String theory and homotopy algebras

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Plan:

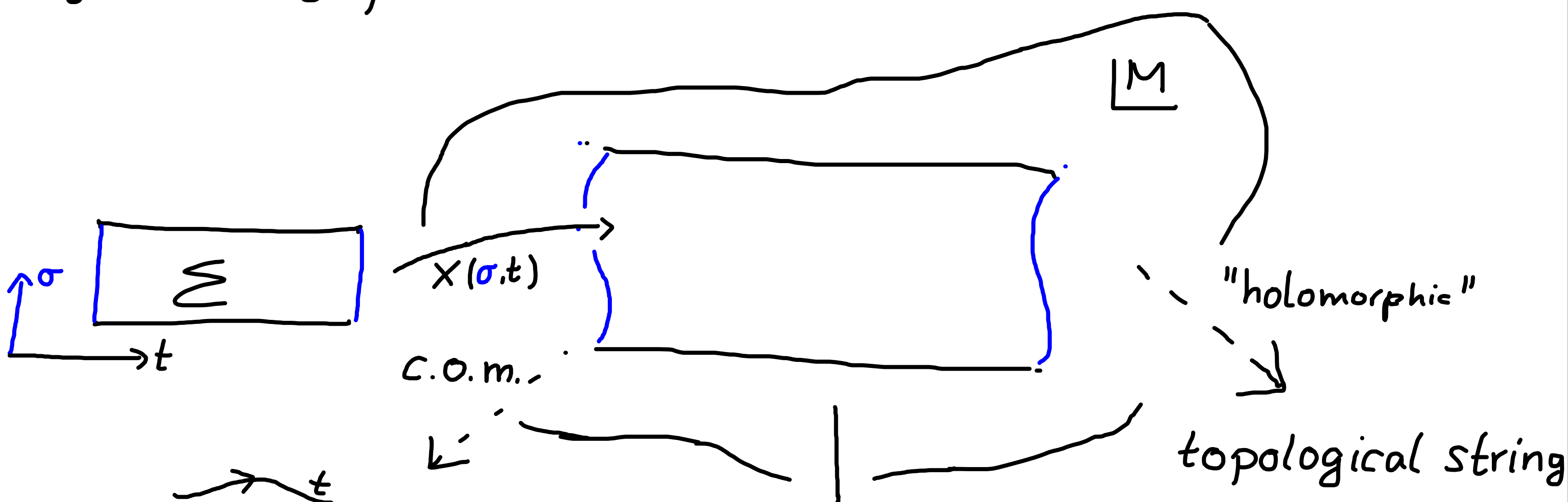
I
- Introduction string theory (open)
- Batalin-Vilkoviski action
- Homotopy (ass) algebras

II
- closed and open/closed strings
- Open-closed homotopy algebra (OCHA)

III
- Quantum BV action
- IBL_∞ algebras
- QOCHA

1) Open strings:

1+1 dim'l <u>world sheet</u>, Σ (surface) embedded in (M, g).
eg. $M = \mathbb{R}^n$, g pseudo Riemannian.



• **Dynamics:** pt. particle

geodesic motion:

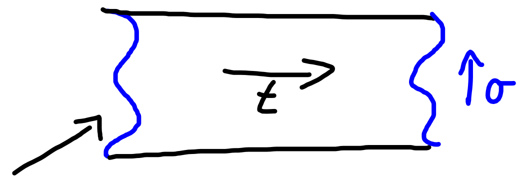
$$A = \int \left[\frac{1}{e}\, g(\dot{x},\dot{x}) + e\, m^2\right] dt$$

$$e = \sqrt{h_{tt}}$$

minimal area: $A = \int g(dx, \wedge * dx)$

$X : \Sigma \longrightarrow M$, (Σ, h) world sheet

Excitations:



Hilbert space: $\mathcal{H}_I \cong L^2(C^\infty(I,M))$, $\sigma \in I = [0,\pi]$, $\partial_\sigma X|_{\sigma=0,\pi} = 0$

time evolution: $U_t : \mathcal{H}_I \to \mathcal{H}_I$ defined formally through

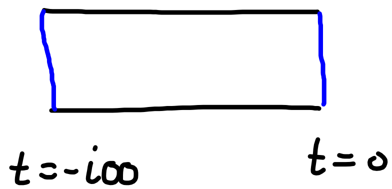
$$U_t[X_2, X_1] \overset{\text{form}}{=} \int D[X] D[h] \, e^{\frac{i}{\alpha'} A[X]}$$

$$X : I \times [0,t] \to M$$
$$X(\sigma, 0) = X_1(\sigma)$$
$$X(\sigma, t) = X_2(\sigma_2)$$

$\sim$ (monoidal) functor on cathegory of intervals to cath. of Hilbert spaces.

α': formal expansion parameter (dimension length^2)

(Wick) analytic contiuation: $t = i\tau$ $\quad \begin{cases} w = \sigma + t = \sigma + i\tau \\ \bar{w} = \sigma - t = \sigma - i\tau \end{cases}$



then for g = Euclidean metric

$$D[x]\, e^{-\frac{1}{\alpha'} A[h,X]} \overset{\text{form}}{=} \text{Gaußian measure}$$

Furthermore, if $\dim(M) = 26$ then the worldsheet metric, h enters only through its conformal structure.

Redundancy in $\int D[h]$: can be taken care of by BRST formalism:

__Illustration__: pt. particle $\qquad t \to t + \delta t \;, \delta t = \varepsilon(t)$ reparam.

$$\delta e = -(\varepsilon e)^{\cdot}$$

$$\int D[e]\, e^{-A[x,e]} \to \int D[e] D[B] D[b] D[c] \; e^{-A[x,e,B,b,c]}$$

$$A[x,e,B,b,c] = \int \left[\frac{1}{e} g(\dot{x},\dot{x}) + e m^2 + iB(e-1) - e b \dot{c}\right]$$

- $c(t)$ and $b(t)$ form an anticommutative algebra Λ

$$b \otimes c = - c \otimes b$$

Elimination of B enforces $e \equiv 1$. The resulting action $A[x,b,c]$ has a nilpotent symmetry: $\quad \delta_Q x = c\dot{x}$

$$\delta_Q b = (-\dot{x}^2 + m^2)$$

δ_Q generated by $Q = cH$, $H = (-\dot{x}^2 + m^2)$ (Hamiltonian)

$\therefore H = 0$ physical (on mass shell) condition.

<u>Def</u>:

$Q\psi = 0$, $\psi \in \mathcal{H}$ is the physical state condition.

$\psi \sim \psi + Q\varphi$, $\varphi \in \mathcal{H}$, ψ and $\psi + Q\varphi$ are gauge equivalent.

