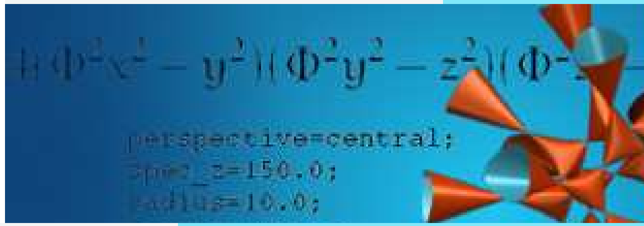
Philipps Universität Marburg

G-structures and their remarkable spinor fields



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Srní, January 2015

– joint work with Simon Chiossi, Thomas Friedrich, and Jos Höll –

Observation:

- $\exists$ multitude of different spinorial field equations, related to different geometric <u>structures</u> and geometric <u>questions</u>

Goal:

- Uniform description of different types of spinor fields
- applications

The Riemannian Dirac operator

(M^n, g): compact Riemannian spin mnfd, Σ: spin bdle

Classical Riemannian Dirac operator D^g:

$$\text{Dfn}: \qquad D^g : \Gamma(\Sigma) \longrightarrow \Gamma(\Sigma), \quad D^g\psi := \sum_{i=1}^n e_i \cdot \nabla^g_{e_i}\psi$$

Properties:

- D^g is elliptic differential operator of first order, essentially self-adjoint on $L^2(\Sigma)$, pure point spectrum
- Of equal fundamental importance than the Laplacian
- In dimension 4: $\mathrm{index}(D^g) = \sigma(M^4)/8$ [Atiyah-Singer, $\sim$ 1963]
- Schrödinger (1932), Lichnerowicz (1962): $\boxed{(D^g)^2 = \Delta + \frac{1}{4}\mathrm{Scal}^g}$

$\sim$ "‘root of the Laplacian"’ for $\mathrm{Scal}^g = 0$

Spinors and Riemannian eigenvalue estimates

SL formula $\Rightarrow$ EV of $(D^g)^2$: $\quad \lambda \ \geq \ \frac{1}{4}\,\mathrm{Scal}^g_{\min}$

- optimal only for spinors with $\langle \Delta\psi, \psi\rangle = \|\nabla^g\psi\|^2 = 0$, i. e. parallel spinors

Thm. (M,g) has parallel spinors iff $\mathrm{Hol}_0(M) = \mathrm{SU}(n), \mathrm{Sp}(n), G_2, \mathrm{Spin}(7)$, and then $\mathrm{Ric}^g = 0$. [Wang, 1989]

Thm. Optimal EV estimate: $\lambda \geq \dfrac{n}{4(n-1)}\mathrm{Scal}^g_{\min}$ [Friedrich, 1980]

- "=" if there exists a Killing spinor (KS) ψ: $\nabla^g_X\psi = \mathrm{const}\cdot X\cdot\psi \quad \forall X$

Link to special geometries:

Thm. $\exists$ KS $\Leftrightarrow n = 5 : (M,g)$ is Sasaki-Einstein mnfd [$\in$ contact str.]

$\Leftrightarrow n = 6 : (M,g)$ nearly Kähler mnfd

$\Leftrightarrow n = 7 : (M,g)$ nearly parallel G_2 mnfd

[Friedrich, Kath, Grunewald. . .]

Killing spinors and submanifolds

Thm. Suppose (M, g) is Sasaki-Einstein ($n = 5$), nearly Kähler ($n = 6$), or nearly parallel G_2 (n=7). Then the metric cone

$$(\bar{M}, \bar{g}) := (M \times \mathbb{R}^+, \tfrac{1}{4} r^2 g^2 + dr^2)$$

has a ∇^g-parallel spinor; in particular, it is Ricci-flat of Riemannian holonomy $\mathrm{SU}(3)$, G_2, resp. $\mathrm{Spin}(7)$. [Bryant 1987, B-Salamon 1989, Bär 1993 (+ Wang 1989)]

Observe: Construction relies on existence of a Killing spinor

Thm. Let (M, g) be a spin manifold with a ∇^g-parallel spinor ψ, $N \subset M$ a codimension one hypersurface. Then $\varphi := \psi\big|_N$ is a generalized Killing spinor on N, i. e. $\nabla^g_X \varphi = A(X) \cdot \varphi$ for a symmetric endomorphism A (Weingarten map). [Friedrich 1998, Bär-Gauduchon-Moroianu 2005]

Observe: Generalizes the Weierstraß representation of minimal surfaces, based on ideas of Eisenhardt (1909)

Parallel spinors and G-structures

Observe: Sasaki-Einstein , nearly Kähler, or nearly parallel G_2-manifolds are *not* the most general $\mathrm{SU}(2)$-, $\mathrm{SU}(3)$- or G_2-manifolds.

