Algebraic Structures Arising in String Topology

Kai Cieliebak,
joint work with Kenji Fukaya,
Janko Latschev and Evgeny Volkov

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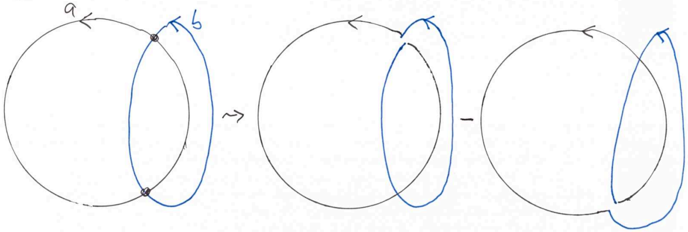
1. String topology

String topology of a surface

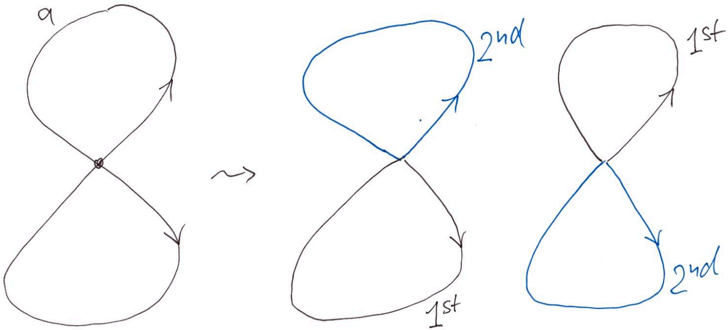
M closed oriented surface of genus $g \geq 2$.

$$\begin{aligned} \Sigma &:= \{\text{unparametrized noncontractible loops on } M\} \\ &=: \text{string space of } M \\ H_0(\Sigma) &:= \mathbb{R}\{\text{connected components of } \Sigma\} \\ &\cong \mathbb{R}\{\text{isotopy classes of unparametrized} \\ &\qquad\qquad \text{noncontractible loops on } M\} \\ &\cong \mathbb{R}\{\text{unparametrized closed geodesics on } M\} \\ &\ni a, b \end{aligned}$$

The Goldman bracket (1986) $\mu(a, b)$



The Turaev cobracket (1991) $\delta(a)$



Involutive Lie bialgebras

Proposition. $H_0(\Sigma)$ with the operations

$$\mu = [\ ,\ ] : H_0(\Sigma) \otimes H_0(\Sigma) \to H_0(\Sigma)$$

and

$$\delta : H_0(\Sigma) \to H_0(\Sigma) \otimes H_0(\Sigma)$$

is an **involutive Lie bialgebra**.

Definition. A **Lie bracket** on a vector space V is a linear map

$$\mu = [\ ,\ ] : V \otimes V \to V$$

satisfying the following two properties:

(skew-symmetry) $\mu(\mathbb{1} + \sigma) = 0 : V^{\otimes 2} \to V$;
(Jacobi identity) $\mu(\mathbb{1} \otimes \mu)(\mathbb{1} + \tau + \tau^2) = 0 : V^{\otimes 3} \to V$,

with the permutations

$$\begin{aligned} \sigma : V \otimes V \to V \otimes V, &\qquad a \otimes b \mapsto b \otimes a, \\ \tau : V \otimes V \otimes V \to V \otimes V \otimes V, &\qquad a \otimes b \otimes c \mapsto c \otimes a \otimes b. \end{aligned}$$

Involutive Lie bialgebras

Dually, a **Lie cobracket** on V is a linear map

$$\delta : V \to V \otimes V$$

satisfying the following two properties:

(skew-symmetry) $(\mathbb{1} + \sigma)\delta = 0 : V \to V^{\otimes 2}$;
(co-Jacobi identity) $(\mathbb{1} + \tau + \tau^2)(\mathbb{1} \otimes \delta)\delta = 0 : V \to V^{\otimes 3}$.

