Towards New Formulation of Quantum field theory: Geometric Picture for Scattering Amplitudes

Part 1

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Work with Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, arxiv: 1212.5605

Work with Nima Arkani-Hamed, arxiv: 1312.2007
Motivation

- One of the most important challenges of theoretical physics: Quantum gravity.
- Method 1: Solve the problem. Most promising candidate: String theory.
- Method 2: Detour - take the inspiration from history of physics. Reformulate Quantum field theory.
- Standard formulation of Quantum field theory: space-time, path integral, Lagrangian, locality, unitarity.
- Perturbative expansion using Feynman diagrams.
- Ultimate goal: Find the reformulation of Quantum field theory where these words emerge as derived concepts from other principle.
Motivation

- This is an extremely hard problem with no guarantee of success. To have any chance we should be able to do it in the simplest set-up.
- We consider the simplest Quantum field theory: $\mathcal{N} = 4$ Super-Yang Mills theory in planar limit.
- We choose one set of objects: on-shell scattering amplitudes.
- In the process of reformulation we make a connection with active area of research in combinatorics and algebraic geometry: Positive Grassmannian $G_+(k,n)$.
- The final result is formulated using a new mathematical object – Amplituhedron which is a significant generalization of the Positive Grassmannian.
Plan of lectures

Lecture 1: Introduction to scattering amplitudes

Lecture 2: Positive Grassmannian

Lecture 3: The Amplituhedron
Very brief introduction to Scattering Amplitudes
On-shell scattering amplitudes

- Fundamental objects in any quantum field theory that describe interactions of particles.

\[ M \sim \langle \text{in} | \text{out} \rangle \]

- Each particle is characterized by the four-momentum \( p_\mu \) and also by spin information.

- The relevant fields have spin \( \leq 2 \), non-gravitational theories have spin \( 0, \frac{1}{2}, 1 \). The information is captured for spin \( \frac{1}{2} \) by spinor while for spin 1 by a vector. Quantum numbers: \( s, m = (-s, \ldots, s) \).

- On-shell: \( p_i^2 = m_i^2 \), in many cases we consider \( m_i = 0 \).

- For massless amplitudes \( p_\mu \) has three degrees of freedom and \( m \) is replaced by helicity \( h = (-s, +s) \).
Kinematics

- Massless momentum $p_\alpha$ can be written in 2x2 matrix as

$$p_{\dot{a}a} = \sigma_{\dot{a}a}^\alpha p_\alpha$$

- The fact that $p^2 = 0$ is reflected in $\det p_{\dot{a}a} = 0$. Therefore $p_{\dot{a}a}$ can be written as a product of two spinors $\lambda_a$ and $\tilde{\lambda}_{\dot{a}}$.

$$p_{\dot{a}a} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

where in (2,2) signature $\lambda$, $\tilde{\lambda}$ are real and independent while in (3,1) signature they are complex and conjugate.

- Scalar products

$$\langle 12 \rangle = \epsilon^{ab} \lambda_{1a} \lambda_{2b}, \quad [12] = \epsilon^{\dot{a}\dot{b}} \tilde{\lambda}_{\dot{1}a} \tilde{\lambda}_{\dot{2}b}$$

are related to the original scalar product $p_1 \cdot p_2$ as

$$(p_1 + p_2)^2 = 2(p_1 \cdot p_2) = \langle 12 \rangle [12]$$
Scattering amplitudes

- The amplitude $\mathcal{M}$ is a function of $p_\mu$ and spin information and is directly related to the probabilities in scattering experiment given by cross sections,

$$\sigma \sim \int d\Omega |\mathcal{M}|^2$$

- Despite the physical observable is $\sigma$, the amplitude $\mathcal{M}$ itself satisfies many non-trivial properties from QFT.

- Studying scattering amplitudes was crucial for developing QFT in hands of Dirac, Feynman, Schwinger, Dyson and others.