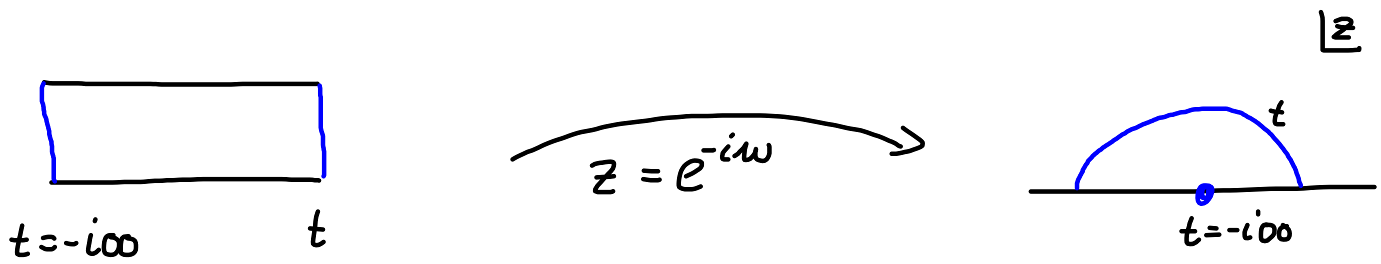
$\therefore Q^2 = 0$, δ_Q is a nilpotent symmetry.

The same procedure applies for the (CFT) of the string with the resulting Hilbert space as a <u>graded</u> vector space $H_I = L^2(C^\infty(I, M))$, M graded manifold, and

$Q = \sum_{n \in \mathbb{Z}} c_n L_{-n}$, $L_n \mathrel{\hat{=}} z^{n+1}\partial_z$ generate holomorpic (conf.) reparametrizations

$\therefore$ The space of physical states is $\mathcal{H}_{phys} = coh(Q)_{deg=1}$

Operator state correspondence:



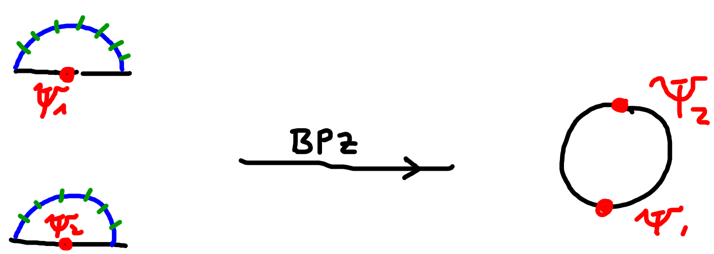
$$H_I \ni \quad \Psi[X] = \int_{X'(t,\sigma)=X(\sigma)} D[X'] \; e^{-A[X']} \, \Psi[X'](0) \,, \qquad (CFT)$$

radial ordering

with "local operator": $\quad \Psi[X'](0) = R[X', \partial X', \partial^2 X', \dots](0)$

(BPZ) inner product $\quad (\Psi_1 , \Psi_2)_{BPZ} = \lim_{z\to 0} \langle (I_* \Psi_1)(z) \;\; \Psi_2(z) \rangle_{CFT}$

(graded symm.) $\qquad \left(I(z) = -\frac{1}{z} \right)$

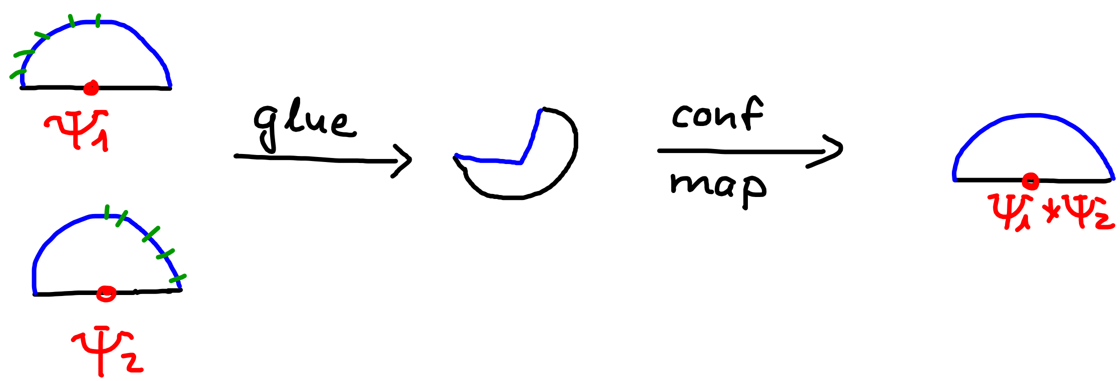


$\therefore$ Due to the $SL(2,\mathbb{R})$ isom. of the punctured disk the $[dc]$ – measure has 3 odd zero modes which we denote by $c_0, c_{\pm 1}$.

- The physical vacum is defined as $|I\rangle = c_1|0\rangle$.

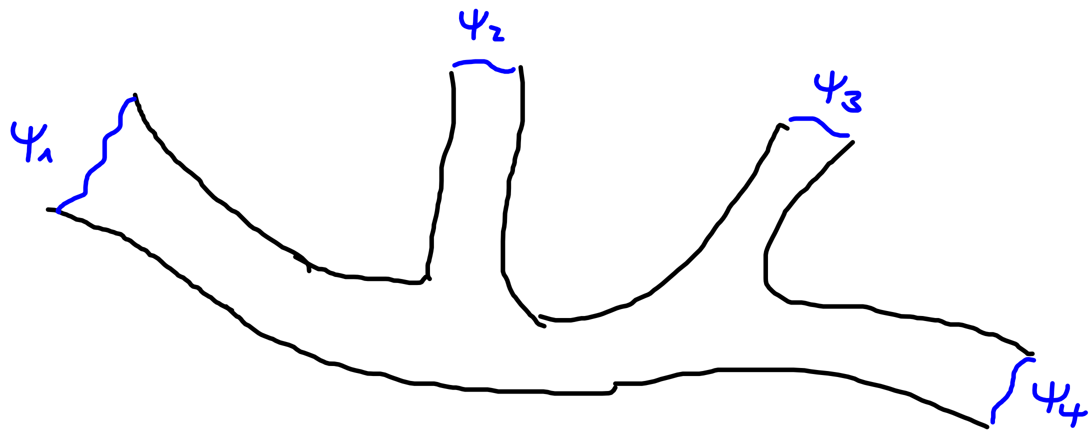
$$\Rightarrow \mathcal{H}_{phys} \simeq \text{operator } coh(Q)\big|_{\deg 0}$$

<u>Product</u> on $\mathcal{H}_I \otimes \mathcal{H}_I$:



$\therefore$ non-commutive but associative.

Interactions:



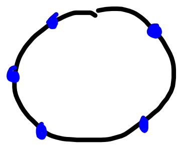
$\psi_i \in \mathcal{H}_{I_i}$

since there is no preferred interaction points we should integrate.

→ decomposition of the moduli space $\mathcal{M}$ of Riemann surfaces with boundaries ($\overset{\text{conf}}{\sim}$ disks with punctures) into vertices and strips (propagators).

Let $\hat{P}_n = \mathcal{M}_n$ together with a coordinate curve around each puncture.