Involutive Lie bialgebras

A **Lie bialgebra structure** on V is a pair (μ, δ) of a Lie bracket and a Lie cobracket (both of degree 0) satisfying

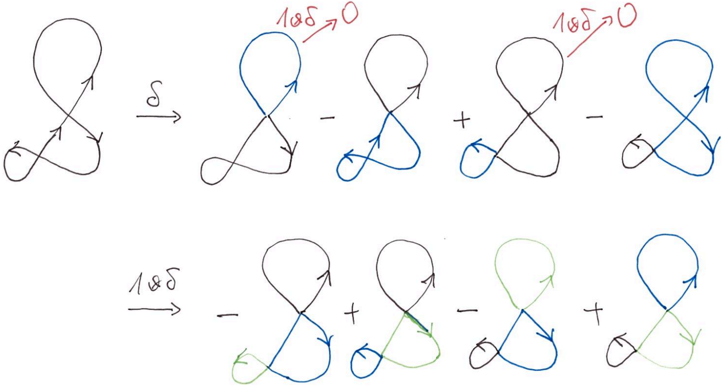
(compatibility)
$\delta\mu = (\mathbb{1} \otimes \mu)(\mathbb{1} + \tau^2)(\delta \otimes \mathbb{1}) + (\mu \otimes \mathbb{1})(\mathbb{1} + \tau)(\mathbb{1} \otimes \delta) : V^{\otimes 2} \to V^{\otimes 2}$.

An **involutive Lie bialgebra (IBL)** structure (μ, δ) is a Lie bialgebra structure which in addition satisfies

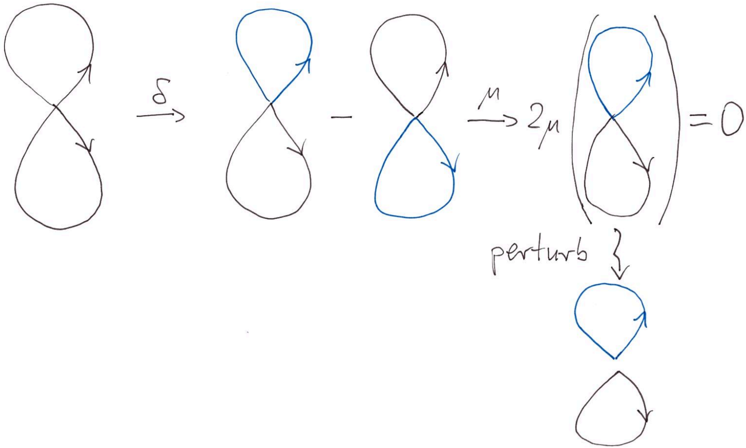
(involutivity) $\mu\delta = 0 : V \to V$.

Proof of co-Jacobi

Cyclic permutations give 12 terms that cancel pairwise:
(black = 1st, blue = 2nd, green = 3rd)



Proof of involutivity



The Chas-Sullivan operations (1999)

Generalization to families of strings in higher dimensional manifolds.

M closed oriented manifold of arbitrary dimension n.

$$\begin{aligned} \Sigma &:= \{\text{unparametrized loops on } M\} \\ &:= \{\text{piecewise smooth maps } S^1 \to M\}/S^1 \\ &=: \text{string space of } M \\ H_i(\Sigma) &= i\text{-th homology of } \Sigma \text{ with } \mathbb{R}\text{-coefficients,} \\ &\qquad \text{modulo the constant strings} \end{aligned}$$

The Chas-Sullivan operations (1999)

Consider two smooth chains

$$a : K_a \to \Sigma, \quad b : K_b \to \Sigma,$$

K_a, K_b manifolds with corners of dimensions i, j. If the evaluation map

$$\mathrm{ev} : S^1 \times S^1 \times K_a \times K_b \to M \times M, \quad (s, t, x, y) \mapsto \Big(a(x)(s), b(y)(t) \Big)$$

is transverse to the diagonal $\Delta \subset M \times M$, then

$$K_{\mu(a,b)} := \mathrm{ev}^{-1}(\Delta)$$

is a manifold with corners of dimension $i + j + 2 - n$.
Concatenation of strings yields a new smooth chain

$$\mu(a, b) : K_{\mu(a,b)} \to \Sigma, \quad (s, t, x, y) \mapsto a(x)_s \#_t b(y)$$