- Two main approaches:
  - Analytic S-matrix program: the amplitude as a function can be fixed using symmetries and consistency constraints.
  - Feynman diagrams: expansion of the amplitude using pieces that represent physical processes with virtual particles.

- In history of physics the second approach was the clear winner, demonstrated most manifestly in development of QCD.
Feynman diagrams

- Theory is characterized by the Lagrangian $\mathcal{L}$, for example
  \[
  \mathcal{L}_{\phi^4} = \frac{1}{2} (\partial_\mu \phi)(\partial^\mu \phi) + \lambda \phi^4
  \]
- Standard QFT approach: generating functional $\rightarrow$ correlation function $\rightarrow$ on-shell scattering amplitude.
- Diagrammatic interpretation: draw all graphs using fundamental vertices derived from Lagrangian, and evaluate them using certain rules.
- Perturbative expansion: tree-level (classical) amplitudes and loop corrections.
Feynman diagrams

- At tree-level the amplitude is a rational function with simple poles of external momenta and spin structure,

\[ M_0 = \frac{N(p_i, s_i)}{p_1^2 p_2^2 p_3^2 \cdots p_k^2} \]

where the poles are of the form \( p_j^2 = (\sum_k p_k)^2 \).

- At loop level the amplitude is an integral over the rational function,

\[ M_L = \int d^4 \ell_1 \cdots d^4 \ell_L \frac{N(p_i, s_i, \ell_j)}{p_1^2 \cdots p_k^2} \]

where the poles now also depend on \( \ell_i \).

- The class of functions we get for \( M_L \) is not known in general.
Simple amplitudes

- Amplitudes are much simpler than could be predicted from Feynman diagram approach.
  - Original calculation: \(2 \rightarrow 4\) tree-level scattering
  - Most complicated process calculated by that time.
  - Result written on 16 pages using small font.
  - Final result simplifies to one-line expression.

\[
M = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 45 \rangle \langle 56 \rangle \langle 61 \rangle}
\]

- The simplicity generalizes to all ”MHV” amplitudes, invisible in Feynman diagrams.
- This started a new field of research in particle physics, many new methods and approaches have been developed. The progress rapidly accelerated in last few years.
Simple amplitudes

- Feynman diagrams work in general for any theory with Lagrangian, however, the results for amplitudes are artificially complicated.

- Moreover, in many cases there are hidden symmetries for amplitudes which are invisible in Feynman diagrams and are only restored in the sum.

- Advantages from both approaches: perturbative QFT and analytic theory for S-matrix.
  - We use perturbative definition of the amplitude using Feynman diagrams and it also serves like a reference result.
  - On the other hand we can use properties of the S-matrix to constrain the result: locality, unitarity, analyticity and global symmetries.

- In our discussion we focus on the tree-level amplitudes and integrand of loop amplitudes.
Other aspects

- Integrated amplitudes: there is a recent activity in classifying functions one can get for amplitudes.
- In certain theories we have a good notion of transcendentality related to the loop order of the amplitude: symbol of the amplitude.
- Relation to multiple zeta values and motivic structures.
- In many theories there are also important non-perturbative effects not seen in the standard expansion.
- This is completely absent in the theory I am going to discuss now – $\mathcal{N} = 4$ SYM in planar limit.
- Despite it is a simple model, it is still an interesting 4-dimensional interacting theory, closed cousin of Quantum Chromodynamics (QCD).
Toy model for gauge theories

$\mathcal{N} = 4$ Super Yang-Mills theory in planar limit.

