}_{>0}$ apply (4) to $f = \underbrace{[y^{1/2}]\varphi_0}_{e^{-\pi y x^2}}$

$$\theta(iy) = \sum_{n \in \mathbb{Z}} ([y^{1/2}]\varphi_0)(n) = y^{-1/2} \sum_{n \in \mathbb{Z}} ([y^{-1/2}]\varphi_0)(n) = y^{-1/2}\, \theta\left(-\frac{1}{iy}\right)$$

$f(\tau) := \theta(\tau) - \left(\frac{\tau}{i}\right)^{-1/2} \theta\left(-\frac{1}{\tau}\right)$ holomorphic in $\mathcal{H}$, $f|_{i\mathbb{R}_{>0}} = 0 \Rightarrow f = 0$.

Why oscillator representation?

classical mechanics (in dim $= 1$): space $= \mathbb{R} \xrightarrow{V} \mathbb{R}$, $x \mapsto V(x)$ potential energy

kinetic energy $T = \frac{p^2}{2m}$

$$\boxed{H = T + V = \frac{p^2}{2m} + V(x)}$$ $x =$ coordinate, $p =$ momentum, $m =$ mass

quantum mechanics: operators $p \rightsquigarrow \hat{p} = \frac{\hbar}{i} \frac{d}{dx}$ $\quad \hbar = \frac{h}{2\pi}$

$$H \rightsquigarrow \hat{H} = \frac{\hat{p}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\left(\frac{d}{dx}\right)^2 + V(x)$$

Schrödinger equation: $\hat{H}\psi = -\frac{\hbar}{i} \frac{\partial \psi}{\partial t}$ $\quad \psi = \psi(x,t)$

try $\psi(x,t) = e^{-\frac{iE}{\hbar}t} \varphi(x)$ $\quad \boxed{\hat{H}\varphi = E\varphi}$

Harmonic oscillator: $V(x) = \frac{ax^2}{2}$ $\quad (a > 0)$

$$\boxed{\left(\frac{h^2}{2m}\left(\frac{1}{2\pi i}\frac{d}{dx}\right)^2 + \frac{a}{2}x^2\right)\varphi(x) = E\varphi(x)}$$

Heisenberg Lie algebra: $\mathbb{R}\cdot P + \mathbb{R}\cdot Q + \mathbb{R}\cdot E$, $[P,Q] = E$, $[E,\cdot] = 0$

want: representations s.t. $E \longmapsto (2\pi i)1$

Standard ("Schrödinger") representation:

$$P \longmapsto \hat{P} = \frac{d}{dx}, \quad Q \longmapsto \hat{Q} = 2\pi i x, \quad E \longmapsto \hat{E} = 2\pi i$$

$(\exp(t\hat{P})f)(x) = f(x+t)$, $(\exp(t\hat{Q})f)(x) = e^{2\pi i t x} f(x)$

$(\exp(t\hat{E})f)(x) = e^{2\pi i t} f(x)$ $\quad$ unitary operators on $L^2(\mathbb{R})$, $\|f\|^2 = \int |f|^2 dx$

Action on the gaussian $\varphi_0 = e^{-\pi x^2}$: $\hat{P} = \frac{d}{dx}$, $\hat{Q} = 2\pi i x$

$\hat{P}\varphi_0 = -2\pi x\,\varphi_0 = i\,\hat{Q}\varphi_0 \implies \boxed{(\hat{P} - i\hat{Q})\varphi_0 = 0}$ $\quad \boxed{A_-\varphi_0 = 0}$

$A_- = \hat{P} - i\hat{Q} = \frac{d}{dx} + 2\pi x$

$A_+ = \hat{P} + i\hat{Q} = \frac{d}{dx} - 2\pi x$

$[A_+, A_-] = 4\pi\cdot 1$

Representation of $\mathfrak{sl}(2)$:

$$X \longmapsto \frac{1}{8\pi}A_+^2 \qquad H \longmapsto -\frac{1}{8\pi}(A_+A_- + A_-A_+) =$$
$$Y \longmapsto -\frac{1}{8\pi}A_-^2 \qquad = \frac{1}{2} - \frac{1}{4\pi}A_+A_- = \hat{H}$$

Standard action of $\mathfrak{sl}(2)$ on $\mathbb{R}A_+ + \mathbb{R}A_-$:

$[H, A_\pm] = \pm A_\pm$, $[X, A_-] = A_+$, $[Y, A_+] = A_-$
$[X, A_+] = 0$, $[Y, A_-] = 0$

Def: $\varphi_n = A_+^n \varphi_0$ $(n \geq 0)$

$\varphi_0 = e^{-\pi x^2}$, $\varphi_1 = (-4\pi x)\,e^{-\pi x^2}$

$$\boxed{\begin{array}{l} Y\varphi_0 = Y\varphi_1 = 0 \\ H\varphi_0 = \frac{1}{2}\varphi_0, \quad H\varphi_1 = \frac{3}{2}\varphi_1 \end{array}}$$

lowest weight vectors for $\mathfrak{sl}(2)$

$$\begin{tikzcd} \varphi_n \arrow[r, bend left, "X"] & \varphi_{n+2} \arrow[l, bend left, "Y"] \end{tikzcd}$$

(up to a mult. const. $\neq 0$)

$L^2(\mathbb{R})$ has orthogonal basis $\varphi_0, \varphi_1, \varphi_2, \ldots$

$\boxed{H\varphi_n = (n + \frac{1}{2})\varphi_n}$

Formula for φ_n : $A_+ = e^{\pi x^2} \circ \frac{d}{dx} \circ e^{-\pi x^2}$

$$\varphi_n = \left(e^{\pi x^2} \circ \left(\frac{d}{dx}\right)^n \circ e^{-\pi x^2}\right)\left(e^{-\pi x^2}\right) = f_n(x)\,e^{-\pi x^2}$$

$$f_n(x) = e^{2\pi x^2}\left(\frac{d}{dx}\right)^n\left(e^{-2\pi x^2}\right) = (-1)^n (2\pi)^{n/2} H_n(x\sqrt{2\pi})$$

Hermite polynomials: $H_n(t) = (-1)^n e^{t^2}\left(\frac{d}{dt}\right)^n\left(e^{-t^2}\right)$

Fourier transform on $L^2(\mathbb{R})$: $\mathcal{F}\circ\mathcal{F} = [-1]$, $([r]f)(x) = f(rx)$

$$(\mathcal{F}f)(x) = \int_{\mathbb{R}} e^{2\pi i x y} f(y)\,dy$$

$$\left.\begin{array}{l} \hat{P}\circ\mathcal{F} = \mathcal{F}\circ\hat{Q} \\ \hat{P}\circ[-1] = -[-1]\circ\hat{P} \\ \hat{Q}\circ[-1] = -[-1]\circ\hat{Q} \end{array}\right\} \implies \mathcal{F}\circ\hat{P} = -\hat{Q}\circ\mathcal{F} \implies \boxed{\mathcal{F}\circ A_\pm = \pm i\, A_\pm \circ \mathcal{F}}$$

A_+^n

$\mathcal{F}\varphi_0 = \varphi_0 \implies \mathcal{F}\varphi_n = \mathcal{F}A_+^n\varphi_0 = i^n \mathcal{F}\varphi_0 = i^n \varphi_n$ $\quad (\forall n \geq 0)$

$H\varphi_n = (n + \frac{1}{2})\varphi_n \implies e^{\pi i H/2}\varphi_n = i^n e^{\pi i/4}\varphi_n = e^{\pi i/4}\mathcal{F}\varphi_n$

$$\boxed{\mathcal{F} = e^{-\pi i/4}\, e^{\pi i H/2}}$$

Next time : given $u : \mathbb{Z} \longrightarrow \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{C}$

$$\left.\begin{array}{l} \theta_{1/2,u}(\tau) = \sum_{n\in\mathbb{Z}} u(n)\,e^{\pi i n^2 \tau}, \quad u(-n) = u(n) \\ \theta_{3/2,u}(\tau) = \sum_{n\in\mathbb{Z}} u(n)\,n\,e^{\pi i n^2\tau}, \quad u(-n) = -u(n) \end{array}\right\} \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{2N^2}$$

$$\theta_{\lambda,u}\left(\frac{a\tau + b}{c\tau + d}\right) = (\text{const.})\,(c\tau + d)^\lambda\,\theta_{\lambda,u}(\tau)$$

Question : $\lambda = \frac{5}{2}, \frac{7}{2}, \ldots$??

Intertwining operators $P = \frac{d}{dx}$, $Q = 2\pi i x$

$$\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}\circ\mathcal{F} = \mathcal{F}\circ\begin{pmatrix}\hat Q\\ -\hat P\end{pmatrix} = \mathcal{F}\circ\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix} \iff \boxed{\mathcal{F}^{-1}\circ\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}\circ\mathcal{F} = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}}$$

Any $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in M_2(\mathbb{R})$: $[a\hat P + b\hat Q\,,\ c\hat P + d\hat Q] = (ad-bc)\underbrace{[\hat P,\hat Q]}_{2\pi i}$

$\forall\ g = \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{R})$ $\begin{pmatrix}P\\ Q\end{pmatrix}\longmapsto\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}$ representation of the Heisenberg Lie algebra

$E\longmapsto(2\pi i)\mathbb{1}$

Is it equivalent to the standard one $P\mapsto\hat P$, $Q\mapsto\hat Q$?

$\Updownarrow$

$\exists\ F_g\in U(\mathcal{H})$ $\boxed{\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}\circ F_g = F_g\circ\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}\hat P\\ \hat Q\end{pmatrix}}$? $\mathcal{H} = L^2(\mathbb{R})$

(1) $g = \begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}$: $F_g = (\text{const})\cdot[a]$: $[a]\circ\hat Q = a\hat Q\circ[a]$, $\hat P\circ[a] = a[a]\circ\hat P$

$\boxed{F_g = |a|^{1/2}u_g\,[a]\ ,\ u_g\in U(1)}$ $\|[a]f\|^2 = |a|^{-1}\|f\|^2$

(2) $g = \begin{pmatrix}1&b\\0&1\end{pmatrix}$: $f\in C^\infty(\mathbb{R})$ $\hat Q\circ f = f\circ\hat Q$, $\hat P\circ f = f\circ\hat P + f'$

want: $f' = (2\pi i)xbf \Longrightarrow f = (\text{const.})e^{\pi i b x^2}$, $\boxed{F_g = u_g e^{\pi i b x^2}\ ,\ u_g\in U(1)}$

(3) $g = \begin{pmatrix}a&b\\c&d\end{pmatrix}$, $c\neq 0$, $ad-bc=1$: $(F_g f)(x) = \int_{\mathbb{R}} K(x,y)\underline{f(y)}\,dy$ rapidly decreasing

$$((\hat P\circ F_g)f)(x) = \int_{\mathbb{R}}\frac{\partial K}{\partial x}(x,y)f(y)dy$$

$$(F_g\circ\hat P f)(x) = \int_{\mathbb{R}}K(x,y)f'(y)dy = \overbrace{[K(x,y)f(y)]_{y=-\infty}^{y=+\infty}}^{0} - \int_{\mathbb{R}}\frac{\partial K}{\partial y}(x,y)f(y)\,dy$$

$$\left.\begin{aligned}\frac{\partial K}{\partial x} &= -a\frac{\partial K}{\partial y} + (2\pi i b)yK\\ 2\pi i x K &= -c\frac{\partial K}{\partial y} + (2\pi i d)yK\end{aligned}\right\}\quad\begin{aligned}\partial K/\partial y &= 2\pi i\left(\frac{-x+dy}{c}\right)K\\ \partial K/\partial x &= 2\pi i\left(\frac{ax-y}{c}\right)K\end{aligned}$$

$$\boxed{K = (\text{const.})e^{\pi i\left(\frac{ax^2-2xy+dy^2}{c}\right)},\quad (\text{const.}) = |c|^{-1/2}u_g,\quad u_g\in U(1)}$$

the Operators $F_g\in U(\mathcal{H})$ are unique up to $U(1)$ $\Longrightarrow$

$SL_2(\mathbb{R})\longrightarrow U(\mathcal{H})/U(1) = PU(\mathcal{H})$ is a group homomorphism

$g\longmapsto F_g\ (\text{mod } U(1))$

does this lift? $- - - \to U(\mathcal{H})$

(to a group homomorphism)?

$\Updownarrow$

$\exists\, u_g\in U(1)$ s.t. $\forall\, g,g'\in SL_2(\mathbb{R})$ $F_g F_{g'} = F_{gg'}$?   NO!

Projective representations and 2-cocycles

Given: $G \xrightarrow{r} PGL(V) = GL(V)/\mathbb{C}^*\cdot 1_V$, G group, V $\mathbb{C}$-v.space
r = group homomorphism

Choose:
$$\begin{tikzcd} G \arrow[r, "r"] \arrow[dr, "\tilde r"'] & PGL(V) \\ & GL(V) \arrow[u, "pr"'] \end{tikzcd}$$

$\forall g_i \in G$
$$\tilde r(g_1)\tilde r(g_2) = \underset{\mathbb{C}^*}{\underset{\uparrow}{c(g_1,g_2)}}\,\tilde r(g_1 g_2)$$

$\tilde r$ = map s.t. $pr\circ\tilde r = r$

Associativity: $(\tilde r(g_1)\tilde r(g_2))\tilde r(g_3) = \tilde r(g_1)(\tilde r(g_2)\tilde r(g_3))$

$$\Longrightarrow \boxed{\begin{array}{c} c(g_1,g_2)\,c(g_1g_2,g_3) = c(g_1,g_2g_3)\,c(g_2,g_3) \\ \text{2-cocycle identity}: \quad c\in Z^2(G,\mathbb{C}^*) \end{array}}$$

Change of $\tilde r$: $\tilde r \rightsquigarrow \tilde r' = \tilde r\cdot d$, $d: G\longrightarrow \mathbb{C}^*$ (map)

$$\boxed{c'(g_1,g_2) = c(g_1,g_2)\,\frac{d(g_1)\,d(g_2)}{d(g_1g_2)}}$$

Assume: the values of c' lie in a subgroup $A\subset\mathbb{C}^*$

Def: group $\tilde G = \{(g,a) \mid g\in G, a\in A\}$
$$(g_1,a_1)(g_2,a_2) = (g_1g_2,\ a_1a_2\,c'(g_1,g_2))$$

$$a \longmapsto (e,a)$$

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "i"] \arrow[d, hook] & \tilde G \arrow[r, "pr"] \arrow[d, "\rho"] & G \arrow[r] \arrow[d, "r"] & 1 \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] & GL(V) \arrow[r] & PGL(V) \arrow[r] & 1 \end{tikzcd}$$

$i(A)\subset Z(G)$
central extension

$$\rho((g,a)) = a\,\tilde r'(g)$$

Homogeneous vs non-homogeneous cocycles

Given: group G acting on a set X, map $f: X^3\longrightarrow A$ (for an abelian group A) s.t. $f(gx_1,gx_2,gx_3) = f(x_1,x_2,x_3)$ $\forall g\in G$.
Fix $x\in X$.