The Chas-Sullivan operations (1999)

The (partially defined) operations on chains

$$\begin{aligned}\mu &: C_i(\Sigma)\otimes C_j(\Sigma)\to C_{i+j+2-n}(\Sigma),\\ \delta &: C_k(\Sigma)\to C_{k+2-n}(\Sigma\times\Sigma)\end{aligned}$$

induce operations on homology

$$\mu : H_i(\Sigma)\otimes H_j(\Sigma)\to H_{i+j+2-n}(\Sigma),$$

$$\delta : H_k(\Sigma)\to H_{k+2-n}(\Sigma\times\Sigma)\overset{\text{Künneth}}{\cong}\bigoplus_{i+j=k+2-n} H_i(\Sigma)\otimes H_j(\Sigma)$$

The Chas-Sullivan operations (1999)

Theorem (Chas-Sullivan). The **string homology**

$$\mathbf{H} := \bigoplus_{i=2-n}^{\infty} \mathbf{H}_i, \qquad \mathbf{H}_i := H_{i+n-2}(\Sigma, \mathrm{const})$$

with the operations

$$\mu : \mathbf{H}_i \otimes \mathbf{H}_j \to \mathbf{H}_{i+j},$$
$$\delta : \mathbf{H}_k \to \bigoplus_{i+j=k} \mathbf{H}_i \otimes \mathbf{H}_j.$$

is an involutive Lie bialgebra.

Proof

The same pictures prove the identities on the transverse chain level up to

- reparametrization of strings,
- diffeomorphism of domains.

This induces the identities on homology.

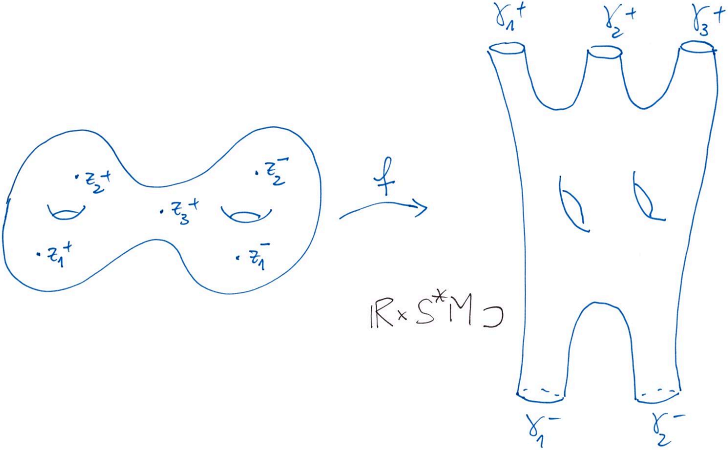
What is the expected structure on the chain level?