Q: What can be said for more general G-manifolds?

Given a mnfd M^n with G-structure ($G \subset \mathrm{SO}(n)$), replace ∇^g by a *metric connection* ∇ *with torsion that preserves the geometric structure!*

$$\text{torsion:} \quad T(X,Y,Z) \ := \ g(\nabla_X Y - \nabla_Y X - [X,Y], Z)$$

Special case: require $T \in \Lambda^3(M^n)$ ($\Leftrightarrow$ same geodesics as ∇^g)

$$\Rightarrow \quad g(\nabla_X Y, Z) \ = \ g(\nabla^g_X Y, Z) + \tfrac{1}{2} T(X,Y,Z)$$

• If existent, such a connection is unique and called the *'characteristic connection'* [Fr-Ivanov 2002, A-Fr-Höll 2013]

• If G is contained in the stabilizer of a generic spinor field, then there exists a ∇-parallel spinor, $\nabla\psi = 0$ ($n = 5: \ \mathrm{SU}(2)$, $n = 6: \ \mathrm{SU}(3)$, $n = 7: \ G_2$, $n = 8: \ \mathrm{Spin}(7)$).

Spin structures and topology in dimension 6 and 7

Observation:

Any 8-dim. real vector bundle over a n-dimensional manifold ($n = 6, 7$) admits a section of length one

$\Rightarrow$ a 6-dim. oriented Riemannian manifold admits a spin structure iff it admits a reduction from $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$ to $\mathrm{SU}(3)$

$\Rightarrow$ a 7-dim. oriented Riemannian manifold admits a spin structure iff it admits a reduction from $\mathrm{Spin}(7)$ to G_2

Use this to give a uniform spinor description of $\mathrm{SU}(3)$-manifolds and G_2-manifolds!

Spin linear algebra in dimension 6 and 7

• In $n = 6, 7$, the spin representations are real and $2^3 = 8$-dimensional, they coincide as vector spaces, call it $\Delta := \mathbb{R}^8$.

$\underline{n = 6}$ [A-Fr-Chiossi-Höll, 2014]

• Δ admits a $\mathrm{Spin}(6)$-invariant cplx structure j (because $\mathrm{Spin}(6) \cong \mathrm{SU}(4)$)

• any real spinor $0 \neq \phi \in \Delta$ decomposes Δ into three pieces,

$$\Delta = \mathbb{R} \cdot \phi \oplus \mathbb{R} \cdot j(\phi) \oplus \underbrace{\{X \cdot \phi \,:\, X \in \mathbb{R}^6\}}_{\cong \mathbb{R}^6,\ \text{the base space}} \qquad (*)$$

• the following formula defines an orthogonal cplx str. on the last piece,

$$J_\phi(X) \cdot \phi \ := \ j(X \cdot \phi)$$

• the spinor defines a 3-form by $\psi_\phi(X, Y, Z) := -(X \cdot Y \cdot Z \cdot \phi, \phi)$.

Exa. Consider $\phi = (0, 0, 0, 0, 0, 0, 0, 1) \in \Delta = \mathbb{R}^8$. Then:

$$J_\phi = -e_{12} + e_{34} + e_{56}, \quad \psi_\phi = e_{135} - e_{146} + e_{236} + e_{245}.$$

Spin linear algebra in dimension 6 and 7

Thm. The following is a 1-1 correspondence: (well-known)

- $\mathrm{SU}(3)$-structures on $\mathbb{R}^6 \longleftrightarrow$ real spinors of length one $(\mathrm{mod}\, \mathbb{Z}_2)$,

$$\mathrm{SO}(6)/\mathrm{SU}(3) = \{\mathrm{SU}(3)\text{-structures on } \mathbb{R}^6\} = \mathbb{P}(\Delta) = \mathbb{RP}^7.$$

$\underline{n = 7}$

- any real spinor $0 \neq \phi \in \Delta$ decomposes Δ into two pieces,

$$\Delta = \mathbb{R} \cdot \phi \oplus \underbrace{\{X \cdot \phi \,:\, X \in \mathbb{R}^7\}}_{\cong \mathbb{R}^7,\ \text{the base space}} \qquad (**)$$