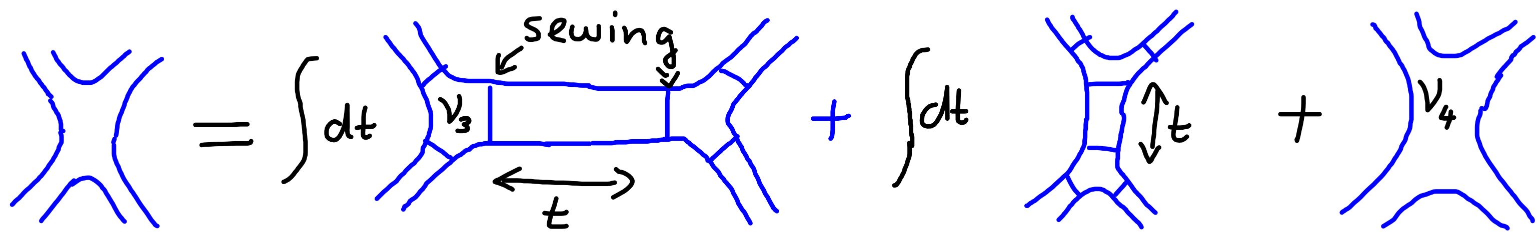
Sewing: 

 $= \mathcal{V}_{k=5} \in C^\bullet(\hat{\mathcal{P}}_n)$, chain complex with

$\deg = \operatorname{codim}\ (\dim \mathcal{M}_n - \dim \mathcal{V}_k)$

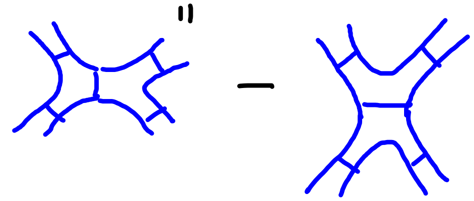
boundary op $\partial : C^k \to C^{k+1}$ $\deg(\partial) = 1$

<u>Decomposition</u>: $t \sim$ moduli



Consistency cond.: ($\mathcal{M}_4$ has no boundary)

$$\Longrightarrow \quad \partial \mathcal{V}_4 + \underbrace{(\mathcal{V}_3 , \mathcal{V}_3)}_{=} = 0 \qquad \boxed{\text{B-V equation}}$$



∴ These operations endow the set of vertices $\{\mathcal{V}_k\}$ with the structure of a BV-algebra.

∴ For disks with n punctures we can choose $\mathcal{V}_3 = pt., \partial\mathcal{V}_3 = 0$ and $(\mathcal{V}_3, \mathcal{V}_3) = 0 \Rightarrow \mathcal{V}_4 = \mathcal{V}_5 = \cdots = 0$.
However, other choices are possible (stubs).

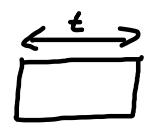
BV-action on $\mathcal{H}_I$: We can use (CFT) to realize a B-V structure on $\mathcal{H}_I$, more precisely on $\mathrm{Hom}(\mathcal{H}_I^{\otimes n}, \mathbb{C})$.

Schematically:

$$\partial \longleftrightarrow Q$$
$$(\cdot,\cdot) \longleftrightarrow (\cdot,\cdot)_{BPZ}$$
$$\nu_3 \longleftrightarrow *_{CFT}$$

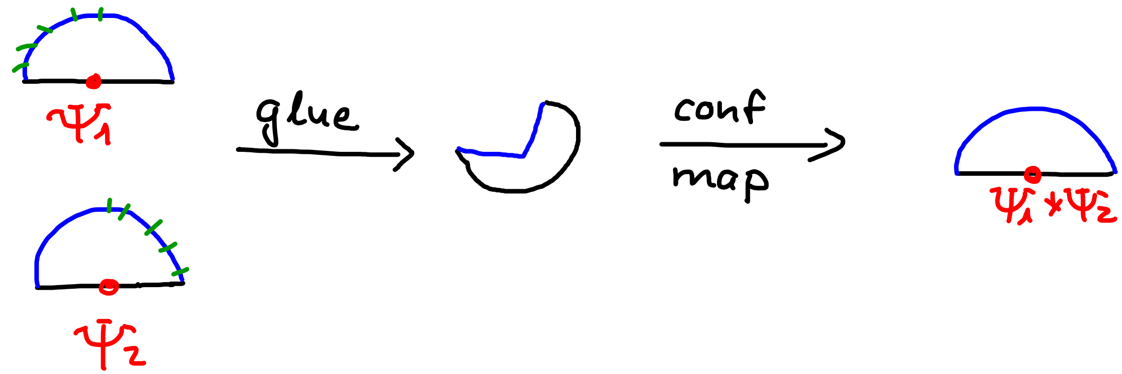
$\underline{\partial}$: Let Q^+ be a homotopy for Q on $\mathcal{H}_{phys}^{\perp}$:

$$QQ^+ + Q^+Q = 1 - \underset{\text{Proj. on } \mathcal{H}_{phys}}{P}$$

e.g. $Q^+ = b_0 L_0^{-1} = -b_0 \int_0^\infty e^{-tL_0}\, dt \;\hat{=}\; \int dt$ 

vertices:

$V_3 \xrightarrow{CFT}$



$\underline{(\ ,\ )}$: Let $\hat{m}_2(\Psi_1, \Psi_2) = \Psi_1 \overset{CFT}{*} \Psi_2$

Then $(V_3, V_3) = 0 \xrightarrow{CFT} \hat{m}_2(\Psi_1, \hat{m}_2(\Psi_2, \Psi_3))$

$$- \hat{m}_2(\hat{m}_2(\Psi_1, \hat{m}_2(\Psi_2, \Psi_3)) = 0$$

Thus: (CFT) provides a morphism between BV-algebras:

$$V_3\,,\ \partial\,,\ (\,,\,) \xrightarrow{CFT} m_2 = *\,,\ Q\,,\ \text{associator}$$

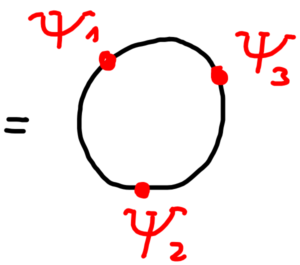
$\therefore\ (Q, *, \mathcal{H}_I)$ is a DGA (Diff. grad. alg.)

alternative decompositions of $\mathcal{M}_n$ with $V_k \neq 0$, $k > 3$ are mapped through CFT to homotopy associative alg's.

$$\hat{m}_2(\hat{m}_2(\cdot,\cdot),\cdot) - \hat{m}_2(\cdot,\hat{m}_2(\cdot,\cdot)) = Q\,\hat{m}_3(\cdot,\cdot,\cdot) + \hat{m}_3(Q\cdot,\cdot,\cdot) + \cdots$$

Thm: every consistent open string theory has the structure of a hom. ass. alg. (Gaberdiel & Zwiebach)

BV action:

Let $C_3(\Psi_1, \Psi_2, \Psi_3) =$ 

$= (\Psi_1 , \Psi_2 * \Psi_3)_{BPZ}$

then the decomposition of M_n derives from the action

$$S[\Psi] = \frac{1}{2}(\Psi, Q\,\Psi)_{BPZ} + \frac{1}{3} C_3(\Psi, \Psi, \Psi)$$