2. IBL_∞-structures

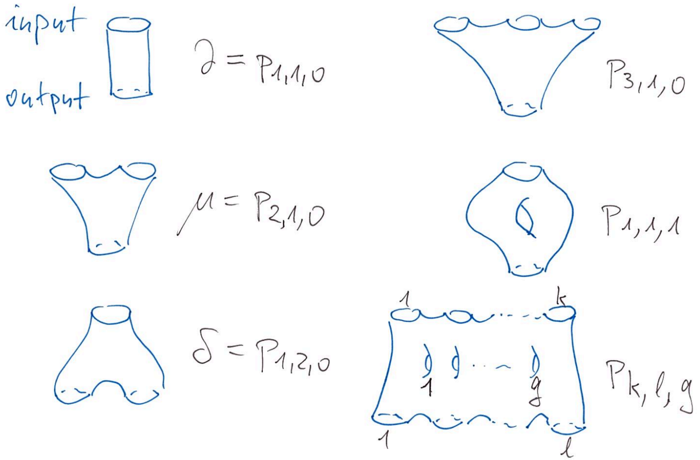
Symplectic field theory

- T^*M cotangent bundle of a manifold M;
- $\omega = \sum_i dp_i \wedge dq_i$ canonical symplectic form;
- $S^*M \subset T^*M$ unit cotangent bundle (with respect to some Riemannian metric on M);
- $T^*M \setminus M \cong \mathbb{R} \times S^*M$ "symplectization of S^*M";
- J suitable almost complex structure on $\mathbb{R} \times S^*M$;
- $(\dot{S}, j)$ closed Riemann surface with finitely many points removed ("punctures");
- $f : \dot{S} \to \mathbb{R} \times S^*M$ holomorphic: $df \circ j = J \circ df$.
- f asymptotic to $\pm$ cylinders over closed geodesics at punctures.

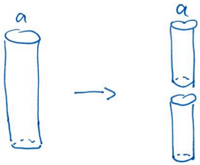
A punctured holomorphic curve



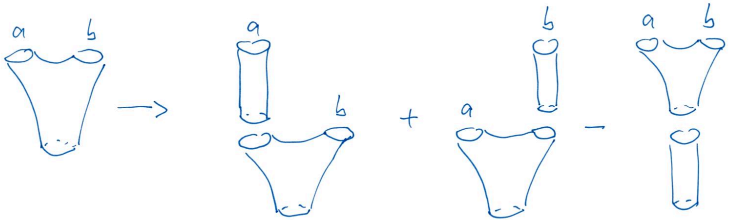
Operations from punctured holomorphic curves



Codimension 1 degenerations of holomorphic curves

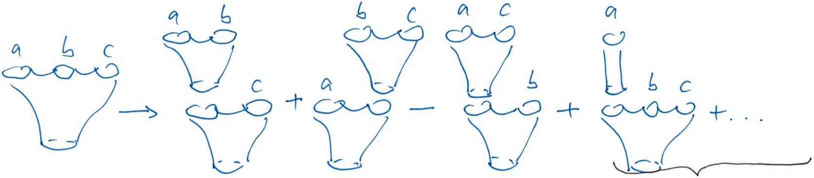


$$0 = \partial(\partial a)$$

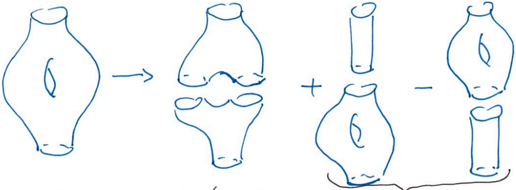


$$0 = \mu(\partial a, b) + \mu(a, \partial b) - \partial\mu(a,b)$$

Codimension 1 degenerations of holomorphic curves



$$0 = \mu(\mu(a,b),c) + \mu(a,\mu(b,c)) - \mu(\mu(a,c),b) + [\hat{\partial}, p_{3,1,0}](a,b,c)$$



$$0 = \delta(\mu a) + [\partial, p_{1,1,1}](a)$$

Algebraic relations

$$
\begin{aligned}
0 &= \partial \circ \partial = \mathfrak{p}_{1,1,0} \circ \mathfrak{p}_{1,1,0},\\
0 &= [\widehat{\partial}, \mu] = [\widehat{\mathfrak{p}}_{1,1,0}, \mathfrak{p}_{2,1,0}],\\
0 &= \mu \circ \widehat{\mu} + [\widehat{\partial}, \mathfrak{p}_{3,1,0}] = \mathfrak{p}_{2,1,0} \circ \widehat{\mathfrak{p}}_{2,1,0} + [\widehat{\mathfrak{p}}_{1,1,0}, \mathfrak{p}_{3,1,0}],\\
0 &= \delta \circ \mu + [\partial, \mathfrak{p}_{1,1,1}] = \mathfrak{p}_{1,2,0} \circ \mathfrak{p}_{2,1,0} + [\mathfrak{p}_{1,1,0}, \mathfrak{p}_{1,1,1}],
\end{aligned}
$$

where operations are extended to higher tensor products by

$$\widehat{\partial}(a \otimes b) := \partial a \otimes b + (1)^{|a|} a \otimes \partial b$$

etc.