- the spinor defines again a 3-form ψ_ϕ, which turns out to be *stable* (i. e. open GL-orbit); but no analogue of neither j nor J_ϕ

Thm. The following is a 1-1 correspondence: (well-known)

stable 3-forms ψ of fixed length, with isotropy $\subset \mathrm{SO}(7) \longleftrightarrow \ldots$ (as above),

$$\mathrm{SO}(7)/G_2 \;=\; \mathbb{P}(\Delta) \;=\; \mathbb{RP}^7.$$

Special almost Hermitian geometry

• $\mathrm{SU}(3)$ manifold (M^6, g, ϕ): Riemannian spin manifold (M^6, g) equipped with a global spinor ϕ of length one, j as before, J induced almost cplx str., ω its kähler form, ψ_ϕ induced 3-form, $\psi_\phi^J := J \circ \psi_\phi$.

Decomposition $(*)$ $\Rightarrow \exists_1$ 1-form η and endomorphism S s. t.

$$\boxed{\nabla^g_X \phi = \eta(X) j(\phi) + S(X) \cdot \phi}$$

η: "intrinsic 1-form", S: "intr. endomorphism" (indeed: $\Gamma = S \lrcorner \psi_\phi - \frac{2}{3}\eta \otimes \omega$)

This equation summarizes all spinor eqs. previously known in dim.6!

Thm. $(\nabla^g_X \omega)(Y, Z) = 2\psi^J_\phi(S(X), Y, Z), \quad 8\eta(X) = -(\nabla^g_X \psi^J_\phi)(\psi_\phi).$

This generalizes the classical nK condition $\nabla^g_X \omega(X, Y) = 0 \ \forall X, Y$.

There are 7 basic classes of $\mathrm{SU}(3)$-structures, called $\chi_1, \chi_{\bar{1}}, \chi_2, \chi_{\bar{2}}, \chi_3, \chi_4, \chi_5$.

[Chiossi-Salamon, 2002]

They are a refinement of the classical Gray-Hervella classification of $\mathrm{U}(3)$-structures. Write $\chi_{1\bar{2}4}$ for $\chi_1^+ \oplus \chi_2^- \oplus \chi_4$ etc.

Examples.

- nearly Kähler mnfds: class $\chi_{\bar{1}}$
- half-flat $\mathrm{SU}(3)$-mnfds: class $\chi_{\bar{1}\bar{2}3}$

Next: express Niejenhuis tensor, $d\omega, \delta\omega$ through ψ_ϕ^j, η, S

$\rightarrow$ don't state the formulas

Thm. The classes of $\mathrm{SU}(3)$ str. are determined as follows:

class	description	dimension
χ_1	$S = \lambda \cdot J_\phi$, $\eta = 0$	1
$\chi_{\bar{1}}$	$S = \mu \cdot \mathrm{Id}$, $\eta = 0$	1
χ_2	$S \in \mathfrak{su}(3)$, $\eta = 0$	8
$\chi_{\bar{2}}$	$S \in \{A \in S^2_0(\mathbb{R}^6) \mid AJ_\phi = J_\phi A\}$, $\eta = 0$	8
χ_3	$S \in \{A \in S^2_0(\mathbb{R}^6) \mid AJ_\phi = -J_\phi A\}$, $\eta = 0$	12
χ_4	$S \in \{A \in \Lambda^2(\mathbb{R}^6) \mid AJ_\phi = -J_\phi A\}$, $\eta = 0$	6
χ_5	$S = 0$, $\eta \neq 0$	6

where $\lambda, \mu \in \mathbb{R}$. In particular S is symmetric and $\eta = 0$ if and only if the class is $\chi_{\bar{1}\bar{2}3}$.