- Maximal supersymmetric version of $SU(N)$ Yang-Mills theory, definitely not realized in nature.
- Particle content: gauge fields ”gluons”, fermions and scalars. At tree-level: amplitudes of gluons and fermions identical to pure Yang-Mills theory. Superfield $\Phi$,

\[
\Phi = G_+ + \eta^A \Gamma_A + \frac{1}{2} \eta^A \eta^B S_{AB} + \frac{1}{6} \epsilon_{ABCD} \eta^A \eta^B \eta^C \Gamma^D + \frac{1}{24} \epsilon_{ABCD} \eta^A \eta^B \eta^C \eta^D G_-
\]

- The theory is conformal, UV finite. In planar limit (large N) hidden infinite dimensional (Yangian) symmetry which is completely invisible in any standard QFT approach.
- The theory is integrable: should have an exact solution. In AdS/CFT dual to type IIB string theory on $AdS_5 \times S_5$. 

Properties of amplitudes in toy model

- The theory has \( SU(N) \) symmetry group, in Feynman diagrams we get different group structures. In planar limit only single trace survives

\[
\mathcal{M}_{123...n} = \sum_{\sigma/\pi} \text{Tr} (T^{a_1} T^{a_2} \ldots T^{a_n}) M_{a_1 a_2 \ldots a_n}
\]

We consider the ”color-stripped” amplitude \( M \) which is cyclic.

- New kinematical variables: \( n \) twistors \( Z_i \), points in \( \mathbb{P}^3 \), and a set of Grassmann variables \( \eta_i \). Natural \( SL(4) \) invariants

\[
\langle Z_1 Z_2 Z_3 Z_4 \rangle.
\]

- The loop momentum is off-shell and has 4 degrees of freedom, represented by a line \( Z_A Z_B \) in twistor space.

- The amplitude is then a rational function of \( \langle \cdots \rangle \) with homogeneity 0 in all \( Zs \) with single poles. The pole structure is dictated by locality of the amplitude:

\[
\langle Z_i Z_{i+1} Z_j Z_{j+1} \rangle \text{ or } \langle Z_A Z_B Z_i Z_{i+1} \rangle \text{ or } \langle Z_A Z_B Z_C Z_D \rangle
\]
Properties of amplitudes in toy model

- All amplitudes are labeled by three numbers \( n, k, L \) where \( k \) is a \( k \)-charge of \( SU(4) \) symmetry of the amplitude. It has physical interpretation in terms of helicities of component gluonic amplitudes (number of \(-\) helicity gluons). In fact we better use the label \( k \equiv k' = k - 2 \).

- Feynman diagram approach is extremely inefficient. For example, \( n = 4, k = 0 \):

\[
\begin{array}{c}
\begin{array}{c}
\includegraphics{feynman_diagram}\end{array}
\end{array}
\]

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>940</td>
<td>47.380</td>
<td>(4 \times 10^6)</td>
<td>(6 \times 10^8)</td>
<td>(10^{11})</td>
<td>(10^{13})</td>
<td>(10^{15})</td>
</tr>
</tbody>
</table>
Overview of the program

- Our ultimate goal: to find a geometric formulation of the scattering amplitude as a single object.
- This formulation should make all properties of the amplitude manifest.
- It better does not use any physical concepts which should emerge as derived properties from the geometry.
- We will proceed in two steps:
  - Step 1: We find a new basis of objects which serve as building blocks for the amplitude. It will be an alternative to Feynman diagrams with very different properties. They will have a direct connection to Positive Grassmannian.
  - Step 2: Inspired by that we find a unique object which represents the full scattering amplitude - Amplituhedron - a natural generalization of Positive Grassmannian. The problem of calculating amplitudes is then reduced to the triangulation.
- The final picture involves new mathematical structures which should be understood more rigorously.
Scattering Amplitudes and Positive Grassmannian
Permutations

- Standard permutation: \((1, 2, \ldots n) \rightarrow (\sigma(1), \sigma(2), \ldots \sigma(n))\).

- Scattering process in \(1 + 1\) dimensions.

- Most trivial example: \((1, 2, 3) \rightarrow (3, 2, 1)\).
Permutations

- The picture is not unique: Yang-Baxter move

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\bullet & \bullet & \bullet \\
1 & 2 & 3
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 3 \\
\bullet & \bullet & \bullet \\
1 & 2 & 3
\end{array}
\]

- Unfortunately, this can not be applied to $3 + 1$ dimensions
  - No particle creation/destruction.
  - Fundamental 4pt interactions.