Definition of IBL_∞-structure

- R commutative ring with unit that contains $\mathbb{Q}$;
- $C = \bigoplus_{k\in\mathbb{Z}} C^k$ free graded R-module;
- degree shift $C[1]^d := C^{d+1}$, so the degrees $\deg c$ in C and $|c|$ in $C[1]$ are related by $|c| = \deg c - 1$;
- k-fold symmetric product

$$E_k C := (C[1] \otimes_R \cdots \otimes_R C[1]) \,/\, \sim;$$

- *reduced symmetric algebra*

$$EC := \bigoplus_{k\geq 1} E_k C.$$

- We extend any linear map $\phi : E_k C \to E_\ell C$ to $\hat{\phi} : EC \to EC$ by $\hat{\phi} := 0$ on $E_m C$ for $m < k$, and for $m \geq k$:

$$\hat{\phi}(c_1 \cdots c_m) := \sum_{\rho\in S_m} \frac{\varepsilon(\rho)}{k!(m-k)!} \phi(c_{\rho(1)} \cdots c_{\rho(k)}) c_{\rho(k+1)} \cdots c_{\rho(m)}.$$

Definition of IBL_∞-structure

Definition. An IBL_∞-**structure** of degree d on C is a series of R-module homomorphisms

$$\mathfrak{p}_{k,\ell,g} : E_k C \to E_\ell C, \qquad k, \ell \geq 1,\ g \geq 0$$

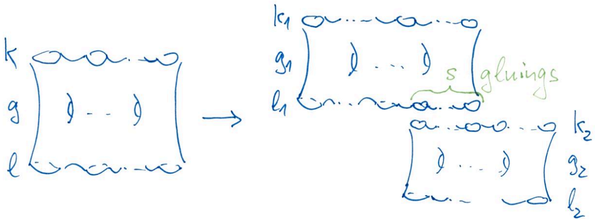
of degree

$$|\mathfrak{p}_{k,\ell,g}| = -2d(k+g-1) - 1$$

satisfying for all $k, \ell \geq 1$ and $g \geq 0$ the relations

$$\sum_{s=1}^{g+1} \sum_{\substack{k_1+k_2=k+s \\ \ell_1+\ell_2=\ell+s \\ g_1+g_2=g+1-s}} (\hat{\mathfrak{p}}_{k_2,\ell_2,g_2} \circ_s \hat{\mathfrak{p}}_{k_1,\ell_1,g_1})|_{E_k C} = 0. \tag{1}$$

This encodes general gluings of *connected* surfaces



$$k_1 + k_2 - s = k$$
$$\ell_1 + \ell_2 - s = \ell$$
$$g_1 + g_2 + s - 1 = g$$

Equivalent definition of IBL_∞-structure

Define the operator

$$\hat{\mathfrak{p}} := \sum_{k,\ell=1}^{\infty} \sum_{g=0}^{\infty} \hat{\mathfrak{p}}_{k,\ell,g} \hbar^{k+g-1} \tau^{k+\ell+2g-2} : EC\{\hbar,\tau\} \to EC\{\hbar,\tau\},$$