Then: f satisfies $\forall x_0,x_1,x_2,x_3\in X$ $\displaystyle\sum_{i=0}^{3}(-1)^i f(x_0,\dots,\hat{x}_i,\dots,x_3) = 0$

$$\Downarrow$$

$c: G\times G\longrightarrow A$, $c(g_1,g_2) = f(x, g_1(x), g_1g_2(x))$
satisfies
$$c(g_1,g_2) + c(g_1g_2,g_3) = c(g_1,g_2g_3) + c(g_2,g_3)$$

Higher-dimensional case

Given: W real v. sp., $\dim(W) = 2n < \infty$, $B : W \times W \to \mathbb{R}$ symplectic (alternating, non-degenerate)

$\exists$ symplectic basis: $W = \bigoplus_{j=1}^{n} (\mathbb{R}P_j \oplus \mathbb{R}Q_j)$ $\quad B(P_j, P_k) = B(Q_j, Q_k) = 0$, $\quad B(P_j, Q_k) = \delta_{jk}$

Heisenberg Lie algebra: $\mathfrak{heis} = W \oplus \mathbb{R}\cdot E$

$[E, \cdot] = 0$, $\quad \forall x, y \in W \quad [x,y] = B(x,y)E$

Heisenberg group: (1) simply connected: exponentiate $[\ ,\ ]$ using Campbell – Hausdorff:

$$\exp(X)\exp(Y) = \exp\left(X + Y + \tfrac{1}{2}[X,Y] + \text{higher } [[\ ,\ ],\ ]\right)$$

$$\{\underbrace{\exp(x+tE)}_{\{x,t\}} \mid x \in W, t \in \mathbb{R}\}$$

$$\{x,t\}\{x',t'\} = \{x+x', t+t' + \tfrac{1}{2}B(x,x')\}$$

(2) push-forward via $\exp(\mathbb{R}E) = \mathbb{R} \xrightarrow{\psi} U(1)$, $\quad t \longmapsto e^{2\pi i t}$

$\mathrm{Heis} = \{(x,a) \mid x \in W, a \in U(1)\}$

$$(x,a)(x',a') = \left(x+x', aa'\psi\left(\tfrac{1}{2}B(x,x')\right)\right)$$

$$1 \longrightarrow U(1) \longrightarrow \mathrm{Heis} \xrightarrow{\pi} W \longrightarrow 0, \quad \pi(x,a) = x, \quad Z = U(1)$$

Schrödinger representation: on $C^\infty(\mathbb{R}^n)$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$

$P_j \longmapsto \hat{P}_j = \frac{\partial}{\partial x_j}$, $\quad Q_j \longmapsto \hat{Q}_j = 2\pi i x_j$, $\quad u \in U(1) \longmapsto u$

$$\left(u \exp\left(\textstyle\sum a_j P_j + b_j Q_j\right) f\right)(x) = u\, e^{2\pi i \left({}^t b x + \frac{1}{2}{}^t b a\right)} f(x+a)$$

irreducible unitary (on $L^2(\mathbb{R}^n)$)

Key Thm (Stone, von Neumann)

An irreducible unitary representation $\rho : \mathrm{Heis} \to U(\mathcal{H})$ ($\mathcal{H}$ = Hilbert space) s.t. $\rho(u) = u \cdot 1$ $\forall u \in U(1)$ is isomorphic to the above representation.

Construction of $\rho : \mathrm{Heis} \to U(\mathcal{H})$, $\rho(u) = u \cdot 1$ $\forall u \in U(1)$:

induced from characters of abelian subgroups containing $U(1)$

$$\left\{\begin{matrix}\text{abelian subgps of} \\ \text{Heis containing } U(1)\end{matrix}\right\} = \left\{ \pi^{-1}(L) \;\middle|\; \begin{matrix} L \subset W \text{ subgroup such that} \\ \forall x, y \in L \quad B(x,y) \in \mathbb{Z} \end{matrix} \right\}$$

L closed $\Longrightarrow$

$\exists$ cont. χ

$$\begin{tikzcd}
1 \arrow[r] & U(1) \arrow[r] \arrow[d, equal] & \mathrm{Heis} \arrow[r] & W \arrow[r] & 0 \\
1 \arrow[r] & U(1) \arrow[r] & \pi^{-1}(L) \arrow[r] \arrow[u, phantom, "\cup"] \arrow[l, bend right, "\chi"'] & L \arrow[r] \arrow[u, phantom, "\cup"] & 0
\end{tikzcd}$$

Def: $\mathrm{Ind}_{\pi^{-1}(L)}^{\mathrm{Heis}}(\chi) = \left\{ \begin{matrix} f : \mathrm{Heis} \to \mathbb{C},\ f(hg) = \chi(h) f(g) \quad \forall h \in \pi^{-1}(L), \forall g \in \mathrm{Heis} \\ f \text{ measurable}, \displaystyle\int_{\pi^{-1}(L)\backslash \mathrm{Heis}} |f|^2 < \infty \end{matrix} \right\}$

$(g * f)(g') = f(g'g)$

Unitary representation of Heis, irreducible if L is maximal.

Formulas: choice of $\chi \Longleftrightarrow$ choice of $\pi^{-1}(L) \leftrightarrows L$, $\chi \circ \sigma = 1$

$\boxed{\sigma(y) = (y, \alpha(y))}$, $\forall y, y' \in L$ $\sigma(y)\sigma(y') = \sigma(y+y') \Longrightarrow \alpha : L \to U(1)$

satisfies $\boxed{\alpha(y)\alpha(y') e^{\pi i B(y,y')} = \alpha(y+y')}$ $\quad \forall y, y' \in L$

$f \in \mathrm{Ind}_{\pi^{-1}(L)}^{\mathrm{Heis}}(\chi) \Longrightarrow f((x,u)) = u f((x,1)) \qquad x \in W,\ u \in U(1)$

write $\quad f(x) := f((x,1)), \quad x \in W$

$f(hg) = \chi(h) f(g) \Longleftrightarrow f(y+x) = \alpha(y)^{-1} e^{-\pi i B(y,x)} f(x) \quad \forall x \in W\ \forall y \in L$

$\int_{\pi^{-1}(L)\backslash \mathrm{Heis}} |f|^2 < \infty \quad \Longleftrightarrow \quad \int_{W/L} |f(x)|^2 dx < \infty$, dx any invariant measu

---

Examples of maximal $L \subset W$ s.t. $B(L,L) \subseteq \mathbb{Z}$:

(1) $L = \bigoplus_{j=1}^{n} (\mathbb{Z}P_j + \mathbb{Z}Q_j)$

(2) $L = \ell$ $\mathbb{R}$-v. subspace s.t. $\ell = \underbrace{\ell^{\perp}}_{\{x \in W \mid \forall y \in \ell\ B(x,y) = 0\}}$ (ℓ = lagrangian subspace

$\ell = \bigoplus_{j=1}^{n} \mathbb{R}P_j$, $\bigoplus_{j=1}^{n} \mathbb{R}Q_j$ $\qquad (\alpha = 1) \quad \underline{\chi((x,u)) = u}$

For ℓ identify W/ℓ with $\ell' = \bigoplus_{j=1}^{n} \mathbb{R}Q_j$. $\qquad$ Schrödinge

$f \in \mathrm{Ind}_{\pi^{-1}(\ell)}^{\mathrm{Heis}}$ is determined by $f|_{\ell'} \in L^2(\mathbb{R}^n) \stackrel{\text{get}}{\Longrightarrow}$ repr.

Intertwining operators

Measures (invariant): $\ell \in \mathrm{Lagr}(W, B) = \{$lagrangian subspaces (of W

$0 \neq e \in \Lambda^n \ell \Rightarrow$ measure on $W/\ell \simeq \ell^* \Longrightarrow H(\ell, e) = L^2(W/\ell)$

Exercise: (1) $\ell_1, \ell_2 \in \mathrm{Lagr}(W) \Longrightarrow ((\ell_1 + \ell_2)/(\ell_1 \cap \ell_2), \overline{B})$ symplectic

$\Longrightarrow$ measure on $(\ell_1 + \ell_2)/(\ell_1 \cap \ell_2)$ $\Big\} \Longrightarrow$ measure on $\ell_2/(\ell_1 \cap \ell_2)$

(2) Given $0 \neq e_i \in \Lambda^n \ell_i$

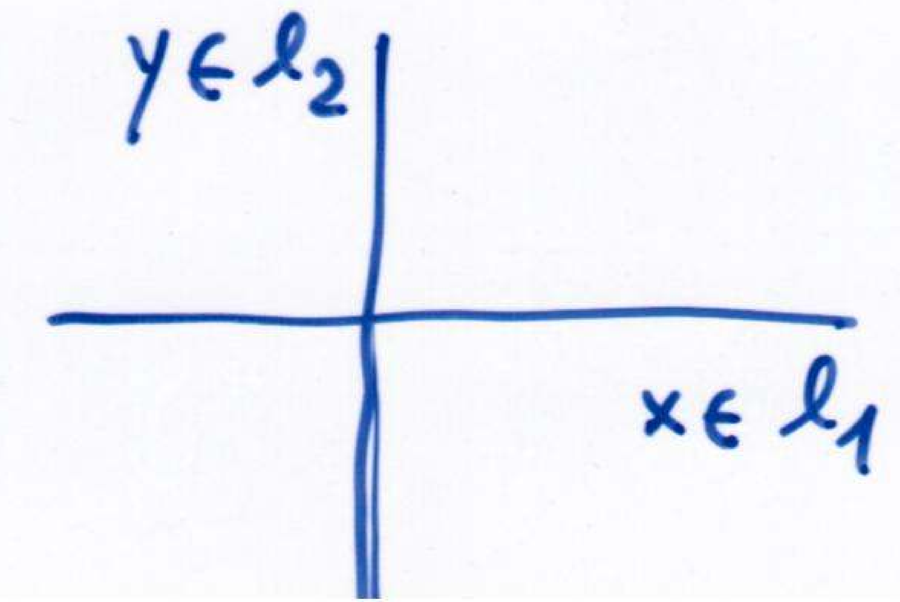
Def: $\ell_1, \ell_2 \in \mathrm{Lagr}(W)$, $0 \neq e_i \in \Lambda^n \ell_i$

$$H(\ell_1, e_1) = \mathrm{Ind}_{\pi^{-1}(\ell_1)}^{\mathrm{Heis}}(\chi) \xrightarrow{F_{\ell_2,\ell_1}} H(\ell_2, e_2)$$

$$f \longmapsto \Big(g \longmapsto \int_{\pi^{-1}(\ell_1 \cap \ell_2)\backslash \pi^{-1}(\ell_2) \simeq \ell_2/(\ell_1 \cap \ell_2)} \chi(h)^{-1} f(hg)\Big)$$

Heis - equivariant

Ex: $\ell_1 \cap \ell_2 = \{0\}$, $\quad H(\ell_1) = L^2(\ell_2) \to H(\ell_2) = L^2(\ell_1)$



$$(F_{\ell_2,\ell_1} f)(x) = \int_{\ell_2} f(\underbrace{(y,1)(x,1)}_{(0,\, e^{2\pi i B(y,x)})(x,1)(y,1)})\, dy$$

$$= \int e^{2\pi i B(y,x)} f(y)\, dy$$

$F_{\ell_2,\ell_1} = \mathcal{F} \circ [$
$\quad = \mathcal{F}^{-}$

<u>General case</u>: $V = \ell_1 + \ell_2$, $B : (W/V)^* \simeq V^\perp = \ell_1 \cap \ell_2$

$$\begin{tikzcd} & & \ell_1 \arrow[dr, hook] & & \\ \{0\} \arrow[r, hook] & V^\perp = \ell_1 \cap \ell_2 \arrow[ur, hook] \arrow[dr, hook] & & V = \ell_1 + \ell_2 \arrow[r, hook] & W \\ & & \ell_2 \arrow[ur, hook] & & \end{tikzcd}$$

dim = $\quad m \quad\quad n \quad\quad 2n-m \quad\quad 2n$

$\ell_i / \ell_1 \cap \ell_2 \in \mathrm{Lagr}(V/V^\perp)$ transversal

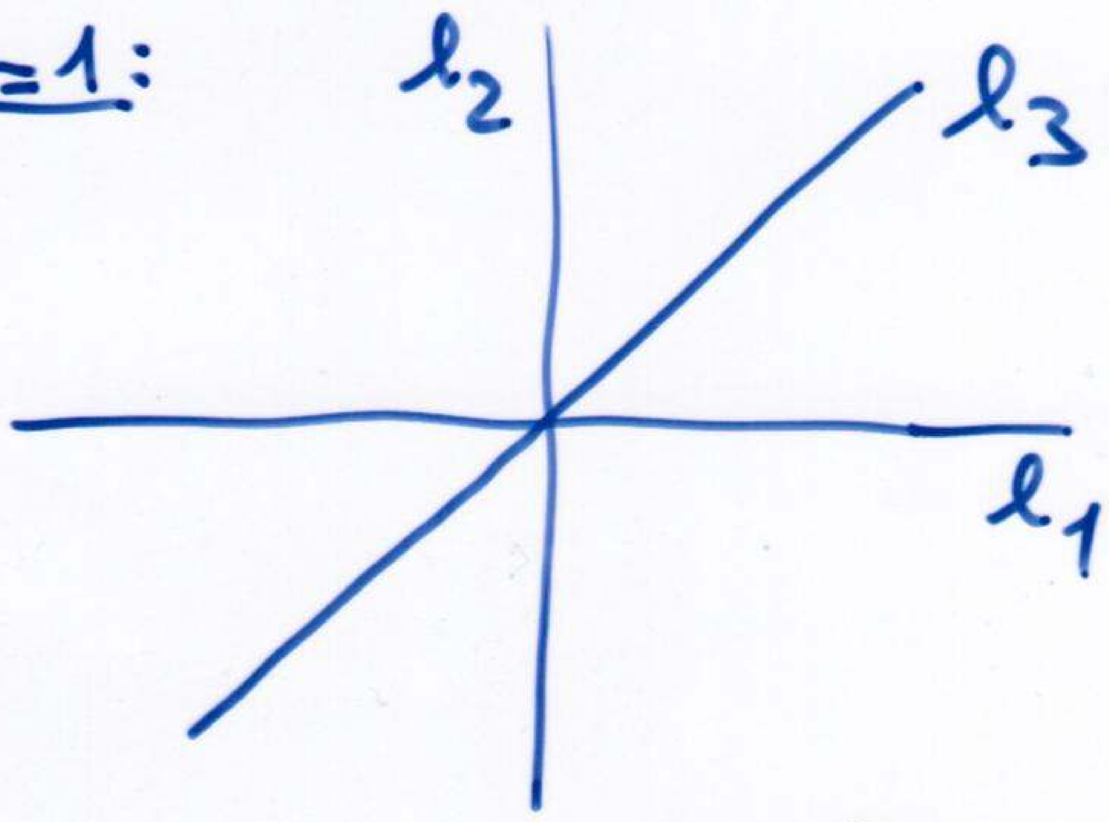
$\Longrightarrow$ $F_{\ell_2,\ell_1} : H(\ell_1) \longrightarrow H(\ell_2)$ is a <u>partial Fourier transform</u> (in $n-m$ variables) $\Longrightarrow$ $\boxed{F_{\ell_1,\ell_2} = F_{\ell_2,\ell_1}^{-1}}$