BV-action

∴ The standard form of S is obtained after a shift $(sA)_i = A_{i-1}$. Then, with $A_0 = s^{-1}\mathcal{H}_I$,

deg

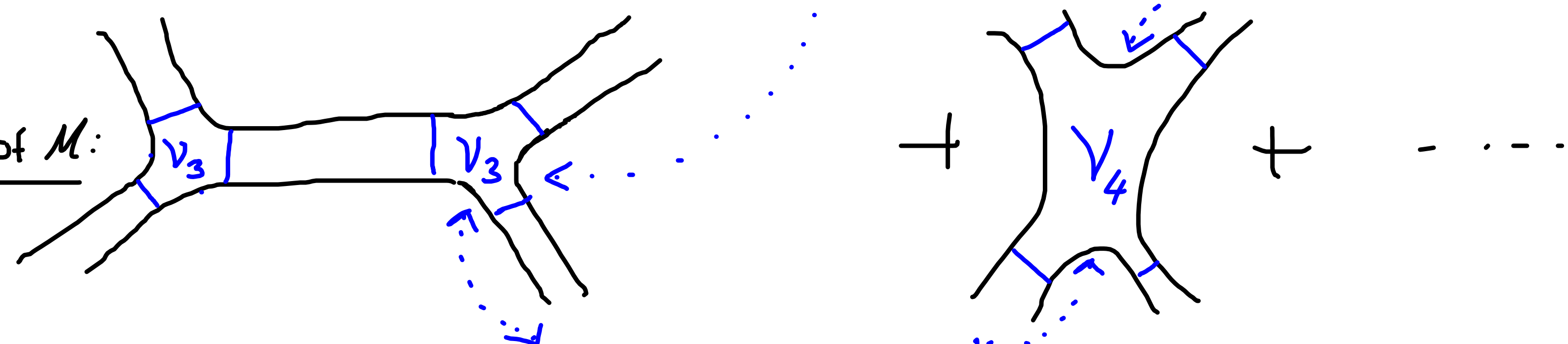
$$\omega(\cdot,\cdot) \equiv (\cdot,\cdot)_{BPZ} \circ (s \otimes s) : A_0 \otimes A_0 \longrightarrow \mathbb{C}$$

is an odd sympl. form of degree -1.

Summary: open string field theory

- action: $S[\Psi] = \frac{1}{2}\omega(\Psi, Q\Psi) + \frac{1}{3!}\omega(\Psi, \hat{m}_2(\Psi,\Psi)) + \frac{1}{4!}\omega(\Psi, \hat{m}_3(\Psi^{\otimes 3})) + \dots$

- dec. of $\mathcal{M}$:



$$m = \underset{\overset{\shortparallel}{Q}}{m_1} \otimes 1\!\!1 \otimes 1\!\!1 \cdots + \underset{\overset{\shortparallel}{s^{-1}\circ \hat{m}_2 \circ (s\otimes s)}}{m_2} \otimes 1\!\!1 \otimes 1\!\!1 \cdots + \underset{\overset{\shortparallel}{s^{-1}\circ \hat{m}_3 \circ (s\otimes s\otimes s)}}{m_3} \otimes 1\!\!1 \otimes 1\!\!1 + \dots$$

$$m : \underbrace{A_0 \otimes A_0 \otimes A_0 \otimes A_0 \cdots}_{\equiv TA_0} \longrightarrow A_0 \otimes A_0 \otimes A_0 \otimes A_0 \cdots$$

$\deg(m) = 1$

- B-V:

$$(S,S) = 0 \iff m^2 = [m,m] = 0 \quad \text{(plus cyclicity)}$$

↖ graded commutator

2) closed strings:

Let us classify possible inf. deformations of a given OSFT (ie. the corresponding A_∞ algebra). This is a cohomology problem:

$$[m+\delta m, m+\delta m] = \underbrace{[m,m]}_{=0 \text{ by ass.}} + 2[m,\delta m] + \cdots = 0$$

∴ The symplectic form ω establishes an isom. from the the cyclic maps $\{\hat{m}_i\}$ and the cyclic chain complex $CC^\bullet$:

$$CC^N \ni C_k(a_1,\cdots a_k) = \omega(a_k, \hat{m}_{k-1}(a_1,\ldots,a_{k-1}))$$

For simplicity we consider the cubic OSFT ($\mathcal{V}_{k\geq 4}=0$)

Then

$$\text{``}[m,\cdot]\text{''} = d_H = \left(\delta - (-1)^k Q\right) \,,$$

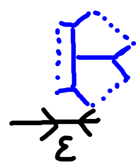
$$Q : CC^k \to CC^k$$

$$C^k(a_1,\cdots,a_k) \longmapsto \sum_{i=1}^{k} (-1)^{|a_1|+\cdots |a_k|} C^k(a_1,\cdots, Qa_i,\cdots a_k)$$

$$\delta : CC^k \to CC^{k+1}$$

$$(\delta C^k)(a_1,\dots,a_{k+1}) = C_k(*a_1,a_2,\cdots,a_k) + \sum_{i=1}^{k-1}(-1)^i C^k(a_1,\dots,a_i * a_{i+1},\cdots,a_{k+1})$$

Non-trivial, consistent def'n of OSFT $\in$ coh(d_H)

Eg. strips: $\mathcal{V}_3 =$ 

$\xrightarrow{CFT} \Delta C^3(\cdot,\cdot,\cdot) = C^3(L_0\cdot,\cdot,\cdot) + \text{cycl.}$

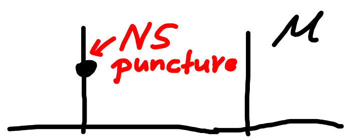
$$= (QC^3)(b\cdot,\cdot,\cdot) + \text{cycl.}$$

.... strips are d_H - exact.

• Super string. super conformal invariance lead extra ghosts (β, γ)

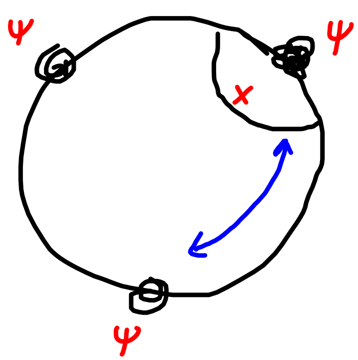
Bosonization: $\beta = \partial\xi\, e^{-\phi}$, $\gamma = \eta\, e^{\phi}$, $[\eta, \xi] = 1$, $\delta(\gamma) = e^{-\phi}$

NS-sector ≙ decomosition of moduli space, $\mathcal{M}$ of super Riemann surfaces



$\xrightarrow{\text{integrate } \theta}$ picture changing op.

$X = Q\xi$ (large Hilbert space, $[\eta, \xi] = 1$)

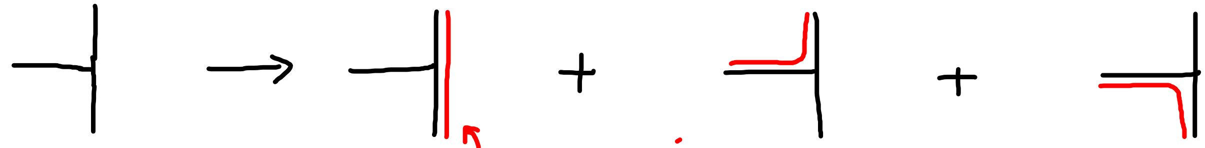


ok on-shell, but not off-shell.