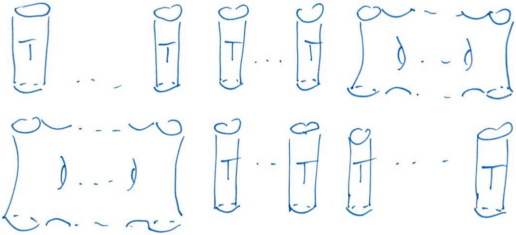
where $\hbar$ and τ are formal variables of degree

$$|\hbar| := 2d, \quad |\tau| = 0,$$

and $EC\{\hbar,\tau\}$ denotes formal power series in these variables with coefficients in EC. Then equation (1) is equivalent to

$$\hat{\mathfrak{p}} \circ \hat{\mathfrak{p}} = 0. \tag{2}$$

This encodes general gluings of *disconnected* surfaces with trivial cylinders



Properties of IBL_∞-structures

- $\partial = \mathfrak{p}_{1,1,0}$ is a boundary operator with homology $H(C) := \ker \partial / \mathrm{im} \partial$.
- $\mu = \mathfrak{p}_{2,1,0}$ and $\delta = \mathfrak{p}_{1,2,0}$ induce the structure of an involutive Lie bialgebra on $H(C)$.
- There are natural notions of IBL_∞-morphism $(e^{\mathfrak{f}}\widehat{\mathfrak{p}} - \widehat{\mathfrak{q}}e^{\mathfrak{f}} = 0)$ and IBL_∞-homotopy equivalence.
- An IBL_∞-morphism which induces an isomorphism on homology is an IBL_∞-homotopy equivalence.
- Every IBL_∞-structure is homotopy equivalent to an IBL_∞-structure on its homology.

All these properties have natural analogues for A_∞- and L_∞-structures.

3. Chain level string topology

Chen's iterated integrals

Idea: Find chain model for string topology of M using the de Rham complex $(A, \wedge, d)$.

- $L := \mathrm{Map}(S^1, M\}$ free loop space;
- $\Sigma = LM/S^1$ string space;
- $A = \{\text{differential forms on M}\}$;
- $B^{cyc}A$ reduced tensor algebra of A modulo cyclic permutations;
- $B^{cyc*}A = \mathrm{Hom}(B^{cyc}A, \mathbb{R})$ cyclic bar complex.

Chen's iterated integrals

Define a linear map

$$I : C_*(\Sigma) \to B^{cyc*}A$$

by

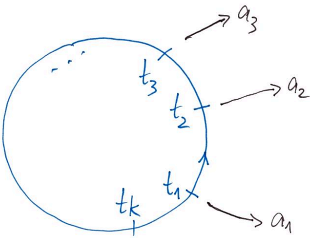
$$\langle If, a_1 \cdots a_k \rangle := \int_{P \times C_k} \mathrm{ev}_f^*(a_1 \times \cdots \times a_k),$$

where

- $f : P \to L$ lift to L of a chain in $C_*(\Sigma)$;
- $a_1, \ldots, a_k \in A$;
- $C_k := \{(t_1, \ldots, t_k) \in (S^1)^k \mid t_1 \leq \cdots \leq t_k \leq t_1\}$;

Chen's iterated integrals

$$
\begin{aligned}
\mathrm{ev} : L \times C_k &\to M^k, \\
(\gamma, t_1, \ldots, t_k) &\mapsto \Big(\gamma(t_1), \ldots, \gamma(t_k)\Big), \\
\mathrm{ev}_f = \mathrm{ev} \circ (f \times \mathbb{1}) : P \times C_k &\to M^k, \\
(p, t_1, \ldots, t_k) &\mapsto \Big(f(p)(t_1), \ldots, f(p)(t_k)\Big),
\end{aligned}
$$



Independence of lift f from Σ to L

For reparametrization $\widetilde{f}(p)(t) = f(p)(t+\sigma(p))$ with $\sigma : P \to S^1$:

$\Longrightarrow \mathrm{ev}_{\widetilde{f}} = \mathrm{ev}_f \circ \rho$ with the orientation preserving diffeomorphism

$$\rho : P \times C_k \to P \times C_k,$$
$$(p, t_1, \ldots, t_k) \mapsto \Big(p, t_1 + \sigma(p), \ldots, t_k + \sigma(p)\Big).$$