The *symmetries of S* translate into a *differential eq. for ϕ*:

$$\begin{aligned} SJ_\phi = \pm J_\phi S \quad &\Longleftrightarrow \quad (J_\phi Y \nabla^g_X \phi, \phi) = \mp(Y\nabla^g_{J_\phi X}\phi, \phi), \\ S \text{ is } \pm\text{-symmetric} \quad &\Longleftrightarrow \quad (X\nabla^g_Y \phi, \phi) = \pm(Y\nabla^g_X\phi, \phi). \end{aligned}$$

Thm. The classification of $\mathrm{SU}(3)$ str. in terms of ϕ is given by $(\lambda := \frac{1}{6}(D^g\phi, j(\phi))$, $\mu := -\frac{1}{6}(D^g\phi, \phi))$: (. . . and similarly for mixed classes)

class	spinorial equation
χ_1	$\nabla^g_X\phi = \lambda X j(\phi)$ for $\lambda \in \mathbb{R}$
$\chi_{\bar{1}}$	$\nabla^g_X\phi = \mu X\phi$ for $\mu \in \mathbb{R}$ (Killing sp.)
χ_2	$(J_\phi Y\nabla^g_X\phi, \phi) = -(Y\nabla^g_{J_\phi X}\phi, \phi)$, $(Y\nabla^g_X\phi, j(\phi)) = (X\nabla^g_Y\phi, j(\phi))$, $\lambda = \eta = 0$
$\chi_{\bar{2}}$	$(J_\phi Y\nabla^g_X\phi, \phi) = (Y\nabla^g_{J_\phi X}\phi, \phi)$, $(Y\nabla^g_X\phi, j(\phi)) = -(X\nabla^g_Y\phi, j(\phi))$, $\mu = \eta = 0$
χ_3	$(J_\phi Y\nabla^g_X\phi, \phi) = (Y\nabla^g_{J_\phi X}\phi, \phi)$, $(Y\nabla^g_X\phi, j(\phi)) = (X\nabla^g_Y\phi, j(\phi))$, and $\eta = 0$
χ_4	$(J_\phi Y\nabla^g_X\phi, \phi) = -(Y\nabla^g_{J_\phi X}\phi, \phi)$, $(Y\nabla^g_X\phi, j(\phi)) = -(X\nabla^g_Y\phi, j(\phi))$ and $\eta = 0$
χ_5	$\nabla^g_X\phi = (\nabla^g_X\phi, j(\phi))j(\phi)$

Corollary. On a 6-dim spin mnfd, $\exists$ spinor of constant length s. t. $D^g\phi = 0$ iff admits a $\mathrm{SU}(3)$ structure of class $\chi_{2\bar{2}345}$ with $\delta\omega = -2\eta$.

Example: twistor spaces as $\mathrm{SU}(3)$-manifolds

• $M^6 = \mathbb{CP}^3$, $\mathrm{U}(3)/\mathrm{U}(1)^3$: twistor spaces of S^4 and $\mathbb{CP}^2$. Both carry metrics $g_t (t > 0)$ and two almost complex structures $\Omega^{\mathrm{K}}, \Omega^{\mathrm{nK}}$ such that

- $(M^6, g_{1/2}, \Omega^{\mathrm{nK}})$ is a nearly Kähler manifold
- $(M^6, g_1, \Omega^{\mathrm{K}})$ is a Kähler manifold

• $\exists$ two real linearly indep. global spinors ϕ_ε in Δ_6 $(\varepsilon = \pm 1)$. **Both spinors** induce the same almost cplx structure J_ϕ ($\Leftrightarrow \Omega^{\mathrm{nK}}$)!

• For $t = 1/2$, ϕ_ε are Riemannian Killing spinors. For general t, define $S_\varepsilon : TM^6 \to TM^6$ by $S_\varepsilon = \varepsilon \sqrt{c} \cdot \mathrm{diag}\left(\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}, \frac{1-t}{2\sqrt{t}}, \frac{1-t}{2\sqrt{t}}\right)$.

Verify: $\nabla^g_X \phi_\varepsilon = S_\varepsilon(X)\phi_\varepsilon$, hence S_ε *is* the intr. endom. and $\eta = 0$.

• Class: $\chi_{\bar{1}\bar{2}}$ for $t \neq 1/2$, $\chi_{\bar{1}}$ for $t = 1/2$.

• For $t = 1$, ϕ_ε are Kählerian Killing spinors, but they *do not* induce the Kählerian cplx str. Ω^{K}! Thus, the Kählerian structure cannot be recovered from the pair of Kählerian Killing spinors (only a $\mathrm{U}(3)$-reduction).

Characteristic connections

Thm. A spin manifold (M^6, g, ϕ) admits a characteristic connection ∇ iff it is of class $\chi_{1\bar{1}345}$ and $\eta = \frac{1}{4}\delta\,\omega$. It satisfies $\nabla\phi = 0$.

For all other classes, an adapted connection ∇ can be defined as well.