↯ cyclicity

(w/ T. Erler, S. Konopka)

ΔV_3 :



$\oint \frac{1}{z} X(z)$ in local coord,

or, in terms of the A_∞ maps:

$$\Delta\hat{m}_2(\cdot,\cdot) = \frac{1}{3}\left(X\hat{m}_2(\cdot,\cdot) + \hat{m}_2(X\cdot,\cdot) + \hat{m}_2(\cdot, X\cdot)\right)$$

$\therefore\ \Delta\hat{m}_2 = [Q,\hat{\mu}_2]$ with $\hat{\mu}_2$ sol'n of $[\eta,\hat{\mu}_2] = \hat{m}_2$:

$$\hat{\mu}_2(\cdot,\cdot) = \frac{1}{3}\left(\xi\,\hat{m}_2(\cdot,\cdot) - \hat{m}_2(\xi\cdot,\cdot) - (-1)^{|\cdot|}\hat{m}_2(\cdot,\xi\cdot)\right)$$

Thus, adding picture is a d_H trivial operation.

$M_2 = m_2 + \Delta m_2$ is the cubic superstring vertex at next order: $\hat{M}_3 = \frac{1}{2}\left(\hat{M}_2(\cdot, \hat{\mu}_2(\cdot \cdot)) + \text{perm.}\right) + \frac{1}{2}[Q, \hat{\mu}_3]$

or, $M_3 = \frac{1}{2}\left([M_2, \mu_2] + [M_1, \mu_3]\right) = \frac{1}{2}[M, \mu]\big|_3$

$$M = M_2 \otimes 1\!\!1 \otimes 1\!\!1 \cdots + M_3 \otimes 1\!\!1 \otimes 1\!\!1 \cdots + \cdots, \quad \mu = \mu_2 \otimes 1\!\!1 \otimes 1\!\!1 + \cdots$$

all orders:

Let $M(t) = \sum_{n=0}^{\infty} t^n M_{n+1}$ $\quad m(t) = \sum_{n=0}^{\infty} t^n m_{n+2}$

$$\hat{\mu}(t) = \sum_{n=0}^{\infty} t^n \hat{\mu}_{n+2}$$

$\rightarrow$ Recurrence relations:

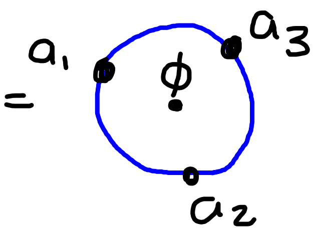
$$\frac{d}{dt}\hat{M}(t) = [\hat{M}(t), \hat{\mu}(t)]\ ; \qquad \frac{d}{dt}\hat{m}(t) = [\hat{m}(t), \hat{\mu}(t)]$$

Thus, the open superstring is (L_∞ gauge) equiv. to the bosonic string

see later

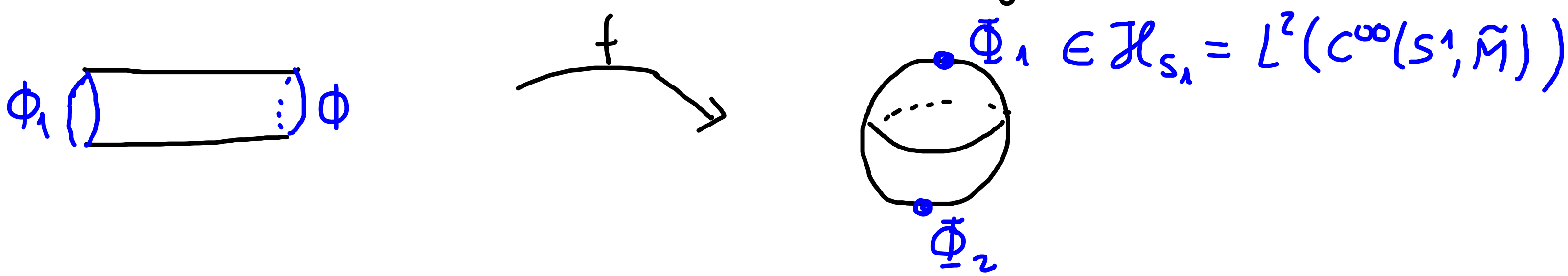
nontrivial def'ns:

It turns out that the only non-trivial A_∞-def'ns are obtained by inserting a closed string puncture:

$\Delta_\phi C^3(a_1, a_2, a_3) =$ 

with $(Q\Delta_\phi C^3) = \Delta_{Q\phi} C^3$

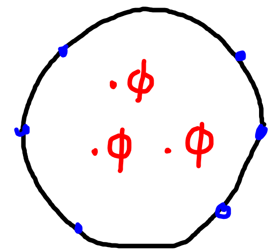
$\therefore$ operator state corr. for closed strings:



Thm: $\mathrm{coh}(d_H) \simeq \mathrm{coh}(Q_c)\big|_{\deg = 2} =$ physical closed string states.

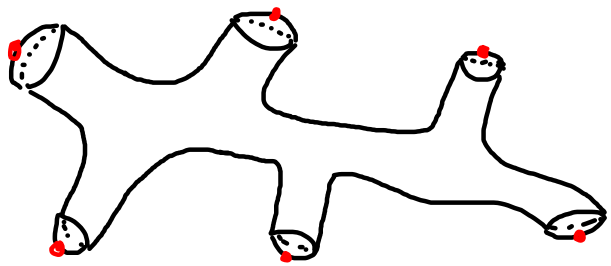
w/ N. Moeller

finite def'ns:



... seems hard to construct from open string world sheet perspective ...

- closed string field theory ~ decomposition of the moduli space of genus zero Riemann surfaces with punctures,



leads again to a BV-equation, $\partial V_4 + (V_3, V_3) = 0$ etc.
but $V_4 = 0$ is not a solution.

$$\partial \longmapsto Q_c$$

world sheet CFT: $V_k \longmapsto \ell_k(\cdot . \cdot)$ graded symmetric

$$(\cdot,\cdot) \longmapsto \text{Jacobiator}$$

with B-V action:

$$S[\phi] = \frac{1}{2}\omega_c(\phi, Q_c\phi) + \frac{1}{3!}\omega_c(\phi, \ell_2(\phi,\phi)) + \frac{1}{4!}\omega_c(\phi, \ell_3(\phi^{\otimes 3})) + \cdots$$

and

$$\underset{\substack{\shortparallel \\ Q_c}}{L = \ell_1} \otimes 1\!\!1 \otimes 1\!\!1 \cdots + \ell_2 \otimes 1\!\!1 \otimes 1\!\!1 \cdots + \ell_3 \otimes 1\!\!1 \otimes 1\!\!1 + \cdots$$

$$L : \underbrace{A_c \wedge A_c \wedge A_c \wedge A_c \cdots}_{\equiv SA_0} \longrightarrow A_c \wedge A_0 \wedge A_c \wedge A_c \cdots$$

$\deg(L) = 1$

• <u>B-V</u>:

$$(S, S) = 0 \iff [L, L] = 0$$

↖ graded commutator

∴ Let $d_c = [L, \cdot]$. Then $\mathrm{coh}(d_c) = \emptyset$

Since inf. def'ns of OSFT are isomorphic to coh(Q_c) it seems natural to claim that finite def'ns correspond to Maurer-Cartan elements of L, $L(e^{\wedge\phi}) = 0$. Let us construct this morphism.