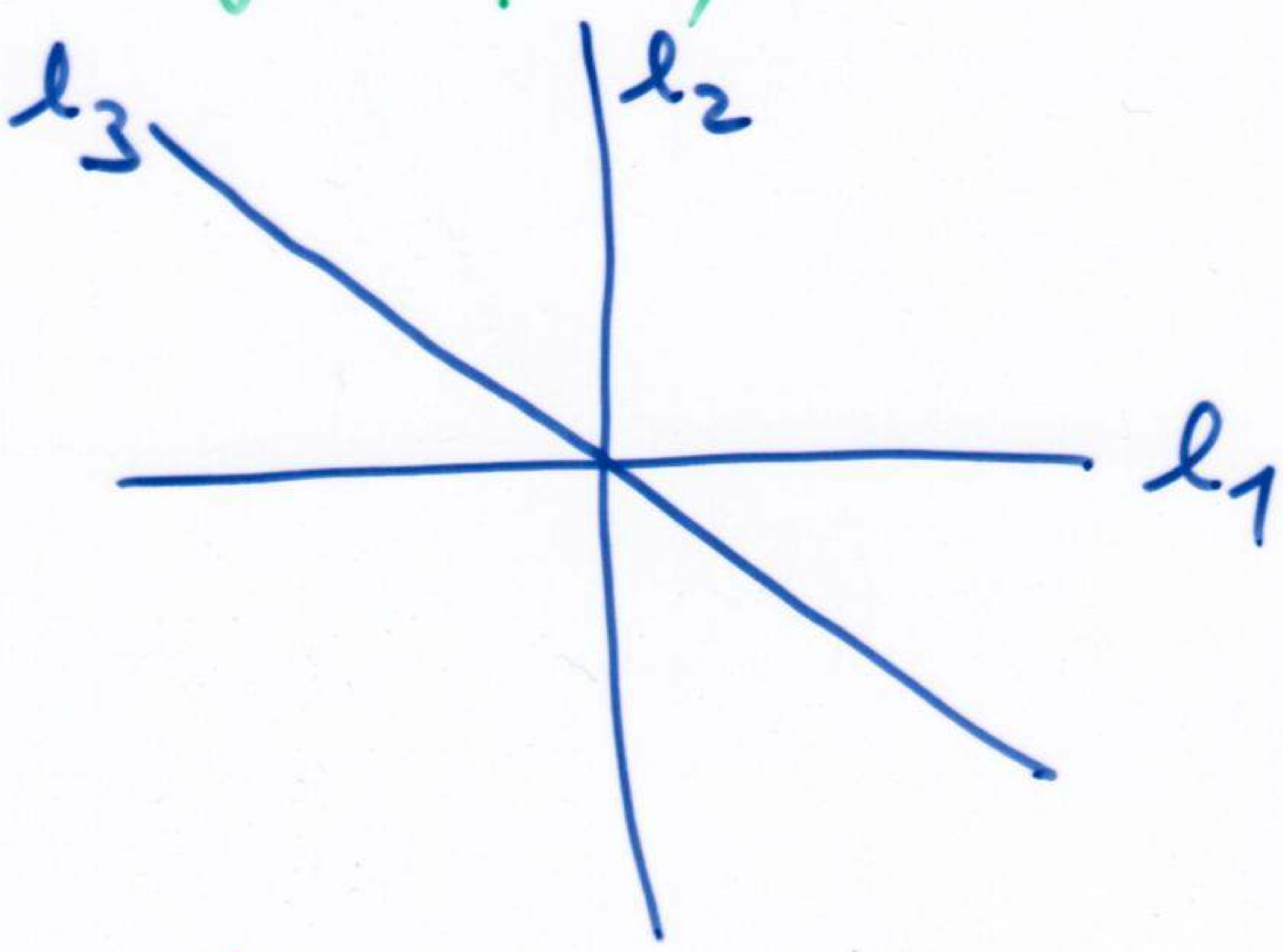
($\Longrightarrow$ unitary)

<u>Invariants of triples of lagrangian subspaces</u>

<u>Basic 2-cocycle</u>: given $\ell_i \in \mathrm{Lagr}(W, B)$, $0 \neq e_i \in \Lambda^n \ell_i$ $\quad (i=1,2,3)$

<u>n=1</u>:





$$\begin{tikzcd} H(\ell_1, e_1) \arrow[rr, "F_{\ell_2,\ell_1}"] & & H(\ell_2, e_2) \arrow[dl, "F_{\ell_3,\ell_2}"] \\ & H(\ell_3, e_3) \arrow[ul, "F_{\ell_1,\ell_3}"] & \end{tikzcd}$$

unitary, Heis-equiv.

$$F_{\ell_1,\ell_3} \circ F_{\ell_3,\ell_2} \circ F_{\ell_2,\ell_1} = \underset{\in}{\overset{U(1)}{}} a(\ell_1, \ell_2, \ell_3) \cdot 1$$

independent of e_1, e_2, e_3

<u>Properties</u>: (1) $a(\ell_1, \ell_2, \ell_3) = a(\ell_2, \ell_3, \ell_1)$

(2) a is a <u>2-cocycle</u>: $\displaystyle\prod_{j=1}^{4} a(\ell_1, -, \hat{\ell}_j, -, \ell_4)^{(-1)^j} = 1$

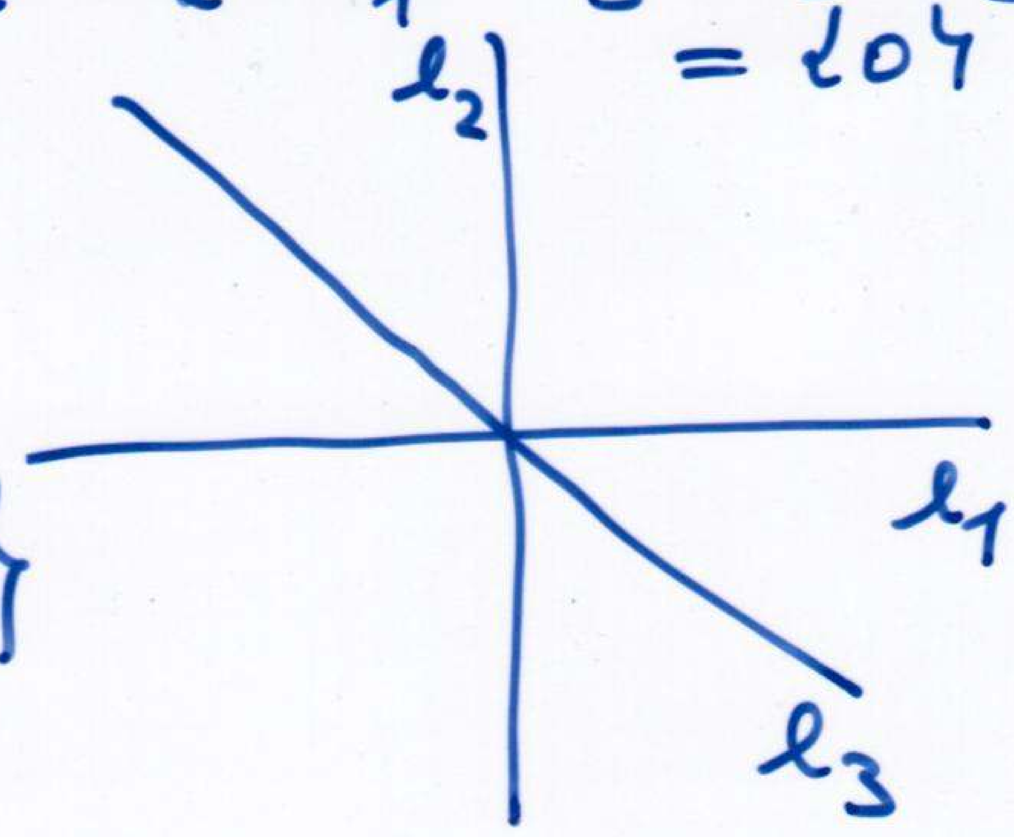
(3) $\forall \gamma \in G = Sp(W, B) = \{ g \in GL(W) \mid B(gx, gy) \ \forall x, y \in W \}$

$$a(\gamma\ell_1, \gamma\ell_2, \gamma\ell_3) = a(\ell_1, \ell_2, \ell_3)$$

<u>Pf of (3)</u>: G acts on Heis $\quad \gamma(x, u) = (\gamma(x), u)$ and on $\{ f : \mathrm{Heis} \to \mathbb{C} \}$, $({}^\gamma f)(g) = f(\gamma^{-1}(g))$, ${}^\gamma x = x$.

$H(\ell, e) \xrightarrow{\gamma} H(\gamma\ell, \gamma e)$, commutes with F_{ℓ_2,ℓ_1}.

Algebraic invariants: **transversal case** $\ell_1 \cap \ell_2 = \ell_1 \cap \ell_3 = \ell_2 \cap \ell_3 = \{0\}$



$B : \ell_2 \xrightarrow{\sim} \ell_1^* \implies W = \ell_1 \oplus \ell_1^*$

$B\left(\begin{pmatrix} x \\ x^* \end{pmatrix}, \begin{pmatrix} y \\ y^* \end{pmatrix}\right) = \langle y^*, x\rangle - \langle x^*, y\rangle$

$\ell_3 = \Gamma_T$ = graph of $T : \ell_1 \xrightarrow{\sim} \ell_1^* = \left\{ \begin{pmatrix} x \\ Tx \end{pmatrix} \mid x \in \ell_1 \right\}$

$\ell_3 \in \mathrm{Lagr} \iff \forall x, y \in \ell_1 \quad \underbrace{B\left(\begin{pmatrix} x \\ Tx \end{pmatrix}, \begin{pmatrix} y \\ Ty \end{pmatrix}\right)}_{\langle Ty, x\rangle - \langle Tx, y\rangle = \langle (T^* - T)x, y\rangle} = 0$

$\underbrace{T = T^*}_{\text{non-degenerate quadratic form on } \ell_1} \iff$

Action of $\{ g \in G \mid g(\ell_i) = \ell_i,\ i = 1, 2\} = \left\{ \begin{pmatrix} a^{-1} & 0 \\ 0 & a^* \end{pmatrix} \mid a \in GL(\ell_1) \right\}$:

$g(\Gamma_T) = \left\{ \begin{pmatrix} a^{-1}x \\ a^*Tx \end{pmatrix} \mid x \in \ell_1 \right\} = \left\{ \begin{pmatrix} y \\ (a^*Ta)y \end{pmatrix} \mid y \in \ell_1 \right\} = \Gamma_{a^*Ta}$

Cor: $G \backslash \left(\mathrm{Lagr}(W)^3\right)^{\text{transversal}} \xrightarrow{\sim} \left\{ \begin{matrix} \text{non-deg. quadratic} \\ \text{forms on } \mathbb{R}^n \end{matrix} \right\} / \mathrm{Isom}$

General case: $\ell_i \in \mathrm{Lagr}(W)$

$Q_{\ell_1, \ell_2, \ell_3} : \ell_1 \oplus \ell_2 \oplus \ell_3 \longrightarrow \mathbb{R}$ quadratic form (on $\simeq \mathbb{R}^{3n}$)

$(x_1, x_2, x_3) \longmapsto B(x_1, x_2) + B(x_2, x_3) + B(x_3, x_1)$

Exercise: in the transversal case $Q \simeq T \perp S \perp (-S)$

Maslov index: $\mathrm{Maslov}(\ell_1, \ell_2, \ell_3) = \mathrm{sgn}(Q_{\ell_1, \ell_2, \ell_3}) \in \mathbb{Z}$

($= \mathrm{sgn}(T)$ in the transversal case).

Key Thm. (0) $\mathrm{Maslov}(\gamma\ell_1, \gamma\ell_2, \gamma\ell_3) = \mathrm{Maslov}(\ell_1, \ell_2, \ell_3) \quad \forall \gamma \in G$

(1) $\sum_{j=0}^{3} (-1)^j \,\mathrm{Maslov}(\ell_0, -, \widehat{\ell_j}, -, \ell_3) = 0$ (1-cocycle)

(2) $a(\ell_1, \ell_2, \ell_3) = \left(e^{-2\pi i/8}\right)^{\mathrm{Maslov}(\ell_1, \ell_2, \ell_3)}$

Pf: (1) WLOG ℓ_0 transversal to ℓ_1, ℓ_2, ℓ_3. Use T above.