Corollary. Whenever ∇ exists,

$$\phi \in \ker D^g \iff T\phi = 0 \iff \text{the } \mathrm{SU}(3)\text{-class is } \chi_3.$$

G_2 geometry

- G_2 manifold (M^7, g, ϕ): Riemannian spin manifold (M^7, g) equipped with a global spinor ϕ of length one, ψ_ϕ induced 3-form.

Decomposition $(**) \Rightarrow \exists_1$ endomorphism S s. t.

$$\boxed{\nabla^g_X \phi = S(X) \cdot \phi}$$

S: "intrinsic endomorphism" (indeed: $\Gamma = -\frac{2}{3} S \lrcorner \psi_\phi$)

Thm. $(\nabla^g_V \psi_\phi)(X, Y, Z) = 2 * \psi_\phi(S(V), X, Y, Z)$.

This generalizes the nearly parallel G_2 condition $\nabla \psi_\phi = d\psi_\phi = c * \psi_\phi$!

There are 4 basic classes of G_2-structures, called $\mathcal{W}_1, \ldots, \mathcal{W}_4$.

[Fernandez-Gray, 1982]

Thm. The classes of G_2 structures are determined as follows:

class	description	dimension
$\mathcal{W}_1$	$S = \lambda\,\mathrm{Id}$	1
$\mathcal{W}_2$	$S \in \mathfrak{g}_2$	14
$\mathcal{W}_3$	$S \in S^2_0\mathbb{R}^7$	27
$\mathcal{W}_4$	$S \in \{V \lrcorner \Psi_\phi \mid V \in \mathbb{R}^7\}$	7

In particular, S is symmetric if and only if $S \in \mathcal{W}_{13}$ and skew iff it belongs in $\mathcal{W}_{24}$.

Corollary. Let (M^7, g, ϕ) be a Riemannian spin manifold with unit spinor ϕ. Then ϕ is harmonic

$$D^g\phi = 0$$

iff the underlying G_2-structure is of class $\mathcal{W}_{23}$.

Thm. The basic classes of G_2-manifolds described in terms of ϕ:

($\lambda := -\frac{1}{7}(D^g\phi, \phi) : M \to \mathbb{R}$ is a real function and $\times$ the cross product relative to Ψ_ϕ)

class	spinorial equation
$\mathcal{W}_1$	$\nabla^g_X \phi = \lambda X \phi$ (Killing spinor)
$\mathcal{W}_2$	$\nabla^g_{X\times Y}\phi = Y\nabla^g_X\phi - X\nabla^g_Y\phi + 2g(Y, S(X))\phi$
$\mathcal{W}_3$	$(X\nabla^g_Y\phi, \phi) = (Y\nabla^g_X\phi, \phi)$ and $\lambda = 0$
$\mathcal{W}_4$	$\nabla^g_X\phi = XV\phi + g(V, X)\phi$ for some $V \in TM^7$
$\mathcal{W}_{12}$	$\nabla^g_{X\times Y}\phi = -14\lambda[Y\nabla^g_X\phi - X\nabla^g_Y\phi + g(Y, S(X))\phi - g(X, S(Y))\phi]$
$\mathcal{W}_{13}$	$(X\nabla^g_Y\phi, \phi) = (Y\nabla^g_X\phi, \phi)$
$\mathcal{W}_{14}$	$\exists V, W \in TM^7 : \nabla^g_X\phi = XVW\phi - (XVW\phi, \phi)$
$\mathcal{W}_{23}$	$S\phi = 0$ and $\lambda = 0$, or $D^g\phi = 0$
$\mathcal{W}_{24}$	$(X\nabla^g_Y\phi, \phi) = -(Y\nabla^g_X\phi, \phi)$

Example: 7-dim. 3-Sasaki mnfds

M^7: 3-Sasaki mnfd, corresponds to $\mathrm{SU}(2) \subset G_2 \subset \mathrm{SO}(7)$.