$\Longrightarrow I\widetilde{f} = If.$

Cyclic invariance

The cyclic permutations

$$\begin{aligned} \tau_C : C_k \to C_k, &\qquad (t_1, \dots, t_k) \mapsto (t_2, \dots, t_k, t_1), \\ \tau_M : M^k \to M^k, &\qquad (x_1, \dots, x_k) \mapsto (x_2, \dots, x_k, x_1) \end{aligned}$$

satisfy

$$\begin{aligned} \mathrm{ev} \circ (\mathbb{1}_L \times \tau_C) = \tau_M \circ \mathrm{ev} : L \times C_k \to M^k, \\ (\mathbb{1}_L \times \tau_C) \circ (f \times \mathbb{1}_C) = (f \times \mathbb{1}_C) \circ (\mathbb{1}_P \times \tau_C) : P \times C_k \to L \times C_k \end{aligned}$$

and

$$\begin{aligned} \tau_M^*(a_1 \times \cdots \times a_k) = (-1)^\eta a_2 \times \cdots \times a_k \times a_1, \\ \eta = \deg a_1(\deg a_2 + \cdots + \deg a_k). \end{aligned}$$

Cyclic invariance

Then

$$\begin{aligned}\langle If, a_2 \cdots a_k a_1 \rangle &= (-1)^{\eta} \int_{P \times C_k} \mathrm{ev}_f^* \tau_M^* (a_1 \times \cdots \times a_k) \\ &= (-1)^{\eta} \int_{P \times C_k} (f \times \mathbb{1}_C)^* (\mathbb{1}_L \times \tau_C)^* \mathrm{ev}^* (a_1 \times \cdots \times a_k) \\ &= (-1)^{\eta + k - 1} \langle If, a_1 \cdots a_k \rangle.\end{aligned}$$

Chain map

Note that

$$\partial C_k = \bigcup_{i=1}^{k} \partial_i C_k, \qquad \partial_i C_k = \{(t_1, \ldots, t_k) \mid t_i = t_{i+1}\} \cong C_{k-1}.$$

Then

$$\langle I\partial f, a_1 \cdots a_k \rangle = \int_{\partial P \times C_k} \mathrm{ev}_f^*(a_1 \times \cdots \times a_k) = I_1 - I_2$$

with

$$\begin{aligned} I_1 &= \int_{\partial(P \times C_k)} \mathrm{ev}_f^*(a_1 \times \cdots \times a_k) \\ &= \int_{P \times C_k} d\Big(\mathrm{ev}_f^*(a_1 \times \cdots \times a_k)\Big) \\ &= \int_{P \times C_k} \mathrm{ev}_f^*\Big(\sum_{i=1}^{k} \pm(a_1 \times \cdots \times da_i \times \cdots \times a_k)\Big) \end{aligned}$$

Chain map

and

$$\begin{aligned} I_2 &= \int_{P\times\partial C_k} \mathrm{ev}_f^*(a_1\times\cdots\times a_k) \\ &= \sum_{i=1}^{k}\int_{P\times\partial_i C_k} \mathrm{ev}_f^*(a_1\times\cdots\times a_k) \\ &= \int_{P\times C_{k-1}} \mathrm{ev}_f^*\Big(\sum_{i=1}^{k} \pm(a_1\times\cdots\times(a_i a_{i+1})\times\cdots\times a_k)\Big) \end{aligned}$$

Thus

$$\langle I\partial f, a_1\cdots a_k\rangle = \langle If, d_H(a_1,\cdots a_k)\rangle$$

with the **Hochschild differential**

$$d_H(a_1 \cdots a_k) := \sum_{i=1}^{k} \pm(a_1 \times \cdots \times da_i \times \cdots \times a_k)$$
$$+ \sum_{i=1}^{k} \pm(a_1 \times \cdots \times (a_i a_{i+1}) \times \cdots \times a_k).$$

So I is a chain map $(C_*(\Sigma), \partial) \to (B^{cyc*}A, d_H)$.