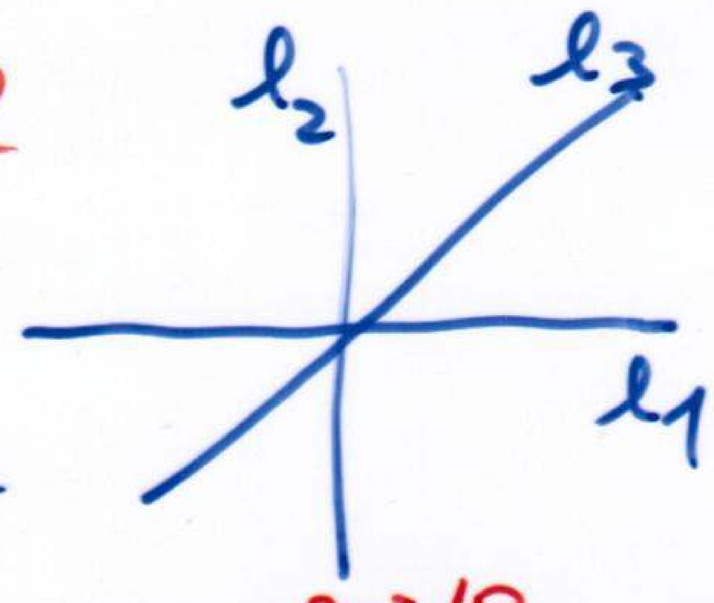
(2) Reduce to the transversal case. Then $\ell_3 = \Gamma_T$,

$T = \sum x_j^2 - \sum y_k^2 \implies$ direct sum of

For $n = 1$, $a(\ell_1, \ell_2, \ell_3)$ combines

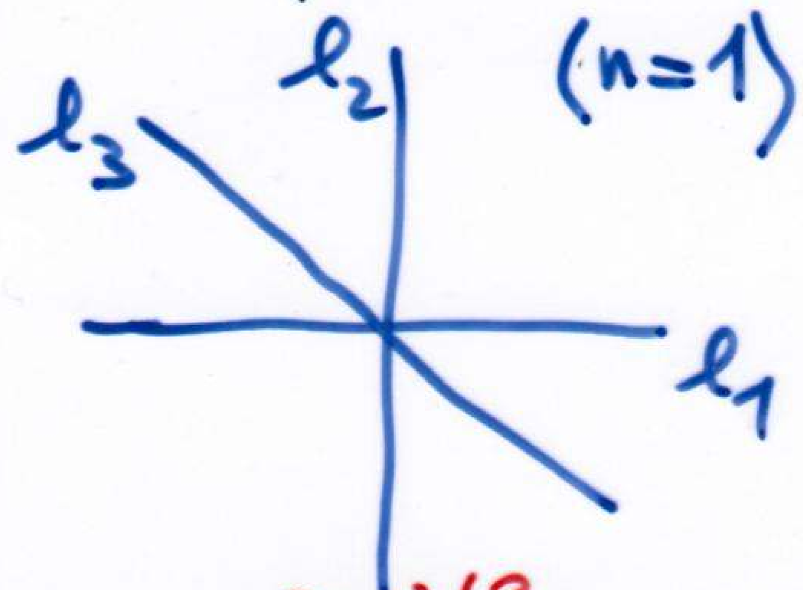
$\mathcal{F}$ and $e^{\pm\pi i x^2}$

Apply them to suitable $f_\tau = e^{\pi i \tau x^2} \implies$



$e^{-2\pi i/8}$

$1 = \mathrm{Maslov}$



$e^{2\pi i/8}$

$\mathrm{Maslov} = -1$

Why "$-$"? For $\psi_t(x) = e^{2\pi i t x}$ $\begin{cases} \text{no change} & \text{if } t = r^2 > 0 \\ \text{complex conj.} & \text{if } t = -r^2 < 0 \end{cases}$

Why 8? Bott periodicity, Clifford algebras

F field (char $\neq 2$): Witt ring $WF = K_0(\text{quadr. forms}/F)/\langle x^2 - y^2\rangle$

$0 \to IF \to WF \xrightarrow{rk} \mathbb{Z}/2\mathbb{Z} \to 0$ sgn: $W\mathbb{R} \xrightarrow{\sim} \mathbb{Z}$

Q quadr. form (non-deg.) $\Longrightarrow$ Clifford algebra $C(Q)$, $\mathbb{Z}/2$-graded

$$C(Q_1 \perp Q_2) \simeq C(Q_1) \hat{\otimes} C(Q_2)$$

$$C : (WF/I^3F, +) \xrightarrow{\sim} ((\text{Clifford algebras over } F)/\sim, \hat{\otimes})$$

$I\mathbb{R} = 2\mathbb{Z}$, $W\mathbb{R}/I^3\mathbb{R} = \mathbb{Z}/8\mathbb{Z}$

Reduced (oriented) Maslov index

Thm: For $\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3 \in \widetilde{Lagr}(W)$ oriented lagrangian subspaces

let $m(\tilde{\ell}_1, \tilde{\ell}_2) = n - \dim(\ell_1 \cap \ell_2) + \begin{cases} 0 & \text{if } B : \ell_1/\ell_1\cap\ell_2 \xrightarrow{\sim} (\ell_2/\ell_1\cap\ell_2)^* \text{ is compatible with orient.} \\ 2 & \text{if not} \end{cases}$

then: $\mathrm{Maslov}(\ell_1, \ell_2, \ell_3) \equiv m(\tilde{\ell}_1, \tilde{\ell}_2) - m(\tilde{\ell}_1, \tilde{\ell}_3) + m(\tilde{\ell}_2, \tilde{\ell}_3) \pmod 4$

Note: $\mathrm{sgn}(Q) \pmod 4 \Longleftrightarrow \mathrm{rk}(Q) \pmod 4$, $\det(Q) \in \mathbb{R}^*/\mathbb{R}^{*2}$

Def. $\widetilde{\mathrm{Maslov}}(\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3) := \underbrace{\mathrm{Maslov}(\ell_1, \ell_2, \ell_3) - m(\tilde{\ell}_1, \tilde{\ell}_2) + m(\tilde{\ell}_1, \tilde{\ell}_3) - m(\tilde{\ell}_2, \tilde{\ell}_3)}_{4\mathbb{Z}}$

$$\tilde{F}_{\tilde{\ell}_2, \tilde{\ell}_1} := (e^{-2\pi i/8})^{m(\tilde{\ell}_1, \tilde{\ell}_2)} F_{\ell_2, \ell_1}$$

Thm:

$$\tilde{F}_{\tilde{\ell}_3, \tilde{\ell}_2} \circ \tilde{F}_{\tilde{\ell}_2, \tilde{\ell}_1} = \underbrace{(e^{-2\pi i/8})^{\widetilde{\mathrm{Maslov}}(\tilde{\ell}_1, \tilde{\ell}_2, \tilde{\ell}_3)}}_{\pm 1} \tilde{F}_{\tilde{\ell}_3, \tilde{\ell}_1}$$

Metaplectic group

Fix: $\tilde{\ell} \in \widetilde{Lagr}(W)$, $0 \neq e \in \Lambda^n \ell \Longrightarrow \forall g \in G = Sp(W)$ $g\tilde{\ell}$, ge

Def:

$$\begin{tikzcd} \mathcal{H} = H(\ell, e) \arrow[r, bend left, "g"] & H(g\ell, ge) \arrow[l, bend left, "F_{\ell, g\ell}"] \end{tikzcd} \qquad R_e(g) := F_{\ell, g\ell} \circ g \in U(\mathcal{H})$$

$$R_e(g_1) R_e(g_2) = c(g_1, g_2) R_e(g_1 g_2), \quad c(g_1, g_2) = (e^{2\pi i/8})^{\mathrm{Maslov}(\ell, g_1\ell, g_1 g_2 \ell)}$$

Def: $\tilde{R}_{\tilde{\ell}}(g) := (e^{2\pi i/8})^{m(\tilde{\ell}, g\tilde{\ell})} R_e(g) \in U(\mathcal{H})$

$$\tilde{R}_{\tilde{\ell}}(g_1) \tilde{R}_{\tilde{\ell}}(g_2) = \underbrace{(e^{2\pi i/8})^{\widetilde{\mathrm{Maslov}}(\tilde{\ell}, g_1\tilde{\ell}, g_1 g_2 \tilde{\ell})}}_{\pm 1} \tilde{R}_{\tilde{\ell}}(g_1 g_2)$$

Cor: $\{\tilde{R}_{\tilde{\ell}}(g)\}$ define a unitary representation of a central extension $1 \to \{\pm 1\} \to \widetilde{Sp}(W, B) \to Sp(W, B) \to 1$

Oscillator (= Weil) representation of the metaplectic group $Mp(2n,\mathbb{R}) = \widetilde{Sp(2n,\mathbb{R})}$ on $L^2(\mathbb{R}^n)$:

$(W,B) = \left(\mathbb{R}^{2n}, \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right)$, $\ell = \begin{pmatrix} \mathbb{R}^n \\ 0 \end{pmatrix}$, $1 \to \{\pm 1\} \to \underbrace{Mp(2n,\mathbb{R})}_{\widetilde{G}} \to \underbrace{Sp(2n,\mathbb{R})}_{G} \to 1$

<u>Formulas:</u> $f \in L^2(\mathbb{R}^n)$, $x \in \mathbb{R}^n$

$$0 < \det(a) \Rightarrow \left(\widetilde{R}\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix} f\right)(x) = \det(a)^{1/2} f({}^t a x)$$

$$ {}^t b = b \Rightarrow \left(\widetilde{R}\begin{pmatrix} I & b \\ 0 & I \end{pmatrix} f\right)(x) = e^{\pi i\, {}^t x b x} f(x)$$

$$\left(\widetilde{R}\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} f\right)(x) = \left(e^{2\pi i/8}\right)^n \int_{\mathbb{R}^n} e^{2\pi i\, {}^t x y} f(y)\, dy$$

<u>Action of $\mathrm{Lie}(\widetilde{G}) = \mathrm{Lie}(G) = \mathfrak{g}$:</u> (1) $\boxed{n=1}$

$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto \pi i x^2$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto x\frac{d}{dx} + \frac{1}{2}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto \frac{i}{4\pi}\left(\frac{d}{dx}\right)^2$

$G = SL_2(\mathbb{R}) \supset K = SO(2) = \left\{ \begin{pmatrix} a & -c \\ c & a \end{pmatrix} \,\middle|\, a^2 + c^2 = 1 \right\} \xrightarrow{\sim} U(1)$

$\begin{pmatrix} a & -c \\ c & a \end{pmatrix} \longleftrightarrow ci + a \in U(1)$

action of $\mathfrak{k} = \mathrm{Lie}(\widetilde{K}) = \mathrm{Lie}(K) = \{ c(-X+Y) \mid c \in \mathbb{R} \} \xrightarrow{\sim} \mathrm{Lie}(U(1))$

$c(-X+Y) \longleftrightarrow u = ci \in \mathrm{Lie}(U(1))$

$\varphi_0 = e^{-\pi x^2}$, $(-X+Y)\varphi_0 = -\frac{i}{2}\varphi_0 \Longleftrightarrow \boxed{u\varphi_0 = -\frac{u}{2}\varphi_0 \quad \forall u \in \mathrm{Lie}(U(1))}$

does NOT integrate to an action of K

$\Longrightarrow$ $1 \to \{\pm 1\} \to \widetilde{K} \to K \to 1$

<u>does not split.</u>

(2) The general case:

$$\boxed{G = Sp(2n,\mathbb{R})}, \quad \mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \,\middle|\, {}^tB = B,\ {}^tC = C \right\}$$

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \longmapsto \pi i ({}^tx B x), \quad \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \longmapsto \frac{i}{4\pi} {}^t\partial C \partial, \quad \begin{pmatrix} A & 0 \\ 0 & -{}^tA \end{pmatrix} \longmapsto {}^tx A \partial + \frac{Tr(A)}{2}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \partial = \begin{pmatrix} \partial/\partial x_1 \\ \vdots \\ \partial/\partial x_n \end{pmatrix}, \quad {}^tB = B, \quad {}^tC = C$$

$$\boxed{K = G \cap O(2n)} = \left\{ \begin{pmatrix} A & -C \\ C & A \end{pmatrix} \,\middle|\, Ci + A \in U(n) \right\} \xrightarrow{\sim} U(n) \ni U$$

$$\mathfrak{k} = Lie(K) = \left\{ \begin{pmatrix} A & -C \\ C & A \end{pmatrix} \,\middle|\, {}^tC = C,\ {}^tA = -A \right\} \xrightarrow{\sim} \left\{ U \in M_n(\mathbb{C}) \,\middle|\, {}^tU = -\overline{U} \right\} = Lie(U(n))$$

$$\begin{pmatrix} A & -C \\ C & A \end{pmatrix} \longleftrightarrow U = Ci + A$$

Action on the gaussian $\varphi_0 = e^{-\pi(x_1^2 + \dots + x_n^2)} = e^{-\pi\, {}^tx x}$

$$\boxed{U\varphi_0 = -\frac{i}{2} Tr(C) \varphi_0 = -\frac{Tr(U)}{2} \varphi_0}$$

$\Longrightarrow 1 \to \{\pm 1\} \to \widetilde{K} \to K \to 1$ does not split.

Siegel upper half space

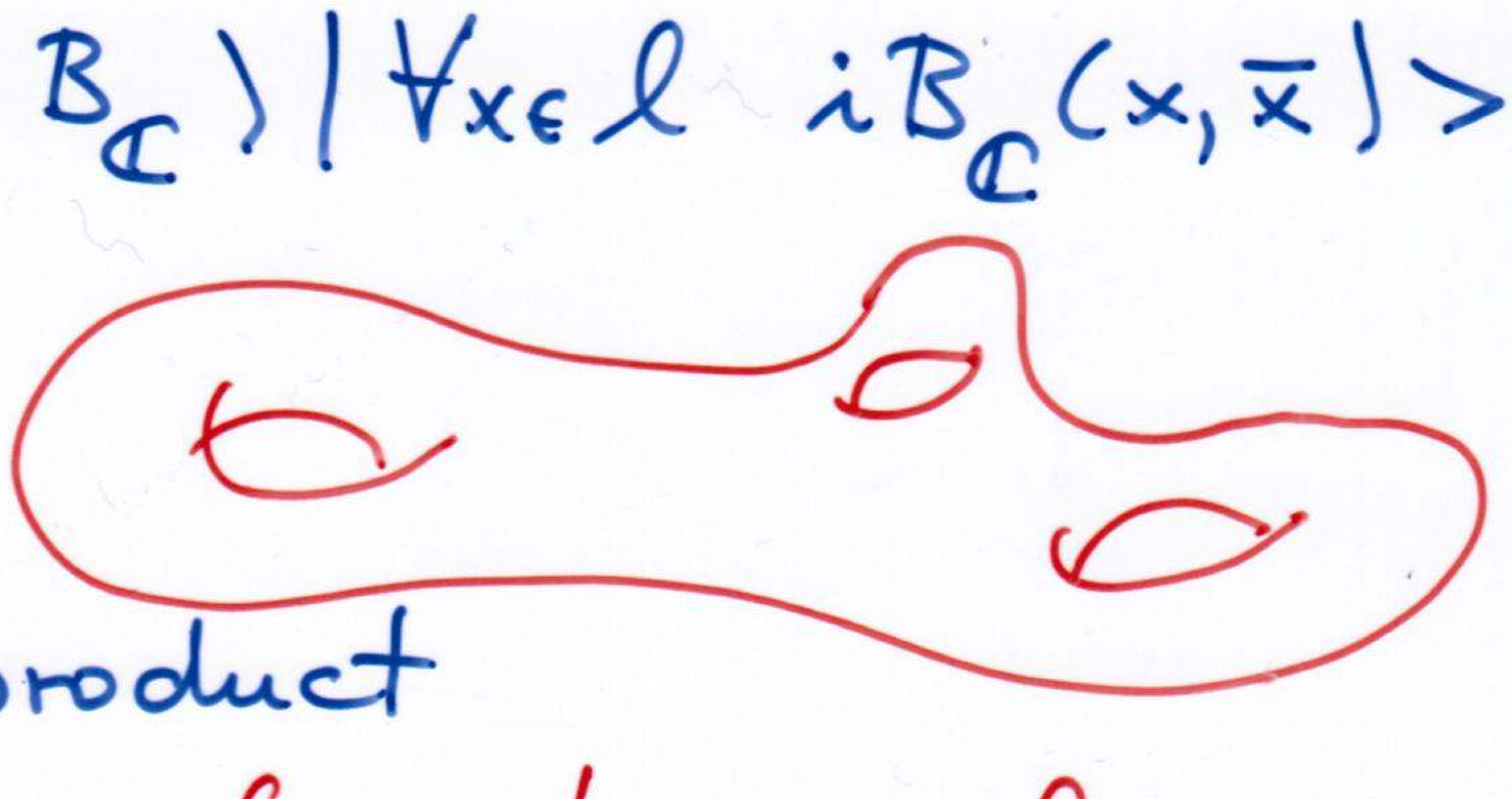
$$G/K \xrightarrow{\sim} \mathcal{H}_n = \{ T \in M_n(\mathbb{C}) \mid {}^tT = T,\ Im(T) > 0 \}$$

$$gK \longmapsto g(i \cdot I), \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix}(T) = (AT + B)(CT + D)^{-1}$$