- 3 orth. Sasaki structures $\eta_i \in T^*M^7$, $[\eta_1, \eta_2] = 2\,\eta_3$, $[\eta_2, \eta_3] = 2\,\eta_1$, $[\eta_3, \eta_1] = 2\,\eta_2$ and $\varphi_3 \circ \varphi_2 = -\varphi_1$ etc. on $\langle \eta_2, \eta_3 \rangle^\perp$
- Known: A 3-Sasaki mnfd is always Einstein and has 3 Riemannian Killing spinors, define $T^v := \langle \xi_1, \xi_2 \xi_3 \rangle$, $T^h = (T^v)^\perp$
- each Sasaki structures η_i induces a characteristic connection ∇^i, but $\nabla^1 \neq \nabla^2 \neq \nabla^3$?!? $\Rightarrow$ Ansatz: $T = \sum_{i,j=1}^{3} \alpha_{ij}\,\eta_i \wedge d\eta_j + \gamma\,\eta_1 \wedge \eta_2 \wedge \eta_3$

Thm. There exists a cocalibrated G_2-structure with char. connection ∇ with parallel spinor ψ on M^7 with the properties:

- ∇ preserves T^v and T^h, and $\nabla T = 0$
- $\xi_i \cdot \psi$ are the 3 Riemannian Killing spinors on M^7 (each defines a nearly par. G_2-str.) [A-Fr, 2010]

Q: What happens if the three structures define only an almost metric 3-contact structure?

Example: Quaternionic Heisenberg group

$N^7 = \mathbb{R}^7$ with basis elements z_1, z_2, z_3, and $\tau_1, \dots, \tau_4$, metric depending on $\lambda > 0$ s. t. $\xi_i := \frac{z_i}{\lambda}$, τ_l are orthonormal, commutator relations

$$[\tau_r, \tau_{1+r}] = \lambda\,\xi_1 \qquad [\tau_r, \tau_{2+r}] = \lambda\,\xi_2 \qquad [\tau_r, \tau_{3+r}] = \lambda\,\xi_3$$
$$[\tau_{2+r}, \tau_{3+r}] = \lambda\,\xi_1 \qquad [\tau_{3+r}, \tau_{1+r}] = \lambda\,\xi_2 \qquad [\tau_{1+r}, \tau_{2+r}] = \lambda\,\xi_3$$

• ξ_1, ξ_2, ξ_3 are Killing vector fields; metric is never Einstein ($\Rightarrow \nexists$ Killing sp.)
η_i: dual form of ξ_i, θ_l: dual form of τ_l

• carries, in standard way, an almost 3-contact metric structure

Thm. The connection ∇ with skew torsion T satisfies

$$T = \eta_1 \wedge d\eta_1 + \eta_2 \wedge d\eta_2 + \eta_3 \wedge d\eta_3 - 4\lambda\eta_{123}$$

• $\nabla T = \nabla \mathcal{R} = 0$, hence it's naturally reductive [Tricerri-Vanhecke]

•Its holonomy algebra is isomorphic to $\mathfrak{su}(2)$, acting irreducibly on $T^v = \mathrm{span}(\xi_1, \xi_2, \xi_3)$ and on T^h.

• ∇ *is* the characteristic connection of the cocalibrated G_2 structure

$$\omega = -\eta_1 \wedge (\theta_{12} + \theta_{34}) - \eta_2 \wedge (\theta_{13} + \theta_{42}) - \eta_3 \wedge (\theta_{14} + \theta_{23}) + \eta_{123}.$$

As such, it admits a parallel spinor field ψ_0, $\nabla\psi_0 = 0$. What about $\xi_i \cdot \psi_0$?

Thm. The spinor fields $\psi_i := \xi_i \cdot \psi_0$, $i = 1, 2, 3$, are generalised Killing spinors satisfying the differential equation

$$\nabla^g_{\xi_i}\psi_i = \frac{\lambda}{2}\,\xi_i\cdot\psi_i, \quad \nabla^g_{\xi_j}\psi_i = -\frac{\lambda}{2}\,\xi_j\cdot\psi_i\ (i \neq j), \quad \nabla^g_X\psi_i = \frac{5\lambda}{4}\,X\cdot\psi_i \ \text{ for } X \in T^h.$$

[A-Ferreira-Storm, 12/2014]

(intrinsic endom. S is diagonal, but not multiple of identity, class $\mathcal{W}_{13}$)

Observe: Only known example where S has three different eigenvalues

Application: cone constructions

• How to construct G_2-str. of any class on cones over $\mathrm{SU}(3)$-manifolds?

Start with (M^6, g, ϕ) with intrinsic torsion (S, η). Choose a function $h = h_1 + ih_2 : I \to S^1$ and define by

$$\phi_t \ := \ h(t)\phi := h_1(t)\phi + h_2(t)j(\phi)$$

a new family of $\mathrm{SU}(3)$-structures on M^6 depending on $t \in I$.

Conformally rescale the metric by some function $f : I \to \mathbb{R}_+$ and consider $M^6_t := (M^6, f(t)^2 g, \phi_t)$. Intrinsic torsion of M^6_t : $(\frac{h^2}{f} S, \eta)$.