- We need fundamental 3pt vertices. Is there a way how to represent a permutation with a diagram which has only 3pt vertices?

- It is not possible to do it with a single 3pt vertex.
Permutations

- Fundamental 3pt vertices:

  
  ![Diagrams showing permutations](image)

  represent permutations \((1, 2, 3) \rightarrow (2, 3, 1)\) and \((1, 2, 3) \rightarrow (3, 2, 1)\).

- Left-Right paths in the graph: left on white vertex, right on black vertex.
Permutations

- Build a 4pt diagram:

- Permutations: $(1, 2, 3, 4) \rightarrow (4, 3, 1, 2)$, resp.
  $(1, 2, 3, 4) \rightarrow (3, 4, 1, 2)$.

- In case $k \rightarrow k$ we draw the lollipop, for
  $(1, 2, 3, 4) \rightarrow (2, 3, 1, 4)$
We can build a diagram and find a permutation.

The permutation is $(1, 2, 3, 4, 5, 6) \rightarrow (5, 4, 6, 1, 2, 3)$. Every permutation can be represented like this!
Permutations

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- The permutation is $(1, 2, 3, 4, 5, 6) \rightarrow (5, 4, 6, 1, 2, 3)$.
- Every permutation can be represented like this!
Permutations

- There exists a different diagram that gives the same permutation

The map diagrams $\leftrightarrow$ permutations is not unique!

- Reduced graphs: minimal number of faces (loops) - they represent permutations.
Permutations

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- Reduced graphs: minimal number of faces (loops) - they represent permutations.
Permutations

- There are two identity moves:
  - merge-expand of black (or white) vertices

- square move

![Diagram of permutation moves](image)
Permutations

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Permutations

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Permutations

- Go back to the Yang-Baxter move. We expand

\[ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1:} \\
\text{Diagram 2:}
\end{array}
\end{array} \]

- We could also use the substitution

\[ \begin{array}{c}
\begin{array}{c}
\text{Diagram 1:} \\
\text{Diagram 2:}
\end{array}
\end{array} \]

and prove the same identity.
Permutations

Then we get

Old diagrams are included as a subset of new diagrams.
We will use affine permutation:

\[ k \rightarrow \sigma(k) \]

where

\[ k + n \geq \sigma(k) \geq k \]

and \( \sigma(k) \mod k \) is a permutation.

1 → 3
2 → 4
3 → 1 + 4 = 5
4 → 2 + 4 = 6
Positive Grassmannian
Configuration of vectors

- Permutations $\leftrightarrow$ Configuration of vectors with consecutive linear dependencies.
- Configuration of $n$ pt in $\mathbb{P}^{k-1}$

$k \rightarrow \sigma(k)$ means that $k \subset \text{span}(k+1, \ldots \sigma(k))$

$1 \subset (2, 34, 5, 6) \rightarrow \sigma(1) = 6, \quad 2 \subset (34, 5) \rightarrow \sigma(2) = 5,$

$3 \subset (4) \rightarrow \sigma(3) = 4, \quad 4 \subset (5, 6, 1, 2) \rightarrow \sigma(4) = 2,$

$5 \subset (6, 1) \rightarrow \sigma(5) = 1, \quad 6 \subset (1, 2, 3) \rightarrow \sigma(6) = 3.$

The permutation is $(1, 2, 3, 4, 5, 6) \rightarrow (6, 5, 4, 8, 7, 9).$
The Positive Grassmannian

- Grassmannian $G(k,n)$: space of $k$-dimensional planes in $n$ dimensions, represented by $k \times n$ matrix modulo $GL(k)$,

$$C = \begin{pmatrix}
* & * & * & \ldots & * & * \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
* & * & * & \ldots & * & * \\
\end{pmatrix}
= \begin{pmatrix}
v_1 \\
\vdots \\
v_k \\
\end{pmatrix}
= \begin{pmatrix}
c_1 & c_2 & \ldots & c_n \\
\end{pmatrix}$$

- We can think about it as collection of $k$ vectors $v_1, \ldots, v_k$ in $n$ dimensions which specify the plane.