Coordinate-free description of $\mathcal{H}_n$:

$$\{ \ell \in Lagr(W_{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}, B_{\mathbb{C}}) \mid \forall x \in \ell \ \ iB_{\mathbb{C}}(x, \overline{x}) > 0 \}$$

Ex ("Hodge" theory):



X cpt Riemann surface

$W = H^1(X, \mathbb{R})$, $B = \cup$ product

$\ell = H^0(X, \Omega^1) \ni \omega, \omega'$ $\quad \int_X \omega \wedge \omega' = 0, \quad i \int_X \omega \wedge \overline{\omega} > 0$

Basic θ-functions on $\mathcal{H}_n$:

$$\boxed{\Theta(T,Z)=\sum_{m\in\mathbb{Z}^n} e^{\pi i\, {}^t m T m + 2\pi i\, {}^t m Z}} \qquad (T\in\mathcal{H}_n,\ Z\in\mathbb{C}^n)$$

Fix: $c:\mathbb{Z}^n\longrightarrow(\mathbb{Z}/M\mathbb{Z})^n\longrightarrow\mathbb{C}$

$$\theta_{c,\text{scalar}}(T)=\sum_{m\in\mathbb{Z}^n} c(m)e^{\pi i\,{}^t m T m},\quad \theta_{c,\text{vector}}(T)=\sum_{m\in\mathbb{Z}^n} c(m)\,m\,e^{\pi i\,{}^t m T m}\in\mathbb{C}^n$$

holomorphic on $\mathcal{H}_n$

Key property: $\exists$ congruence subgroup $\Gamma\subset Sp(2n,\mathbb{Z})$

$$\forall \underbrace{\begin{pmatrix}A&B\\C&D\end{pmatrix}}_{\gamma}\in\Gamma\quad \theta_{c,\text{scalar}}(\gamma(T))=(\text{const.})\sqrt{\det(CT+D)}\ \theta_{c,\text{scalar}}(T)$$

$$\theta_{c,\text{vector}}(\gamma(T))=(\text{const.})\sqrt{\det(CT+D)}\ (CT+D)\,\theta_{c,\text{vector}}(T)$$

Canonical automorphy factor:

$$J: G\times\mathcal{H}_n\longrightarrow GL_n(\mathbb{C})$$
$$g,\ T\longmapsto \boxed{CT+D=J(g,T)}\qquad g=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in G$$

Properties: (1) $\forall\ k\in K\simeq U(n)$ $\quad\boxed{J(k,iI)=k\in U(n)}$

(2) 1-cocycle: $\boxed{J(gg',T)=J(g,g'(T))\,J(g',T)}$

(3) $\widetilde{G}\simeq\{[g\in G,\ \lambda:\mathcal{H}_n\to\mathbb{C}\text{ holomorphic s.t. }\lambda(T)^2=\det J(g,T)]\}$

$$[g,\lambda][g',\lambda']=[gg',(\lambda\circ g')\lambda']$$

the map $\widetilde{G}\longrightarrow\mathbb{C}^*$, $[g,\lambda]\longmapsto\lambda(iI)$ restricts to a group homomorphism "$\det^{1/2}$": $\widetilde{K}\longrightarrow U(1)$ whose square is equal to $\widetilde{K}\xrightarrow{pr}K=U(n)\xrightarrow{\det}U(1)$.

Above: $\qquad St=$ standard repr. of $U(n)$ on $\mathbb{C}^n$

$\left\{\begin{matrix}\theta_{c,\text{scalar}}\\ \theta_{c,\text{vector}}\end{matrix}\right\}$ is a holomorphic Siegel modular form of weight $\left\{\begin{matrix}\text{"}\det^{1/2}\text{"}\\ \text{"}\det^{1/2}\text{"}\otimes St\end{matrix}\right.$

$$\boxed{\text{Automorphy} \quad \text{on } G/K \longleftrightarrow \text{on } \Gamma\backslash G}$$

$G = Sp(2n,\mathbb{R})$

$K \cong U(n)$

$G/K \cong \mathcal{H}_n$

$gK \longmapsto g(iI)$

$$GL_n(\mathbb{C}) \ni \boxed{J\left(\begin{pmatrix} A & B \\ C & D\end{pmatrix}, T\right) = CT + D}$$

Data:

- $\Gamma \subset G$ discrete, $\mathrm{vol}(\Gamma\backslash G/K) < \infty$
- representation $\rho : GL_n(\mathbb{C}) \longrightarrow GL(V)$

---

(1) From G/K to $\Gamma\backslash G$:

if $\boxed{f : G/K \longrightarrow V}$ satisfies

$$\forall \gamma \in \Gamma \quad \boxed{f(\gamma(T)) = \rho(J(\gamma, T))\, f(T)} \quad (T \in G/K = \mathcal{H}_n)$$

$$\Downarrow$$

$$\boxed{\tilde f : G \longrightarrow V, \quad \tilde f(g) = \rho(J(g, iI))^{-1} f(g(iI))}$$

satisfies $\forall \gamma \in \Gamma \;\; \tilde f(\gamma g) = \tilde f(g) \iff \boxed{\tilde f : \Gamma\backslash G \longrightarrow V}$ $(g \in G)$

and $\boxed{\forall k \in K \quad \tilde f(gk) = \rho(k)^{-1} \tilde f(g)}$

(2) From $\Gamma\backslash G$ to G/K: given $T = u + iv \in \mathcal{H}_n$

there is a positive definite $v^{1/2}$ s.t. $(v^{1/2})^2 = v$.

$$i\cdot I \xmapsto{\begin{pmatrix} v^{1/2} & 0 \\ 0 & (v^{1/2})^{-1}\end{pmatrix}} iv \xmapsto{\begin{pmatrix} I & u \\ 0 & I\end{pmatrix}} u + iv = T \qquad (i\cdot I \xmapsto{g_T} T)$$

$$\boxed{g_T := \begin{pmatrix} I & u \\ 0 & I\end{pmatrix}\begin{pmatrix} v^{1/2} & 0 \\ 0 & (v^{1/2})^{-1}\end{pmatrix} \in G}$$

$$\boxed{f(T) = f(g_T(iI)) = \underbrace{\rho(J(g_T, iI))}_{\rho(v^{1/2})^{-1}}\, \tilde f(g_T)}$$

What is an automorphic form?

Def: given • G (a covering of) a semisimple Lie group
• $K \subset G$ maximal compact subgroup
• $\Gamma \subset G$ discrete subgroup s.t. $\mathrm{vol}(\Gamma\backslash G) < \infty$
• V $\mathbb{C}$-v. space, $\dim(V) < \infty$.

A $\mathcal{C}^\infty$ function $\tilde{f} : G \longrightarrow V$ is an automorphic form if:

(1) $\tilde{f}(\gamma g) = \tilde{f}(g)$ $\forall \gamma \in \Gamma$ ($\Longleftrightarrow$ $\boxed{\tilde{f} : \Gamma\backslash G \longrightarrow V}$)

(2) $\boxed{\tilde{f}(gk) = \rho(k)^{-1}\tilde{f}(g)}$ $\forall k \in K$ for some repr. $\rho : K \longrightarrow GL(V)$
"$\tilde{f}$ has weight ρ"

(3) $\tilde{f}$ has "polynomial growth"

(4) $\tilde{f}$ satisfies a suitable system of linear differential equations: G acts on $\mathcal{C}^\infty(\Gamma\backslash G, V)$ by right translations $\Longrightarrow$ action of $U(\mathfrak{g})$.
We require $I \cdot \tilde{f} = 0$, $I \subset Z(\mathfrak{g})$ ideal of $\dim_{\mathbb{C}} Z(\mathfrak{g})/I < \infty$. "$\tilde{f}$ is $Z(\mathfrak{g})$-finite"
(ex: for a ring morphism $Z(\mathfrak{g}) \longrightarrow \mathbb{C}$,
$\mathrm{Ker}(\chi) \cdot \tilde{f} = 0 \Longleftrightarrow \forall z \in Z(\mathfrak{g})$ $z \cdot \tilde{f} = \chi(z)\tilde{f}$).

Variant: replace $\tilde{f}$ by $r \circ \tilde{f}$ for some linear form $V \xrightarrow{r} \mathbb{C}$. (2) is replaced by

(2') Under the action of K on $\tilde{f} \in \mathcal{C}^\infty(\Gamma\backslash G, V)$ (by right multiplication), $\tilde{f}$ generates a finite-dimensional subspace of $\mathcal{C}^\infty(\Gamma\backslash G, V)$
"f is K-finite"

Ex: Given $G \longrightarrow GL(E)$, $\lambda \in \mathrm{Hom}(E, \mathbb{C})^\Gamma$, $e \in E$ s.t. $\dim_{\mathbb{C}}(\mathrm{span}\ K \cdot e) < \infty$, the matrix element $\tilde{f}(g) := \lambda(g \cdot e)$ satisfies (1), (2').

<u>Variant</u>: $\lambda \in \mathrm{Hom}(E, \mathbb{C})^{\Gamma}$, $e \in (E \otimes V)^K$ (diagonal K-action) $\Longrightarrow$

$\tilde{f}(g) = \lambda(g \cdot e) \in V$, $\tilde{f} : \Gamma \backslash G \longrightarrow V$, $\tilde{f}(gk) = \rho_V(k)^{-1} \tilde{f}(g)$

Weil's θ - distributions

$\tilde{G} = Mp(2n, \mathbb{R}) \supset \tilde{K}$

$\tilde{G}$ acts on $\mathcal{H} = L^2(\mathbb{R}^n) = L^2(W/\ell)$

$$\mathcal{H} \supset \mathcal{H}_\infty = \{ f \in \mathcal{H} \mid \underbrace{U(\mathrm{heis}) f \in \mathcal{H}}_{\forall \alpha \ \forall \text{ polynomial } P \quad \underbrace{P(x) D^\alpha f(x)}_{\in L^2(\mathbb{R}^n)} \xrightarrow{|x| \to \infty} 0} \}$$

$$\| $$

$$\mathcal{S}(\mathbb{R}^n) = \{ f \in \mathcal{C}^\infty(\mathbb{R}^n) \mid \text{— " —} \}$$

<u>Schwartz space</u> <u>Ex</u>: $A = {}^tA \in M_n(\mathbb{R})$ positive definite, P polynomial $\Longrightarrow$ $P(x) e^{-\pi \, {}^t x A x} \in \mathcal{S}(\mathbb{R}^n)$.

<u>Thm</u>. There is a congruence subgroup $\Gamma \subset Sp(2n, \mathbb{Z})$ and $\beta : \tilde{\Gamma} \longrightarrow \mu_8$ such that the θ - distribution

$$\lambda = \sum_{u \in \mathbb{Z}^n} \delta_u : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}$$

satisfies $\lambda(\gamma \cdot \varphi) = \beta(\gamma) \lambda(\varphi)$ $\forall \gamma \in \tilde{\Gamma}$, $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$.

Idem for $\lambda_{u_0} = \sum_{u \in \mathbb{Z}^n} \delta_{u + u_0}$ for fixed $u_0 \in \mathbb{Q}^n$ (and smaller Γ).

<u>Rmk</u>. β disappears in the adelic version.