Dfn. *spin cone* over M^6: $(\bar{M}^7, \bar{g}) = (M^6 \times I, f^2(t)g + dt^2)$ with spinor ϕ_t.

Exa. Suppose we want $\bar{M}^7$ to be a nearly parallel G_2-manifold: need h'/h constant, so $h(t) = \exp(i(ct + d))$, $c, d \in \mathbb{R}$. Easiest: sine cone $(M^6 \times (0, \pi), \sin(t)^2 g + dt^2, e^{it/2}\phi)$ [Fernández-Ivanov-Muñoz-Ugarte, 2008; Stock, 2009]

• Similarly, we can construct G_2-manifolds of any desired pure class (construction really uses the spinor!).

To conclude:

Obtained a uniform description of all possible defining spinorial differential eqs. on 6-dim. $\mathrm{SU}(3)$-manifolds and G_2-manifolds, generalizing Killing spinors, generalized Killing spinors, quasi-Killing spinors [Friedrich-Kim, 2000]. . .

So far, all spinors encountered are *generalized Killing spinor with torsion (gKST)*, i. e.

$$\nabla\phi = A(X)\cdot\phi$$

for some endomorphism $A : TM^6 \to TM^6$; but the same eq. can be expressed in different ways.

- Not the differential eq. is the basic object, but rather the G-structure!

Outlook: $n = 8$ and $\mathrm{Spin}(7)$-structures [work in progress – Konstantis]

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Another application:

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* * *

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Application II: eigenvalue estimates with skew torsion

(M, g): mnfd with G-structure and charact. connection ∇^c, torsion T, assume $\nabla^c T = 0$ (for exa., naturally reductive)

$\not{D}$: Dirac operator of connection with torsion $T/3$ (generalizes Dolbeault op. of Hermitian manifolds)

Generalized SL formula: [A-Friedrich, 2003]

$$\not{D}^2 \;=\; \Delta_T + \frac{1}{4}\mathrm{Scal}^g + \frac{1}{8}\,||T||^2 - \frac{1}{4}T^2$$

[$1/3$ rescaling: Slebarski (1987), Bismut (1989), Kostant, Goette (1999), A (2002)]

Split spin bundle into eigenspaces of T, estimate action of T on each subbundle $\Rightarrow$

Corollary (universal estimate). The first EV λ of $\not{D}^2$ satisfies

$$\lambda \geq \frac{1}{4}\mathrm{Scal}^g_{\min} + \frac{1}{8}\|T\|^2 - \frac{1}{4}\max(\mu_1^2, \ldots, \mu_k^2),$$

where $\mu_1, \ldots, \mu_k$ are the eigenvalues of T.

Universal estimate:

- follows from generalized SL formula
- does not yield Friedrich's inequality for $T \to 0$
- optimal iff $\exists$ a ∇^c-parallel spinor:

  This sometimes happens on mnfds with $\mathrm{Scal}^g_{\min} > 0$!

$\longrightarrow$ Results:

[• deformation techniques: yield often estimates quadratic in Scal^g, require subtle case by case discussion, often restriced curvature range]

[A-Friedrich-Kassuba, 2008]

• twistor techniques: estimates always linear in Scal^g, no curvature restriction, rather universal, leads to a twistor eq. with torsion and sometimes to a Killing eq. with torsion

[A-(Becker-Bender)-Kim, 2013]

Twistors with torsion

$m : TM \otimes \Sigma M \to \Sigma M$: Clifford multiplication

$p =$ projection on $\ker m$: $p(X \otimes \psi) = X \otimes \psi + \frac{1}{n}\sum_{i=1}^n e_i \otimes e_i X\psi$

∇^s: $\nabla^s_X Y := \nabla^g_X Y + 2sT(X, Y, -)$

($s = 1/4$ is the "standard" normalisation, $\nabla^{1/4} =$ char. conn.)

twistor operator: $P^s = p \circ \nabla^s$

Fundamental relation: $\|P^s\psi\|^2 + \frac{1}{n}\|D^s\psi\|^2 = \|\nabla^s\psi\|^2$

ψ is called s-twistor spinor $\Leftrightarrow$ $\psi \in \ker P^s$ $\Leftrightarrow$ $\nabla^s_X\psi + \frac{1}{n}XD^s\psi = 0.$

A priori, not clear what the **right value of** s might be:

different scaling in ∇ $\left[s = \frac{1}{4}\right]$ and $D\!\!\!/$ $\left[s = \frac{1}{4\cdot 3}\right]$!