For this purpose it is usefull to view L_∞-maps as coderivations:

Consider a graded coalgebra, A with comultiplication

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow & & \downarrow{\scriptstyle \Delta\otimes 1} \\ A \otimes A & \xrightarrow{1\otimes\Delta} & A\otimes A\otimes A \end{array}$$

A coderivation is a map, $L: A \longrightarrow A$ with Leibnitz rule

$$\begin{tikzcd} A \arrow[r, "L"] \arrow[d, "\Delta"'] & A \arrow[d, "\Delta"'] \\ A \wedge A \arrow[r, "L\otimes 1 + 1\otimes L"] & A \wedge A \end{tikzcd}$$

L is a differential if $[L,L] = 0$, $\deg(L) = 1$

In our case A is the cotensor algebra $SA_c = \sum_{n=0}^{\infty} A_c^{\otimes n} \big/ S$
where S is the graded symmetrization:

$$S(a\otimes b) = a\otimes b - (-1)^{|a||b|}\, b\otimes a$$

and $\Delta(a_1 \wedge \cdots \wedge a_n) = \sum_{i=1}^{n} \sum_{\sigma}{}' (a_{\sigma_1} \wedge a_{\sigma_2} \cdots \wedge a_{\sigma_i}) \otimes (a_{\sigma_{i+1}} \wedge \cdots \wedge a_{\sigma_n})$

where $\sum_{\sigma}{}'$ = sum over perm. s.t. $\sigma_1 < \sigma_2 \cdots < \sigma_i$, $\sigma_{i+1} < \sigma_{i+2} < \cdots < \sigma_n$

$\therefore$ Coder$(SA_c) \cong Hom(SA_c, A_c)$: $\ell = \pi_1 \circ L \in Hom(SA_c, A_c)$

$\ell_n = \ell \circ i_n \qquad L \circ i_n = \sum_{i+j=n} \sum_{\sigma}{}' (\ell_i \wedge 1^{\wedge j}) \circ \sigma$

$\therefore$ An L_∞ alg. is a coderivation L of degree 1 with $[L,L] = 0$

A *Maurer Cartan element* $c \in A_c$ is a constant morphism, ie.

$$L(e^c) = 0 \ , \quad |c| = 0 \ , \quad e^c = \sum_{n=0}^{\infty} \frac{1}{n!} c^{\wedge n} \ , \quad \Delta(e^c) = e^c \otimes e^c$$

L_∞ morphism: (A, L) and (A', L') L_∞ algebras. Then $\mathcal{F} \in Morp(A, A')$ is an L_∞ morphism if

$|F| = 1$

$$\begin{cases} \Delta \circ \mathcal{F} = \mathcal{F} \otimes \tilde{\mathcal{F}} \circ \Delta \\ \mathcal{F} \circ L = L' \circ \mathcal{F} \end{cases}$$

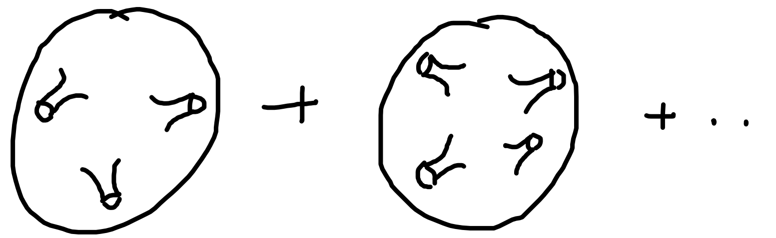
$\therefore$ $\mathcal{F}$ is dermined by $Hom(SA, A) \ni f = \pi_1 \circ \mathcal{F}$ through $\mathcal{F} = \sum_{n=0}^{\infty} \frac{1}{n!} f^{\wedge n} \circ \Delta_n$

Δ_n = n-fold comultiplication.

$\therefore$ In a similar fashion, the set of open string vertices, m_k (disks with k puctures on $\partial\Sigma$) define coderivations on TA_0. The composition $m_i : TA_0 \to TA_0$, $m_1 \circ m_2 = \frac{1}{2}[m_1, m_2]_0$ makes $Coder(TA_0)$ into a graded Lie algebra.

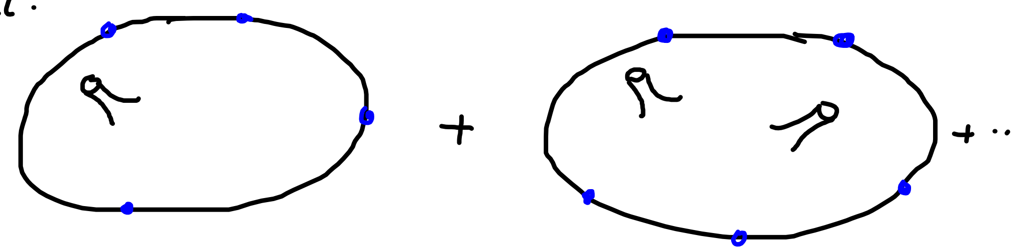
after this excursion let us now get back to string world sheets:

closed string:

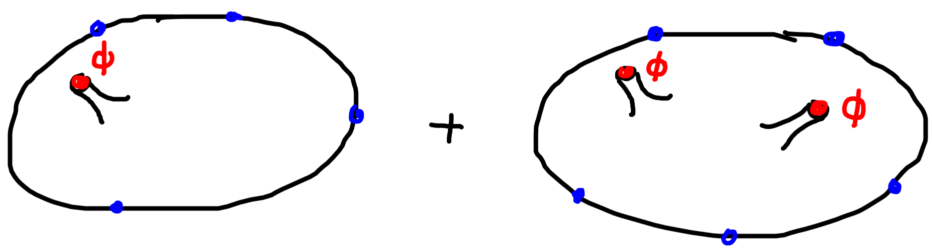


$+ \dots \to L_\infty$ maps

open-closed:

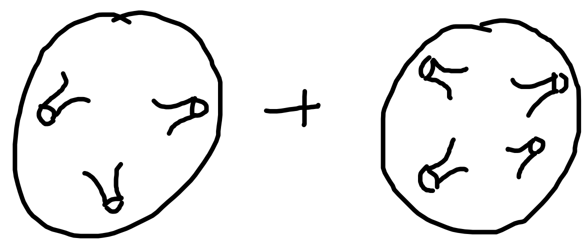


open:



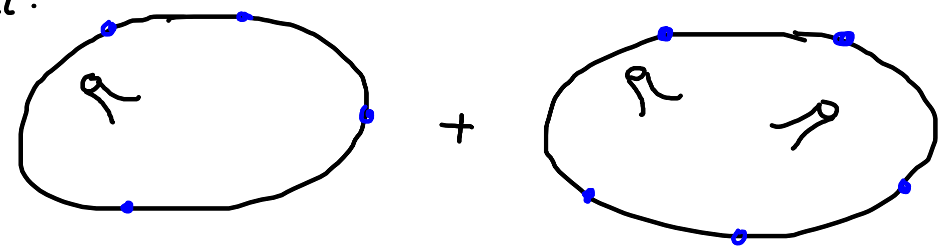
$+ \dots \in \mathrm{Coder}(TA_0)$

closed string :



$+ \dots \to L_\infty$ maps

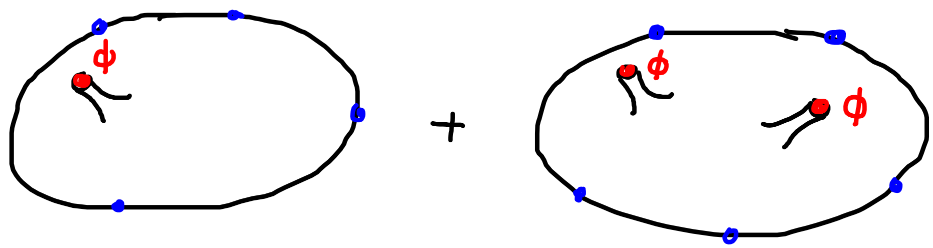
open-closed :



$+ \dots = \mathcal{F}$

L_∞-morphism

open :



$+ \dots \in \mathrm{Coder}(TA_o)$

BV-action :

$$S(\psi, \phi) = \omega_c \circ L(e^{\phi}) + (\omega_o \circ f)(e^{\phi}; e^{\psi})$$ with $f \in \mathrm{Hom}(SA_c, \mathbb{R}) \otimes \mathrm{Hom}(TA_o, A_o)$

$S(\Psi,\phi)$ satisfies the BV equation $\Rightarrow$ $\mathcal{F} = \sum_{n=0}^{\infty} \frac{1}{n!} f^{\wedge n} \circ \Delta_n$

is an L_∞ morphism:

$$(SA_c, L) \xrightarrow{\mathcal{F}} (S(\mathrm{Coder}(TA_o), [\,,\,])$$

open-closed homotopy algebra (Kajiura & Stasheff)

We have already shown that $\mathcal{F}$ is a quasi isomorphism:

$$\mathrm{Coh}(\ell_1 = Q_c) \cong \mathrm{coh}(d_H = [m, \cdot])$$

Then, it follows (Kontsevich) that

$$\mathcal{M}(A, L) \cong \mathcal{M}(\mathrm{Coder}(TA_o), [\,,\,])$$

mod. spaces of Maurer-Cartan elements

i.e. inequivalent OSFT $\cong$ closed string backgrounds

<u>Appendum</u>: rep. consistent OSFT realizes an A_∞ structure

$$: T A_0 \longrightarrow TA_0 \ , \ m^2 = \frac{1}{2}[m, m] = 0$$

$$\bigoplus_{n=0}^{\infty} A_0^{\otimes n} \quad \underline{\rfloor}$$

more generally let: $m \in \mathrm{coder}(TA_0)$, $\deg(m) = 1$, then the composition, $m_1 \otimes m_2 \to m_1 \circ m_2 + m_2 \circ m_1$ defines a Lie bracket on $\mathrm{coder}(TA_0)$ (= set of open string vertices)

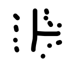
- the set of open-closed vertices defines an L_∞-morphism
  $$(A_c, L) \xrightarrow{L_\infty \text{ morph}} (S(\mathrm{coder}(TA_0)), [\ ,\ ])$$

$$\mathcal{M}(A, L) \cong \mathcal{M}(\mathrm{Coder}(TA_0), [\ ,\ ])$$

mod. spaces of Maurer-Cartan elements = set of MC elements / L_∞ g.t.

i.e. inequivalent OSFT $\cong$ closed string backgrounds

Examples of L_∞ g.t.:

1) adding strips 

2) backgrond shifts: $\Psi \mapsto \Psi + \Psi_0$, $m(e^{\Psi_0}) = m_1(\Psi_0) + m_2(\Psi_0, \Psi_0) + \dots = 0$

3) adding picture (integrate out odd moduli) to the vertices. (superstring)

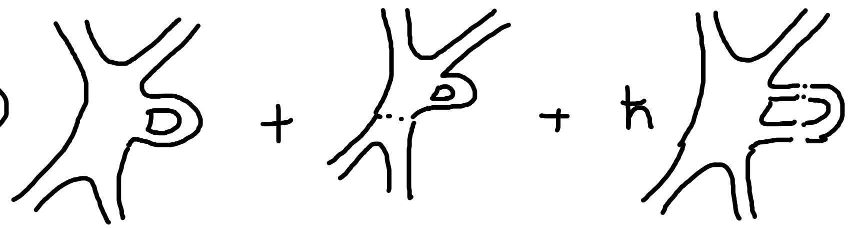
$\therefore$ 1) Leaves correlation fn's on $coh(Q_0)$ (=S-matrix) inv.

2) " " " " " not inv. (because Q_0 varies to $Q_0 + m_2(\Psi_0, \cdot)$)

3) " " " " " not inv. (because it is a large g.t.)

$\to$ generic L_∞ g.t. are physical!

Quantisation: allow for world sheet with higher genus.

QBV: ∂  $= 0$ e.t.c

$$\partial V_4 + (V_3, V_3) + \hbar \Delta V_6 = 0 \qquad ''$$

$$\Big\downarrow \text{(CFT)}$$

Action: $$S[\psi,\phi] = \sum_{g=0}^{\infty} \hbar^{2g-1} (\omega_c \circ \ell^g)(e^{\hbar^{1/2}\phi})$$

$$+ \sum_{b=1}^{\infty}\sum_{g=0}^{\infty} \hbar^{2g+b-1} (\omega_o^{\otimes b} \circ f^{gb})(e^{\hbar^{1/2}\phi}; \underbrace{e^{\psi}, \dots, e^{\psi}}_{b\text{-times}})$$

QBV: $$(S,S) + \hbar \Delta S = (\omega^{-1})^{i\bar{j}} \partial_i S \partial_{\bar{j}} S + \hbar (\omega^{-1})^{i\bar{j}} \partial_i \partial_{\bar{j}} S = 0$$

Smpl. form

In terms of coderivations (M. Markl):

order of the coder. $(= g + \#\text{outputs})$

lift of ω^{-1} to a coder^2

$$= \bigoplus_{n=1}^{\infty} \hbar^n \, \mathrm{Coder}^n(SA)$$

Let $\mathcal{L} = \sum_g \hbar^g L^g + \hbar D(\omega_c^{-1}) \in \mathcal{C}\mathrm{odes}(SA_c, \hbar)$

$$L^g = \ell_1^g \otimes \mathbb{1} \otimes \mathbb{1} \cdots + \ell_2^g \otimes \mathbb{1} \otimes \mathbb{1} \cdots + \mathbb{1} \otimes \ell_3^g \otimes \mathbb{1} \cdots +$$

With this, the QBV-equation becomes $[\mathcal{L}, \mathcal{L}] = 0.$ (+cycl.)