<u>Idea of proof</u>:

$$\begin{tikzcd}
\mathcal{H} = \mathrm{Ind}_{\pi^{-1}(\ell)}^{\mathrm{Heis}}(\chi) \supset \mathcal{H}_\infty = \mathcal{S}(\mathbb{R}^n) \arrow[d, "\Phi"'] \arrow[dd, bend left, "\lambda_{u_0}"] \arrow[loop above, "\tilde{G} \text{ acts}"] \\
\mathcal{H}_{\mathbb{Z}} = \mathrm{Ind}_{\pi^{-1}(\mathbb{Z}^{2n})}^{\mathrm{Heis}}(\chi) \supset (\mathcal{H}_{\mathbb{Z}})_\infty \arrow[d, "\text{value at } u_0"] \\
\mathbb{C}
\end{tikzcd}$$

natural intertwiner for Heis

Γ acts on $\mathbb{Z}^{2n}$

Stone - von Neumann $\Longrightarrow$ Φ intertwines the two $\tilde{\Gamma}$ - actions up to a scalar $\in U(1)$.

$$(\Phi f)(g) = \sum_{\pi^{-1}(L \cap \ell) \backslash \pi^{-1}(L)} \chi(h)^{-1} f(hg) \quad \Big| \quad L = \mathbb{Z}^{2n}, \ L \cap \ell \subset \ell \text{ lattice}$$

Construction of automorphic forms

Data: $\rho : \widetilde{K}$ (= 2-fold cover of $U(n)$) $\longrightarrow GL(V)$

$\varphi \in (\mathcal{S}(\mathbb{R}^n) \otimes V)^{\widetilde{K}}$, $\quad a \in \mathbb{Q}^n$

Define: for $g \in \widetilde{G} = Mp(2n, \mathbb{R})$

$$\tilde{f}(g) = \lambda_a(g \cdot \varphi) \in V, \qquad \tilde{f} : G \longrightarrow V$$

for $T = u + iv \in \mathcal{H}_n$, $\quad g_T = \begin{pmatrix} I & u \\ 0 & I \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & (v^{1/2})^{-1} \end{pmatrix}$

$$f(T) = \underbrace{\rho(J(g_T, iI))}_{\rho(v^{1/2})^{-1}} \tilde{f}(g_T) = \rho(v^{1/2})^{-1} \lambda_a(\underbrace{g_T \cdot \varphi}_{\text{action in the Weil representation}})$$

Then: $\exists$ congruence subgroup $\Gamma_a \subset Sp(2n, \mathbb{Z})$ s.t. $\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_a$ $f((AT+B)(CT+D)^{-1}) =$

$$= \zeta \rho(CT+D) f(T), \qquad \zeta^8 = 1.$$

Ex: (1) $a = 0$, $\varphi = \varphi_0 = e^{-\pi {}^t x x}$ $\xrightarrow{\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}}$ $(\det v)^{1/4} e^{-\pi {}^t x v x}$

$V = \rho = \det^{1/2}$

$$(\det v)^{1/4} e^{-\pi {}^t x v x} \xrightarrow{\begin{pmatrix} I & u \\ 0 & I \end{pmatrix}} \underbrace{(\det v)^{1/4} e^{\pi i {}^t x T x}}_{g_T \cdot \varphi_0}$$

$$\boxed{f(T) = \lambda(e^{\pi i {}^t x T x}) = \sum_{u \in \mathbb{Z}^n} e^{\pi i {}^t u T u}}$$

(2) $\boxed{n=1}$, $\varphi = (\text{const.}) \underbrace{A_+^2 \varphi_0}_{\varphi_2} = \left(x^2 - \frac{1}{4\pi}\right) \varphi_0$, $\quad V = \rho = \det^{5/2}$

$\tau = u + iv \in \mathcal{H}$

$$g_\tau \cdot \varphi_2 = v^{1/4} \left(v x^2 - \frac{1}{4\pi}\right) e^{\pi i \tau x^2}$$

$$\boxed{f(\tau) = \sum_{n \in a + \mathbb{Z}} \left(n^2 - \frac{1}{4\pi v}\right) e^{\pi i n^2 \tau}}$$

NOT HOLOMORPHIC

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_a \quad \underline{f\left(\frac{a\tau + b}{c\tau + d}\right) = \zeta (c\tau + d)^{5/2} f(\tau)}$

$\tilde{f}$ is Eigenvector for the Casimir $\Omega \in Z(\mathfrak{sl}_2)$: $\Omega \tilde{f} = \lambda \tilde{f}$

For which φ is f holomorphic?

$$h : U(1) \longrightarrow K \supset Z(K), \quad h(e^{i\alpha}) = \begin{pmatrix} I\cdot\cos\alpha & -I\cdot\sin\alpha \\ I\cdot\sin\alpha & I\cdot\cos\alpha \end{pmatrix}$$

complex structure J on $\mathfrak{g}/\mathfrak{k}$ is $\mathrm{Ad}\, h(e^{-2\pi i/8})$.

Formulas: $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ + \mathfrak{p}_-$ $\quad (J = i, -i)$

$$\mathfrak{p}_\pm = \left\{ \begin{pmatrix} A & \pm iA \\ \pm iA & -A \end{pmatrix} \middle| A = {}^tA \in M_n(\mathbb{C}) \right\}$$

Properties: (1) $f : G \to \mathbb{C}$ is of the form

$$G \longrightarrow G/K \xrightarrow{\text{holomorphic}} \mathbb{C} \iff (\mathfrak{k} + \mathfrak{p}_-) f = 0.$$

(2) $\mathfrak{p}_-\, J(g, iI) = 0$.

Corollary: $\mathfrak{p}_- \varphi = 0 \implies f$ is holomorphic.

True for $\varphi = \varphi(x) = (\text{pol. of deg} \le 1)\, e^{-\pi x^2} \implies$

$f = \theta_{c,\text{scalar}}, \ \theta_{c,\text{vector}}$.

θ - functions of quadratic forms

Ex: $\theta(\tau)^d = \sum_{u \in \mathbb{Z}^d} e^{\pi i \overbrace{(u_1^2 + \cdots + u_d^2)}_{S(u)} \tau}$, $\quad \theta\left(\frac{a'\tau + b'}{c'\tau + d'}\right)^d = \zeta (c'\tau + d')^{d/2} \theta(\tau)^d$

General case: (V, S) quadratic form (non-deg.), $d = \dim_{\mathbb{R}} V$

(W, B) as before $\implies (V \otimes W, S \otimes B)$ symplectic

Dual pair: $O(V) \times Sp(W) \longrightarrow Sp(V \otimes W)$

$\qquad\qquad h, g \longmapsto h \otimes g$

Weil representation of $Mp(V \otimes W)$: $\dim_{\mathbb{R}} W = 2n$

Fix: $\ell \in \mathrm{Lagr}(W) \implies V \otimes \ell \in \mathrm{Lagr}(V \otimes W)$ ← $O(V)$-stable

$\tilde{R}_{V\otimes\ell}$ acts on $L^2(V \otimes \underbrace{(W/\ell)}_{\ell^*}) \simeq L^2(V^n) \supset \mathcal{S}(V^n)$

$h \in O(V)$ acts by

$$\boxed{(\tilde{R}(h) f)(v) = \alpha(h)\, f(h^{-1} v)} \qquad v \in V^n, \quad \alpha : O(V) \to \{\pm 1\}$$

R. Howe's idea: restriction of $\tilde{R}_{V\otimes \ell}$ to $O(V)\times Mp(W)$ should give a correspondence between some representations of $O(V)$, $Mp(W)$

Case of S positive definite: fix a basis of ℓ and of $V \Rightarrow$ coordinates x_{jp} on $V^n = V\otimes \ell^*$ $(1\leq j\leq n,\ 1\leq p\leq d)$

Def: a polynomial $P : (V^n)_{\mathbb{C}} \longrightarrow \mathbb{C}$ is pluriharmonic (w.r.t. S) if

$$\forall p,q \quad \sum_{j,k=1}^{n} \left(\frac{\partial}{\partial x_{jp}}\frac{\partial}{\partial x_{kq}}\right)(S^{-1})_{pq}\, P = 0$$

$[\ S = {}^tS \in M_d(\mathbb{R})$ pos. def. $]$

Consider: $\varphi = P\cdot\varphi_0 \in \mathcal{S}(V^n)$, $\varphi_0 = e^{-\pi \sum_{j,p,q} x_{pj} S_{pq} x_{qj}}$

Proposition: P pluriharmonic $\Longleftrightarrow p_- \varphi = 0$.

Fix: $O(V)$-stable subspace of pluriharmonic polynomials and its basis $P_1, \dots, P_M$.

Thm: For each $a\in M_{d\times n}(\mathbb{Q})$, $T\in \mathcal{H}_n$

$$\theta_a(T) = \sum_{X\in M_{d\times n}(\mathbb{Z})+a} \begin{pmatrix} P_1\\ \vdots \\ P_M\end{pmatrix}(X)\, e^{\pi i \operatorname{Tr}(T\, {}^tXSX)}$$

is a vector-valued holomorphic automorphic form on $Mp(2n,\mathbb{R})$ (in fact, on $Sp(2n,\mathbb{R})$ if $2 \mid d = \dim(V)$)

Ex: $\boxed{n=1}$ $P : V_{\mathbb{C}} \to \mathbb{C}$ is a <u>harmonic</u> polynomial if $\underbrace{\sum_{p,q=1}^{d} (S^{-1})_{pq} \frac{\partial}{\partial x_p} \frac{\partial}{\partial x_q}}_{\text{(const.) action of } Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathrm{Lie}(\overbrace{SL_2(\mathbb{R})}^{Sp(W)})} P = 0.$

---

(1) <u>$d=1$</u>: $S(x) = x^2$, $P(x) = \boxed{1, x}$ are harmonic

$\Rightarrow$ get $\theta_{1/2}(\tau)$, $\theta_{3/2}(\tau)$.

(2) <u>$d=2$</u>: $S(x) = x_1^2 + x_2^2$, $\boxed{(x_1 \pm i x_2)^M \quad (M \geq 0)}$ are harmonic

$\Rightarrow$ get $f(\tau) = \sum_{n_1, n_2 \in \mathbb{Z} + a} (n_1 + i n_2)^M e^{\pi i (n_1^2 + n_2^2)\tau}$ $\quad (a \in \mathbb{Q})$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (\text{const.})\,(c\tau+d)^{\overbrace{d/2 + \deg(P)}^{M+1}} f(\tau)$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in$ some congruence subgroup of $SL_2(\mathbb{Z})$

$\underline{2^{nd}\text{ approach (Siegel)}}$: use majorants $\tilde{S} > 0$

$\underline{\text{Ex}}$: $S = \underbrace{x_1^2 + \cdots + x_p^2}_{S_+} + \underbrace{(-y_1^2 - \cdots - y_q^2)}_{S_-}$ on $V = \mathbb{R}^{p+q}$

majorant of S

$$\boxed{\tilde{S} = S_+ - S_- = x_1^2 + \cdots + x_p^2 + y_1^2 + \cdots + y_q^2}$$

Action of $\mathrm{Lie}(Mp(W)) = \mathfrak{sl}(2) \subset \mathrm{Lie}(Mp(V \otimes W))$ on $\mathcal{S}(V)$: $(x \in V)$ $(\dim W = 2)$

$$X \longmapsto \pi i \underbrace{({}^t x S x)}_{S}, \quad Y \longmapsto \frac{i}{4\pi}({}^t\partial S^{-1} \partial), \quad H \longmapsto \frac{1}{2}({}^t x \partial + {}^t \partial x)$$

(true for $\underline{\text{any}}$ S)

$\underline{\text{Gaussian of } \tilde{S}}$: $\varphi = e^{-\pi \tilde{S}} \in \mathcal{S}(V)$ $\quad \tilde{S} = (\,,\,)$ on V

$$X\varphi = (\pi i S)\varphi, \quad Y\varphi = \left((\pi i S) + \frac{i}{2}(q-p)\right)\varphi, \quad H\varphi = \left(-2\pi \tilde{S} + \frac{p+q}{2}\right)\varphi$$

$$\Rightarrow \boxed{i(X - Y)\varphi = \frac{-p+q}{2}} \qquad \begin{matrix} K = SO(2) \subset SL_2(\mathbb{R}) = Sp(W) \\ \cong \quad \uparrow \qquad\qquad\qquad \uparrow \\ U(1) \quad \tilde{K} \qquad \subset \qquad Mp(W) \end{matrix}$$

$\Downarrow$

φ is $\tilde{K}$-finite, with $\tilde{K}$ acting by $\det^{-\frac{p-q}{2}}$.

$\underline{\text{General } \theta\text{-machinery}}$: $\boxed{\tau = u + iv} \in \mathcal{H}$, $a_0 \in \mathbb{Q}^{p+q}$.

$$\varphi \xrightarrow{\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix}} v^{\frac{p+q}{4}} e^{-\pi v \tilde{S}} \xrightarrow{\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}} v^{\frac{p+q}{4}} e^{\overbrace{\pi i u S - \pi v \tilde{S}}^{\pi i(\tau S_+ + \bar{\tau} S_-)}}$$

$$g_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} : i \longmapsto \tau$$

$$\underbrace{(v^{-1/2})^{\frac{p-q}{2}} \lambda_{a_0}(g_\tau \cdot \varphi)}_{f(\tau)} = v^{q/2} \sum_{\binom{x}{y} \in \mathbb{Z}^{p+q} + a_0} e^{\pi i(\tau(x_1^2 + \cdots + x_p^2) - \bar{\tau}(y_1^2 + \cdots + y_q^2))}$$

$$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{a_0} \subset SL_2(\mathbb{Z}) \qquad f\left(\frac{a\tau + b}{c\tau + d}\right) = (\text{const.})(c\tau + d)^{\frac{p-q}{2}} f(\tau)$$

Is f an automorphic form?

$$\underline{F(\tau) = v^{-q/2} f(\tau)} \qquad \underline{F\left(\frac{a\tau+b}{c\tau+d}\right) = (\cdot\cdot)(c\tau+d)^{p/2}(c\bar{\tau}+d)^{q/2} F(\tau)}$$

f is NOT $\mathcal{Z}(\mathfrak{sl}_2)$ – finite $\Longrightarrow$ NOT an automorphic form!

$\Omega = XY + YX + \frac{H^2}{2}$ (if $\underline{p, q \geq 1}$)

$\forall n \geq 1 \quad \Omega^n \varphi = \left((-8\pi^2 S_+ S_-)^n + \text{lower terms} \right) \varphi$

<u>Explanation</u>: $\tilde{S}$ is NOT canonical.

$\{\text{majorants of } S\} \longleftrightarrow \underbrace{\{ V = V_+ \perp V_-, \ S|_{V_+} > 0, \ S|_{V_-} < 0 \}}$

symmetric space of $O(V)$ $\simeq O(p,q)/O(p) \times O(q)$

$\operatorname{sgn}(S) = (p,q)$

$Gr_q^-(V) = \left\{ Z \subset V \,\middle|\, \begin{array}{l} \dim(Z) = q \\ S|_Z < 0 \end{array} \right\}$

$\underset{Z}{\Psi}$

majorant $\boxed{\tilde{S}_Z(x) = S(pr_{Z^\perp}(x)) - S(pr_Z(x))}$

<u>Siegel</u>: consider for each $Z \in Gr_q^-(V)$

$$f(\tau, Z) = f_Z(\tau) = (\operatorname{Im} \tau)^{q/2} \sum_{\binom{x}{y} \in \mathbb{Z}^{p+q} + a_0} e^{\pi i (\tau S(pr_{Z^\perp}(x)) + \bar{\tau} S(pr_Z(x)))}$$

and integrate over $Gr_q^-(V)$:

$I(f)(\tau) = \int_{Gr_q^-(V)} f(\tau, Z) \, dZ$ <u>is an automorphic form</u>.

<u>Siegel's Formula</u>: for $p+q > 4$

$I(f)$ is an <u>Eisenstein series</u>: linear combination of functions such as

$$\sum_{\binom{m}{n} \in \mathbb{Z}^2 + b_0}{}' \frac{\operatorname{Im}(\tau)^{q/2}}{(m\tau + n)^{p/2} (m\bar{\tau} + n)^{q/2}}.$$

<u>Theta correspondence</u>: consider

$\tau \longmapsto \int_{Gr_q^-(V)} f(\tau, Z) \, (\text{automorphic form on } O(p,q)) \, dZ$

The work of Kudla – Millson (et al.)