Idea: Use possible improvements of an eigenvalue estimate as a guide to the 'right' twistor spinor

Thm (twistor integral formula). Any spinor φ satisfies

$$\begin{aligned}\int_M \langle \not{D}^2\varphi,\varphi\rangle dM &= \frac{n}{n-1}\int_M \|P^s\varphi\|^2 dM + \frac{n}{4(n-1)}\int_M \mathrm{Scal}^g\|\varphi\|^2 dM \\ &+ \frac{n(n-5)}{8(n-3)^2}\|T\|^2\int \|\varphi\|^2 dM - \frac{n(n-4)}{4(n-3)^2}\int_M \langle T^2\varphi,\varphi\rangle dM,\end{aligned}$$

where $s = \frac{n-1}{4(n-3)}$.

Thm (twistor estimate). The first EV λ of $\not{D}^2$ satisfies $(n > 3)$

$$\lambda \geq \frac{n}{4(n-1)}\mathrm{Scal}^g_{\min} + \frac{n(n-5)}{8(n-3)^2}\|T\|^2 - \frac{n(n-4)}{4(n-3)^2}\max(\mu_1^2,\ldots,\mu_k^2),$$

where $\mu_1,\ldots,\mu_k$ are the eigenvalues of T, and "$=$" iff

- Scal^g is constant,
- ψ is a twistor spinor for $s_n = \frac{n-1}{4(n-3)}$,
- ψ lies in Σ_μ corresponding to the largest eigenvalue of T^2.

- reduces to Friedrich's estimate for $T \to 0$
- estimate is good for $\mathrm{Scal}^g_{\min}$ dominant (compared to $\|T\|^2$)

Ex. (M^6, g) $\mathrm{U}(3)$-mnfd of class $\mathcal{W}_3$ ("balanced"), $\mathrm{Stab}(T)$ abelian

Known: $\mu = 0, \pm\sqrt{2}\|T\|$, no ∇^c-parallel spinors

twistor estimate: $$\lambda \ \geq \ \frac{3}{10}\mathrm{Scal}^g_{\min} - \frac{7}{12}\|T\|^2$$

universal estimate: $$\lambda \ \geq \ \frac{1}{4}\mathrm{Scal}^g_{\min} - \frac{3}{8}\|T\|^2$$

- better than anything obtained by deformation

On the other hand:

Ex. (M^5, g) Sasaki: deformation technique yielded better estimates.

Twistor and Killing spinors with torsion

Thm (twistor eq). ψ is an s_n-twistor spinor $(P^{s_n}\psi = 0)$ iff

$$\nabla^c_X \psi + \frac{1}{n} X \cdot \not{D}\psi + \frac{1}{2(n-3)}(X \wedge T)\cdot\psi \;=\; 0,$$

Dfn. ψ is a Killing spinor with torsion if $\nabla^{s_n}_X \psi = \kappa X \cdot \psi$ for $s_n = \frac{n-1}{4(n-3)}$.

$$\Leftrightarrow \;\; \nabla^c \psi - \left[\kappa + \frac{\mu}{2(n-3)}\right] X \cdot \psi + \frac{1}{2(n-3)}(X \wedge T)\psi \;=\; 0.$$

In particular:

- ψ is a twistor spinor with torsion for the same value s_n
- κ satisfies a quadratic eq. linking it to curvature (but, in general, not Einstein)
- $\mathrm{Scal}^g =$ constant.

In general, this twistor equation cannot be reduced to a Killing equation.

. . . with one exception: $n = 6$

Thm. Assume ψ is a s_6-twistor spinor for some $\mu \neq 0$. Then:

- ψ is a $\not{D}$ eigenspinor with eigenvalue $\not{D}\psi = \frac{1}{3}\left[\mu - 4\frac{\|T\|^2}{\mu}\right]\psi$

- the twistor equation for s_6 is equivalent to the Killing equation $\nabla^s\psi = \lambda X \cdot \psi$ for the same value of s.

Ex. Manifolds with Killing spinors with torsion:

- Odd-dim. Heisenberg groups (naturally reductive!)

- Tanno deformations of arbitrary Einstein-Sasaki manifolds, for example $\mathrm{SO}(n+2)/\mathrm{SO}(n)$ (again naturally reductive!)