$\therefore$ Two equivalent interpretations:

$D(\omega^{-1})$ is induced by the repr. on A_c
↳ loop homotopy alg. (Markl)

$D(\omega^{-1})$ intrinsic part of the algebra
↳ IBL_∞ algebra (Cieliebak, Fukaya, Latschev)

Ex: <u>IBLie</u>: $\mathfrak{h} = L^1 + \hbar L^2$:

$|\mathfrak{h}| = 1$

$$\ell^1 = \pi_1 \circ L^1 : SA \to A$$
$$d = \ell^1 \circ i_1 : A \to A$$
$$[\,,\,] = \ell^1 \circ i_2 : A \wedge A \to A$$
$$\delta = \pi_2 \circ L^2 \circ i_1 : A \to A \wedge A$$

$[\mathfrak{h}, \mathfrak{h}] = 0 \Rightarrow \quad d^2 = 0 \quad$ (diff.)

$d \circ [\,,\,] + [\,,\,] \circ (d \wedge 1 + 1 \wedge d) = 0$ (d der. of $[\,,\,]$)

$(d \wedge 1 + 1 \wedge d) \circ \delta + \delta \circ d = 0$ (d coder for δ)

$\sum_\sigma{}' [\,,\,] \circ ([\,,\,] \wedge 1) \circ \sigma = 0$ ($[\,,\,]$ Jacobi)

$(\delta \wedge 1 + 1 \wedge \delta) \circ \delta = 0$ (δ, co-Jacobi)

$\sum_\sigma{}' ([\,,\,] \wedge 1) \circ \sigma \circ (\delta \wedge 1 + 1 \wedge \delta) + \delta \circ [\,,\,] = 0$ (compat. of δ and $[\,,\,]$)

$[\,,\,] \circ \delta = 0$ (δ and $[\,,\,]$ in involution)

Involutive Lie bialgebra

Applications: R-matrix (intg. models), quantum open string (see later)

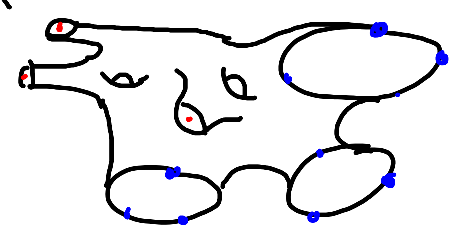
IBL$_\infty$ morphism: $(A, \mathfrak{h})$ and $(A', \mathfrak{h}')$ IBL$_\infty$ algebras. Then $\mathfrak{F} \in Morp(A, A')$ is an L_∞ morphism if

$|\mathfrak{F}| = 1$

$$\begin{cases} \Delta \circ \tilde{\mathfrak{F}} = \cancel{\mathfrak{F}} \otimes \tilde{\mathfrak{F}} \circ \Delta \\ \mathfrak{F} \circ \mathfrak{h} = \mathfrak{h}' \circ \mathfrak{F} \end{cases} \longrightarrow \mathfrak{F} = \sum_{n=0}^{\infty} \frac{1}{n!} \mathfrak{f}^{\wedge n} \circ \Delta_n$$

(cf. Cieliebak, Fukaya, Latschev)

$$\mathcal{F} = \sum_{n=1}^{\infty} \hbar^{n+g-1} F^{n,g}, \quad F^{n,g} : SA \longrightarrow (A')^{\wedge n-g}$$



Quantum open-closed homotopy algebra (w/. K. Münster)

$$(A_c, \mathfrak{h}_c) \xrightarrow{\mathfrak{F}} \left(\mathrm{Coder}^{gcl}(TA_o), \mathfrak{h}_o \right)$$

$$\therefore\ h' = [\,,\,] + \hbar\delta \quad : SCoder(TA_0) \longrightarrow TA_0$$

$$m + m \otimes m \longmapsto [m, m] + \delta(m)$$

$$\bigcirc \longmapsto \circledcirc$$

Thus the QBV algebra on the vector space of open string vertices is that of a IBLie algebra.

<u>Maurer-Cartan elements:</u> (cf. Cieliebak, Fukaya, Latschev)

$$\mathcal{M}(A, L) \cong \mathcal{M}(Coder(TA_0), [\,,\,] + \hbar\delta)$$

mod. spaces of Maurer-Cartan elements

we conclude that to each quantum consistent OSFT there should be a solution of $h(e^{\Phi}) = 0$ with $\Phi \in A_c + A_c \wedge A_c + \cdots$

With $\underline{\Phi} = \phi + d_i \wedge d^i$ ($\{d_i\}$ basis of A_c and d^i the dual basis w.r.t. ω_c)

we get $\quad Q_c\phi + \ell_2(\phi,\phi) + \hbar\,\ell_2(d_i,d^i) + \cdots = 0$

$$\underbrace{(Q_c d_i)\wedge d^i + \ell_2(\phi,d_i)\wedge d^i}_{= Q_c[\phi]\, d_i \wedge d^i} + \omega_c^{-1} = 0 \qquad | \circ \omega_c$$

$$\rightarrow \quad Q_c[\phi] \circ f + f \circ Q_c[\phi] = 1\,, \quad f = d_i \wedge d^i \circ \omega_c$$

$\Rightarrow$ $\mathrm{coh}(Q_c[\phi]) = \emptyset$ unless ω_c is degenerate !

i.e. non-existence of QOSFT $\Longleftrightarrow$ no solution of closed QM-C eqn.

Counter example: Topological string: $N=2$ superstring (= super Riemann surface with two odd coordinates)

Translation inv. in odd directions gives rise to two odd generators:

$G_\pm \in \Gamma \to \Sigma$ (spinor bdle). $[J, G_\pm] = \pm G_\pm$, L_n, $\bar{L}_n$

"topological twist": $L_n \mapsto L_n + \frac{1}{2} n J_n \Rightarrow$ $G_+ \in \otimes^{2,0} T\Sigma \longrightarrow b$

$G_- \in \mathrm{Hom}(\mathcal{H}_0, \mathcal{H}_0) \longrightarrow Q$

$\therefore$ $\mathrm{coh}(b) = \mathrm{coh}(G_+) = \{\text{phys. states}\} \Rightarrow \{c, b\} = 1 - P$

$\longrightarrow$ ω_c is degenerate on shell.

$\to$ $Q_c \cdot f + f \circ Q_c = 1 - P$ Propagator equation.

Conclusion: For the top. string the QMC equation has a solution $\Rightarrow$ QOSFT is well defined.

Summary:

Homotopy ass (or Lie) algebras:

- elucidat the algebraic structure of string theory.
- allow to identify the closed string coh. in open string field theory
- simplify the construction of super string field theory (decomp. of super moduli space.