Positive semi-definite <u>symmetric matrices</u> parameterise <u>two</u> kinds of objects:

(1) Fourier coefficients of Siegel modular forms:

$$f(T) = \sum_{B \in \mathrm{Sym}_n(\frac{1}{N}\mathbb{Z})_{\geq 0}} e^{2\pi i \,\mathrm{Tr}(Bu)} \times \underbrace{(\text{function of } B, v)}_{a(B)\, e^{2\pi i\,\mathrm{Tr}(Biv)} \text{ if } f \text{ holomorphic}} \qquad T = u + iv \in \mathcal{H}_n$$

(2) special cycles in $D = \mathrm{Gr}_q^-(V) \simeq O(p,q)/O(p) \times O(q)$.

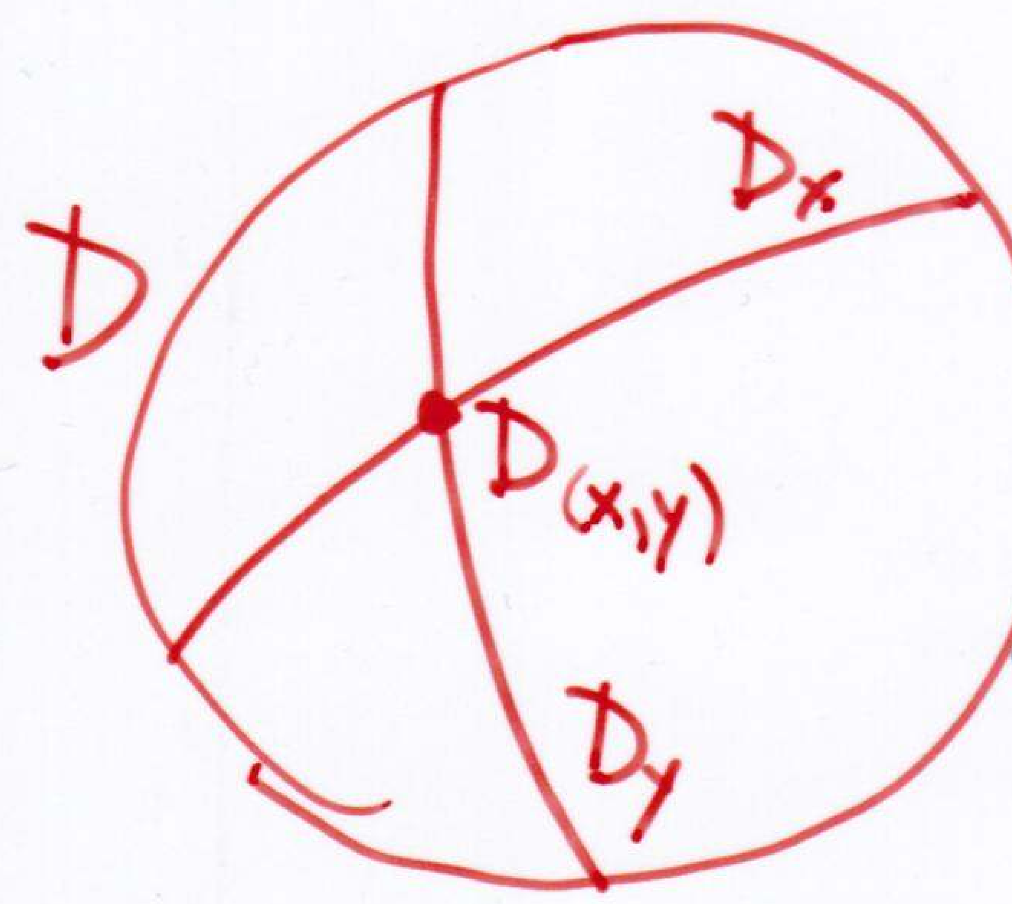
Special cycles in D

<u>Recall</u>: V, $(\,,\,)$ quadratic space over $\mathbb{R}$, $\mathrm{sgn}(V) = (p,q)$, $\boxed{p, q \geq 1}$, $D = \mathrm{Gr}_q^-(V) = \{ Z \subset V \mid \dim(Z) = q,\ (\,,\,)|_Z < 0 \} \ni \sigma$

$G = O(V)$ $(\simeq O(p,q))$, $K = G_\sigma$ $(\simeq O(p) \times O(q))$, $G/K \xrightarrow{\simeq} D$

$\boxed{\dim_{\mathbb{R}}(D) = pq}$

<u>Def</u>: $\boxed{x \in V \Rightarrow D_x := \{ Z \in D \mid Z \perp x \}} \subset D$



(0) $(x,x) < 0 \Rightarrow D_x = \emptyset$

(1) $\underline{(x,x) > 0} \Rightarrow D_x = \mathrm{Gr}_q^-(\underbrace{x^\perp}_{\mathrm{sgn}(p-1,q)}) \Rightarrow \mathrm{codim}_{\mathbb{R}}(D_x) = q$

<u>Def</u>: $X = (x_1, \dots, x_n) \in V^n \Rightarrow \boxed{D_X := D_{x_1} \cap \dots \cap D_{x_n} \subset D}$

$\underline{\underline{X}} := \mathrm{span}_{\mathbb{R}}(x_1, \dots, x_n) \subset V$, $(X, X) = ((x_j, x_k)) \in \mathrm{Sym}_n(\mathbb{R})$

(2) If $\underline{(X,X) > 0}$ (pos. def.) $\Rightarrow D_X = \mathrm{Gr}_q^-(\underbrace{\underline{\underline{X}}^\perp}_{\mathrm{sgn}(p-n,q)})$ $\quad \mathrm{codim}_{\mathbb{R}}(D_X) = nq$

(3) If $\left.\begin{array}{l} \dim_{\mathbb{R}}(\underline{\underline{X}}) = t \leq n \\ (\,,\,)|_{\underline{\underline{X}}} > 0 \end{array}\right\} \Rightarrow D_X = \mathrm{Gr}_q^-(\underbrace{\underline{\underline{X}}^\perp}_{\mathrm{sgn}(p-t,q)}) \Rightarrow \mathrm{codim}_{\mathbb{R}}(D_X) = tq$

Special cycles in $\Gamma\backslash D$

Arithmetic subgroups $\Gamma \subset G = O(V)$:

(1) if $\exists$ lattice $L \subset V$ s.t. $(L, L) \subseteq \mathbb{Z} \Rightarrow$

$\Gamma(L) = \{g \in G \mid gL \subseteq L\} \subset G$ discrete, $vol(\Gamma\backslash D) < \infty$

$\Gamma(L)\backslash D$ not compact $\Leftrightarrow \exists\, 0 \neq x \in L\ (x, x) = 0 \Leftarrow p+q > 4$.

(2) Can get $\Gamma \subset G$ with $\Gamma\backslash D$ compact from $(\,,\,)$ over a totally real number field $F \neq \mathbb{Q}$ when $sgn((\,,\,)) = (p, q) \times (p+q, 0)^{[F:\mathbb{Q}]-1}$.

Assume: $\exists L$; fix $\Gamma \subset \Gamma(L)$ congruence subgroup.

Consider: $X \in L^n$ as in (3) above.

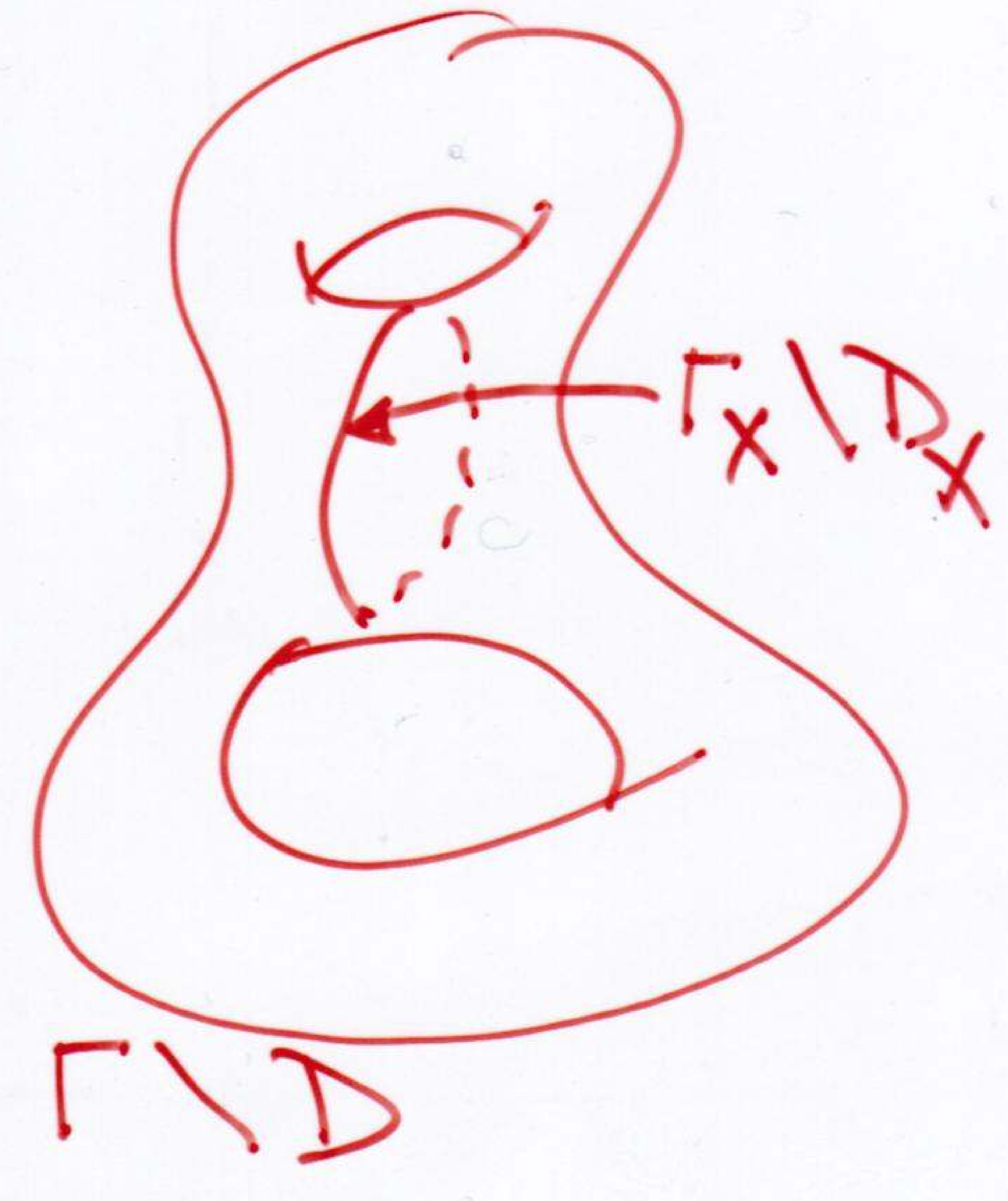
Def: $G_X = \{g \in G \mid g(\underline{\underline{X}}) = \underline{\underline{X}}\}$ (depends only on $\underline{\underline{X}} = \mathrm{span}_{\mathbb{R}}(\{x_i\})$)

$\Gamma_X = \Gamma \cap G_X$. Then $G_X \simeq O(p-t, q)$ acts on $D_X \simeq \dfrac{O(p-t, q)}{O(p-t) \times O(q)}$.

Basic fact: if $\Gamma \subset$ suitable Γ', then:

(a) $\Gamma_X \backslash D_X$ is orientable

(b) $i_X : \Gamma_X \backslash D_X \longrightarrow \Gamma\backslash D$ is injective. (codim$_{\mathbb{R}} = tq$)



Fundamental class:

$$\boxed{[\Gamma_X \backslash D_X] \in H^{tq}(\Gamma\backslash D, \mathbb{R})}$$

(depends only on $\Gamma\cdot X$)

Euler class: tautological $rk = q$ bundle (oriented, G-equivariant) on $D \Rightarrow$ Euler class $\boxed{e_q \in H^q(\Gamma\backslash D, \mathbb{R})}$

$2 \nmid q \Rightarrow e_q = 0$

Main Theorem of Kudla – Millson (Publ. IHES, 1990)

Fix $N \geq 1$, $v \in L$; set $\mathcal{L} = N\cdot L + v$. Then:

$$f(T) = \sum_{B \in Sym_n(\mathbb{Z})_{\geq 0}} \left(\sum_{\substack{X \in \Gamma \backslash \mathcal{L}^n \\ \frac{(X,X)}{2} = B}} [\Gamma_X \backslash D_X] \cup e_q^{\,n-rk(B)} \right) e^{2\pi i \, Tr(BT)} \qquad (T \in \mathcal{H}_n)$$

finite set

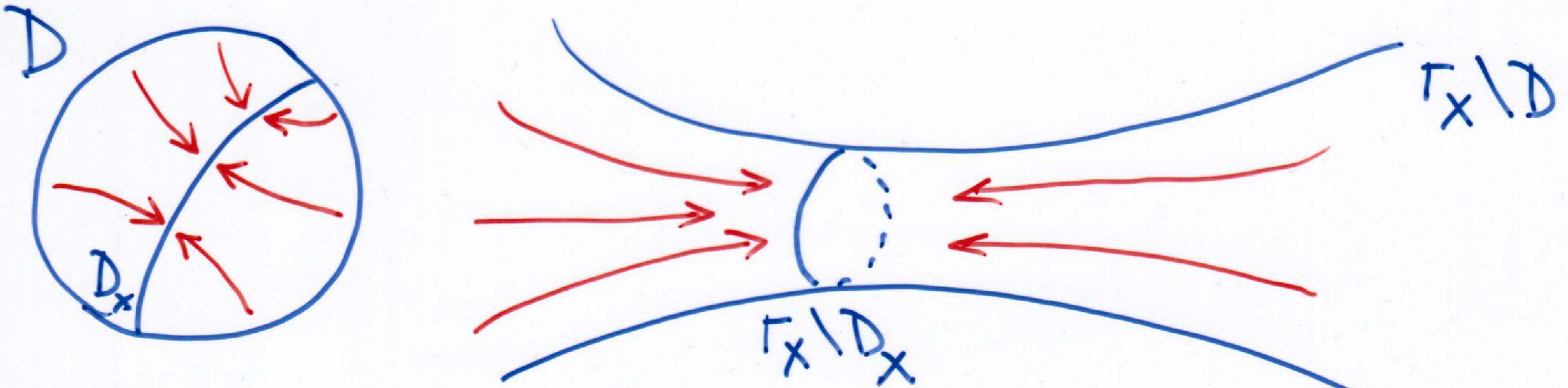
is a **holomorphic Siegel modular form** of weight $\frac{p+q}{2}$ with coefficients in $H^{nq}(\Gamma\backslash D, \mathbb{R})$