- We consider a positive part of $G(k,n)$ which is a space with boundaries.
The Positive Grassmannian

- Positive part:

\[ C = [c_1 \ c_2 \ \ldots \ c_n] \]

All minors

\[(c_{i_1} \ldots c_{i_k}) > 0 \quad \text{for} \quad i_1 < i_2 < \ldots < i_k.\]

- Cyclic structure: \( c_1 \rightarrow c_2, \ c_2 \rightarrow c_3, \ldots, \ c_n \rightarrow (-1)^{k+1}c_1.\)
We can think about $C$ as collection $n$ points in $\mathbb{P}^{k-1}$.

Back to 6pt example:

$$C = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & c_{16} \\
0 & 1 & 0 & 0 & c_{25} & a \cdot c_{25} \\
0 & 0 & 1 & c_{34} & c_{35} & a \cdot c_{35}
\end{pmatrix}$$

Five-dimensional configuration in $G(3, 6)$. 
The Positive Grassmannian

- Positive part of $G(k, n)$: convex configurations of points.
- Top cell in the Grassmannian (no constraint imposed) $\rightarrow$ configuration of $n$ generic points in $\mathbb{P}^{k-1}$.
- Stratification of the space is nicely provided by imposing linear dependencies between consecutive points.

This corresponds to sending minors of $G_+(k, n)$ to zero.
- Boundaries preserve convexity: all minors of $G_+(k, n)$ stay positive (except the ones sent to zero).
Equivalence

Reduced graphs (mod identity moves)

\[ \iff \]

Permutations

\[ \iff \]

Configurations of vectors with linear dependencies

\[ \iff \]

Cells of Positive Grassmannian
Plabic graphs and Positive Grassmannian
Plabic graphs

- These diagrams are known in the literature as "plabic graphs" and were extensively studied by Alexander Postnikov (math/0609764).
- He established the connection to the positive Grassmannian and showed how to construct explicitly a matrix for each reduced diagram.
- There is a precise definition what the "reduced" means but in practice it means that the diagram does not have any bubbles.
- Bubble reduction:

\[
\begin{align*}
\text{bubble} & \Rightarrow \text{reduced diagram}
\end{align*}
\]
Plabic graphs

Example:

1 2

4

3

Postnikov proved isomorphism between permutations and reduced plabic graphs (modulo identity moves).

In order to find the Grassmannian matrix for each reduced diagram we have to choose variables.

- Edge variables.
- Face variables.

Orientation: choose an arrow for each edge.
Plabic graphs

Example:

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- In order to find the Grassmannian matrix for each reduced diagram we have to choose variables.
  - Edge variables.
  - Face variables.
- Orientation: choose an arrow for each edge.
Once we have given a perfect orientation, the system of equations $C \cdot \tilde{\lambda}$ becomes trivial to construct: each vertex can be viewed as giving an equation which expands the $\tilde{\lambda}$'s of the vertex's sources in terms of those of its sinks. Combining all such equations then gives us an expansion of the external sources' $\tilde{\lambda}$'s in terms of those of the external sinks. Notice that when identifying two legs, $(I_{\text{in}}, I_{\text{out}})$ during amalgamation the degree of freedom lost in the process is accounted for via the replacement of the pair $(\alpha_{I_{\text{in}}}, \alpha_{I_{\text{out}}})$ with the single variable $\alpha_{I} \equiv \alpha_{I_{\text{in}}}\alpha_{I_{\text{out}}}$.