$\Rightarrow \exists\ \Gamma \subset Sp(2n, \mathbb{Z})$ congruence subgroup s.t.

$$\forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma \quad f\left((AT+B)(CT+D)^{-1}\right) = (\text{const.}) \det(CT+D)^{\frac{p+q}{2}} f(T) \qquad (T \in \mathcal{H}_n)$$

Main ideas of proof:

Geometry: "geodesic flow"



get $\Gamma_X \backslash D \simeq$ the normal bundle of $\Gamma_X \backslash D_X$

oriented real vector bundle of $rk = tq$

$$\begin{tikzcd} \Gamma_X\backslash D \arrow[d, "pr_X"', shift right] \arrow[dr, "p"] & \\ \Gamma_X\backslash D_X \arrow[u, "i"', shift right, hook] \arrow[r, hook, "i_X"'] & \Gamma\backslash D \end{tikzcd}$$

Fundamental class of the 0-section:

$$\begin{array}{c} E \\ \pi \big\downarrow \big\uparrow i \\ B \end{array}$$ oriented $rk_{\mathbb{R}} = n$ bundle ($\mathcal{C}^\infty$)

<u>Thom form</u>: $\omega \in A^n(E)$, $d\omega = 0$, <u>cpt supp along the fibres</u>, $\underbrace{\pi_*}_{\text{integral along fibres.}} \omega = 1$

(so $\forall b \in B$ $\quad H^n_c(\pi^{-1}(b), \mathbb{R}) \xrightarrow[\sim]{\int} \mathbb{R}$, $\quad \omega|_{\pi^{-1}(b)} \longmapsto 1$)

<u>Then</u>: $\forall \eta \in A_c^{\dim(B)}(E)$, $d\eta = 0$

$$\int_E \omega \wedge \eta = \int_B (\pi_* \omega) \wedge i^*(\eta) = \int_{i(B)} \eta \implies [\omega] = [i(B)] \in H^n(E, \mathbb{R})$$

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(1) One needs a more general version of this for

$$\begin{array}{c} \Gamma_X \backslash D \\ \big\downarrow pr_X \\ \Gamma_X \backslash D_X \end{array}$$

with ω only "rapidly decreasing".

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(2) Construction of $\varphi \in \underbrace{(\mathcal{S}(V^n) \otimes A^{nq}(G/K))^G}_{(\mathcal{S}(V^n) \otimes \Lambda^{nq} \mathfrak{p}^*)^K}$, $d\varphi = 0$

$\mathfrak{g} = Lie(G) = \mathfrak{k} \oplus \mathfrak{p}$

$\mathfrak{p} = T_o D \simeq V_+ \otimes V_-^*$, $\dim_{\mathbb{R}} \mathfrak{p} = pq$

at o $\quad V = V_+ \oplus V_-$ $\qquad \varphi = \varphi_{[nq]} = \overbrace{\varphi_{[q]} \wedge \cdots \wedge \varphi_{[q]}}^{n \text{ times}}$

<u>$n=1$</u>: $\varphi_{[q]} \in (\mathcal{S}(V) \otimes \Lambda^q \mathfrak{p}^*)^K$

At o: majorant of $S = (x,x) = \sum_1^p x_j^2 - \sum_{p+1}^{p+q} x_k^2$ is $\tilde{S} = \sum_1^{p+q} x_j^2$

$\varphi^+ = e^{-\pi \tilde{S}}$

<u>Ex</u>: (1) $sgn(V) = (1,q)$: $\varphi_{[q]} = \underbrace{\left(x_1 - \frac{1}{2\pi} \frac{\partial}{\partial x_1}\right)^q \varphi^+}_{\text{Hermite pol.} \times \varphi^+} \otimes (\text{basis of } \Lambda^{max} \mathfrak{p}^*)$

(up to a constant)

(2) $sgn(V) = (p,1)$: $\varphi_{[1]} = \sum_{\alpha=1}^p \underbrace{\left(x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha}\right) \varphi^+}_{2x_\alpha \varphi^+} \otimes \omega_\alpha$ } basis of $\mathfrak{p}^*$ dual to x_α

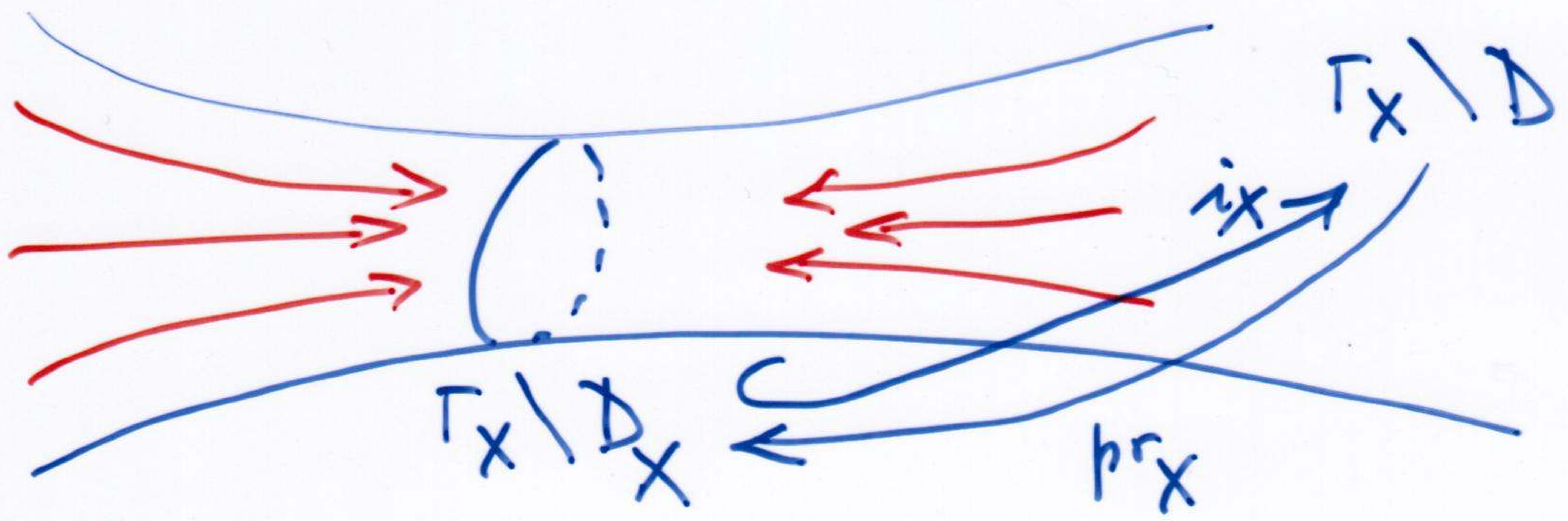
Properties of $\varphi \in (\mathcal{S}(V^n) \otimes A^{nq}(G/K))^G$:

value at $X \in V^n$: $\varphi_X \in A^{nq}(G/K)^{G_X}$, $d\varphi_X = 0$

$\forall g \in G \quad g^* \varphi_X = \varphi_{g^{-1}X}$

$\boxed{\text{From now on: } \begin{matrix} X \in L^n \\ (X,X) > 0 \end{matrix}}$

(3) Integrals along fibres:



$\boxed{(pr_X)_* \varphi_X = 1}$

Cor: $\varphi_{\Gamma\cdot X} := \sum_{Y \in \Gamma\cdot X} \varphi_Y = \sum_{\gamma \in \Gamma/\Gamma_X} \gamma^*(\varphi_X)$ satisfies

$[\varphi_{\Gamma\cdot X}] = [\Gamma_X \backslash D_X] \in H^{nq}(\Gamma\backslash D)$

Pf: $\forall\, \eta \in A^{(p-n)q}(\Gamma\backslash D)$, $d\eta = 0$, η bounded

$$\int_{\Gamma\backslash D} \varphi_{\Gamma\cdot X} \wedge \eta = \int_{\Gamma_X\backslash D} \varphi_X \wedge \eta \overset{(1)}{=} \int_{\Gamma_X\backslash D_X} (pr_X)_* \varphi_X \wedge i_X^*(\eta) \overset{(3)}{=} \int_{\Gamma_X\backslash D_X} i_X^*(\eta)$$

(4) θ – machinery: $\widetilde{G'} = Mp(2n, \mathbb{R}) \supset \widetilde{K'}$

$G \times \widetilde{G'} \subset Mp(V \otimes \underset{\mathbb{R}^{2n}}{\underline{W}})$ acts on $L^2(V^n) \supset \mathcal{S}(V^n)$

by the Weil representation.

Facts: (a) $\widetilde{K'}$ acts on φ by $\det^{-(p+2)/2}$.

(b) $\lambda_{\mathcal{L}^n} = \sum_{X \in \mathcal{L}^n} \delta_X : \mathcal{S}(V^n) \longrightarrow \mathbb{C}$ is $\widetilde{\Gamma'}$-invariant

(for suitable $\Gamma' \subset Sp(2n, \mathbb{Z})$)

Cor: the function $\underset{\substack{\uparrow \\ \widetilde{G}'}}{g'} \mapsto \lambda_{\mathcal{L}^n}(\widetilde{R}(g')\cdot\varphi) = \sum_{X\in\mathcal{L}^n} (\widetilde{R}(g')\varphi)_X$

non-holomorphic

gives rise to a function $F_\varphi(T)$ $(T\in\mathcal{H}_n)$ with values in $A^{nq}_{cl}(G/K)$ s.t.

$\forall \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \Gamma'$ $F_\varphi((AT+B)(CT+D)^{-1}) = (\text{const.})\det(CT+D)^{\frac{p+q}{2}} F_\varphi(T)$

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(5) As $(\widetilde{R}(\overbrace{\begin{pmatrix} I & u \\ 0 & I\end{pmatrix}}^{g'})\varphi)_X = e^{\pi i \,\mathrm{Tr}(u(X,X))}\varphi_X$, $\quad u\in \mathrm{Sym}_n(\mathbb{R})$

we get that the cohomology class

$$[F_\varphi(u+i\cdot I)] = \sum_{X\in\mathcal{L}^n} e^{\pi i\,\mathrm{Tr}(u(X,X))}[\varphi_X] = \sum_{\substack{X\in\mathcal{L}^n \\ (X,X)>0}} + \sum_{\text{rest}} =$$

$$= \sum_{\substack{X\in\Gamma\backslash\mathcal{L}^n \\ \text{finite}}} \underbrace{[\varphi_{\Gamma\cdot X}]}_{[\Gamma_X\backslash D_X]} e^{2\pi i\,\mathrm{Tr}(uB)} + \text{rest}$$

$B = \frac{(X,X)}{2} \in \mathrm{Sym}_n(\mathbb{Z})_{>0}$

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(6) $\underbrace{\mathrm{Lie}(\widetilde{G}')}_{sp(2n)} = k'\oplus p'$, $p'_{\mathbb{C}} = p'_+ \oplus p'_-$.

Fact: $p'_-\varphi \in \mathcal{S}(V^n)\otimes dA^{nq-1}(G/K)$ is exact.

Cor: $[F_\varphi(T)]$ is a holomorphic Siegel modular form of weight $\frac{p+q}{2}$ with values in $H^{nq}(G/K)$.

$\Downarrow$ (5)

$\forall B\in\mathrm{Sym}_n(\mathbb{Z})_{>0}$ the B-th Fourier coefficient of $[F_\varphi(T)]$ is equal to $\sum_{\substack{X\in\Gamma\backslash\mathcal{L}^n \\ (X,X)=B}} [\Gamma_X\backslash D_X]$.

(7) More work is needed to treat $X \in \mathcal{L}^n$ with $(X, X) \geq 0$ but not $(X, X) > 0$ (those with (X, X) not ≥ 0 do not contribute to $[F_\varphi(T)]$: for $n > 1$ by the Koecher principle, for $n = 1$ since $[F_\varphi(T)]$ has polynomial growth).

"QED".

Refinements

(a) similar result for $G \times G' = U(p, q) \times U(n, n)$ (K–M, Publ. IHES 1990)

(b) cohomology with coefficients (Funke – Millson)

(c) Bergeron – Millson – Moeglin:

$\forall\ n < \frac{1}{2}\left(m - \left[\frac{m}{2}\right] - 1\right) \qquad (m = p+q)$

$H^{nq}(\Gamma \backslash D, \mathbb{C})$ is generated by $[\Gamma_X \backslash D_X]$ if compact (even with coefficients)

(d) $\mathrm{sgn}(V) = (p, 2)$: D is hermitian,

$\Gamma \backslash D$ is a Shimura variety

(quasi-projective algebraic variety, smooth if Γ is small, defined over a small number field).

Thm (Bergeron – Millson – Moeglin)
If $\mathrm{sgn}(V) = (p, 2)$ and $\Gamma \backslash D$ is compact, then the Hodge conjecture holds for $\underbrace{H^{n,n}(\Gamma \backslash D) \cap H^{2n}(\Gamma \backslash D, \mathbb{Q})}$ if $n < \frac{1}{2}\left[\frac{p+1}{2}\right]$

any element is a $\mathbb{Q}$-linear combination of classes of algebraic subvarieties of the Shimura variety $\Gamma \backslash D$ (of $\mathrm{codim}_{\mathbb{C}} = n$).

Kudla's programme (arithmetic case)

If $\Gamma \backslash D$ is a Shimura variety ($G = O(p, 2)$ or $U(p, q)$) there should be an arithmetic refinement of the main result of Kudla-Millson, with $H^{\bullet}(\underbrace{\Gamma \backslash D}_{Y})$ replaced by

- Chow groups $CH^r(Y)$
  ($CH^1(Y)$ = divisor class group)
- Arakelov Chow groups $\widehat{CH}^r(Y_{\mathbb{Z}})$

of a model $Y_{\mathbb{Z}}$ of Y over $\mathbb{Z}$.

Green's currents appear!
Solutions of $dd^c G - \delta_{\Gamma_X \backslash D_X} = \varphi_{\text{Kudla-Millson}}$
Related to "Mock θ-functions".

References:

- Mumford – Tata lectures on Theta, III
- Lion, Vergne – The Weil representation, Maslov index and theta series
- Kudla, Millson – Publ. IHES (1990)
- Kudla – Séminaire Bourbaki No 876, 1999–2000
  – in MSRI, vol. 49