If we denote the external sources of a graph by \{a_1, ..., a_k\} ≡ A, then the final linear relations imposed on the $\tilde{\lambda}$'s can easily be seen to be given by,

$$\tilde{\lambda}_A + \sum_{\Gamma \in \{A \rightarrow a\}} \prod_{e \in \Gamma} \alpha_e = 0,$$

(4.56)

with

$$\sum_{\Gamma \in \{A \rightarrow a\}} \prod_{e \in \Gamma} \alpha_e = -\sum_{\Gamma \in \{A \rightarrow a\}} \prod_{e \in \Gamma} \alpha_e,$$

(4.57)

and where $\Gamma \in \{A \rightarrow a\}$ is any (directed) path from $A$ to $a$ in the graph. (If there is a closed, directed loop, then the geometric series should be summed—we will see an example of this in (4.64).) The entries of the matrix $C_Aa$ are called the "boundary measurements" of the on-shell graph. The on-shell form on $C(\alpha) \in G(k,n)$ can then be written in terms of the variables $C_Aa$ according to:

$$\prod_{\text{vertices}} \text{vol}(\text{GL}(1)_v) \prod_{\text{edges}} d\alpha e \alpha_e \delta_k \times 4 (C \cdot \tilde{\eta}) \delta_k \times 2 (C \cdot \tilde{\lambda}) \delta_2 \times (n - k) (\lambda \cdot C_{\perp}).$$

(4.58)

Let us consider a simple example to see how this works. Consider the following perfectly oriented graph:

(4.59)

Using the equations for each directed 3-particle vertex, we can easily expand the $\tilde{\lambda}$ of each source—legs 1 and 2—in terms of those of the sinks—legs 3 and 4; e.g.,

$$\tilde{\lambda}_2 = \alpha_2 \alpha_6 (\alpha_3 \tilde{\lambda}_3 + \alpha_7 (\alpha_4 \tilde{\lambda}_4)).$$

(4.60)

Such expansions obviously result in (4.57): the coefficient $C_Aa$ of $\tilde{\lambda}$ in the expansion of $\tilde{\lambda}_A$ is simply (minus) the product of all edge-variables $\alpha_e$ along any path $\Gamma \in \{A \rightarrow a\}$. Doing this for all the $C_Aa$ of our example above, we find,

$$C_{13} = \alpha_1 \alpha_5 \alpha_6 \alpha_3$$

$$C_{14} = \alpha_1 \alpha_5 \alpha_6 \alpha_7 \alpha_4 + \alpha_1 \alpha_8 \alpha_4$$

$$C_{23} = \alpha_2 \alpha_6 \alpha_3$$

$$C_{24} = \alpha_2 \alpha_6 \alpha_7 \alpha_4 - \alpha_1 \alpha_4 \alpha_6 \alpha_7.$$

For this example (positive matrix for fixed signs of $\alpha_i$):

$$C = \begin{pmatrix}
1 & 0 & -\alpha_1 \alpha_3 \alpha_5 \alpha_6 & -\alpha_1 \alpha_4 \alpha_5 \alpha_6 \alpha_7 - \alpha_1 \alpha_4 \alpha_8 \\
0 & 1 & -\alpha_2 \alpha_3 \alpha_6 & -\alpha_2 \alpha_4 \alpha_6 \alpha_7
\end{pmatrix}$$
Face variables

- Variables associated with faces.

- "Gauge invariant" (fluxes) associated with faces of the graph. Only one condition $\prod f_i = -1$.

- The rule for entries of the $C$ matrix,

$$C_{iJ} = - \sum_{\text{paths } i \to J} \prod (-f_j) \quad \text{faces left to the path}$$

- For this example:

$$C = \begin{pmatrix} 1 & 0 & \frac{f_0}{f_3}f_4 & -f_0f_4 + f_4 \\ 0 & 1 & -f_0f_1f_3f_4 & -f_0f_1f_4 \end{pmatrix}$$
Face variables

▶ Moves and face variables

▶ Reduction: eliminate irrelevant variable

▶ Face (or edge) variables are cluster variables and these are cluster transformations.