Theorem (Chen). If M is simply connected, then I induces an isomorphism to the **cyclic cohomology** of A

$$I_* : H_*(\Sigma) \to H_*(B^{cyc*}A, d_H).$$

Let $(A, \wedge, d, \langle\ ,\ \rangle)$ be any **cyclic DGA**, i.e.,

$$\langle da, b\rangle + (-1)^{|a|}\langle a, db\rangle = 0,$$
$$\langle a, b\rangle + (-1)^{|a||b|}\langle b, a\rangle = 0.$$

Suppose first that A is finite dimensional. Let e_i be a basis with dual basis e^i and set $g^{ij} := \langle e^i, e^j\rangle$.

Proposition. $B^{cyc*}A$ carries a canonical dIBL-structure, i.e. an IBL_∞-structure with only 3 operations $\mathfrak{p}_{1,1,0}$, $\mathfrak{p}_{2,1,0}$ and $\mathfrak{p}_{1,2,0}$, defined as follows:

$$\mathfrak{p}_{1,1,0}(\varphi)(a_1,\dots,a_k) := \sum_{i=1}^{k} \pm\varphi(a_1,\dots,da_i,\dots,a_k),$$

$$\mathfrak{p}_{2,1,0}(\varphi,\psi)(a_1,\dots,a_{k_1+k_2})$$

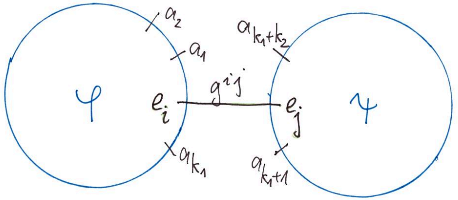
$$:= \sum_{i,j} \sum_{c=1}^{k_1+k_2} \pm g^{ij}\varphi(e_i,a_c,\dots,a_{c+k_1-1})\psi(e_j,a_{c+k_1},\dots,a_{c-1}),$$

$$\mathfrak{p}_{1,2,0}(\varphi)(a_1\cdots a_{k_1}\otimes b_1\cdots b_{k_2})$$

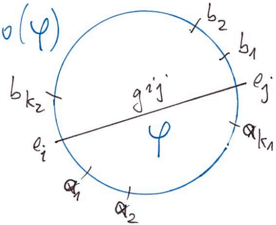
$$:= \sum_{i,j}\sum_{c=1}^{k_1}\sum_{c'=1}^{k_2} \pm g^{ij}\varphi(e_i,a_c,\dots,a_{c-1},e_j,b_{c'},\dots,b_{c'-1}).$$

dIBL-structure on the cyclic bar complex

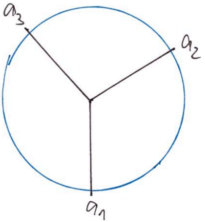
$P_{2,1,0}(\varphi, \psi)$



$P_{1,2,0}(\varphi)$



$\mathcal{M}_{1,0}$



Moreover, $\mathfrak{m}_{1,0} \in B_3^{cyc*}$ defined by the triple intersection product

$$\mathfrak{m}_{1,0}(a_1, a_2, a_3) := \langle a_1 \wedge a_2, a_3 \rangle.$$

satisfies the Maurer-Cartan equation $\widehat{\mathfrak{p}}(e^{\mathfrak{m}}) = 0$, or equivalently,

$$\mathfrak{p}_{1,1,0}(\mathfrak{m}_{1,0}) + \frac{1}{2}\mathfrak{p}_{2,1,0}(\mathfrak{m}_{1,0}, \mathfrak{m}_{1,0}) = 0,$$
$$\mathfrak{p}_{1,2,0}(\mathfrak{m}_{1,0}) = 0.$$

It induces a twisted differential which agrees with the Hochschild differential:

$$\mathfrak{p}^{\mathfrak{m}}_{1,1,0} := \mathfrak{p}_{1,1,0} + \mathfrak{p}_{2,1,0}(\mathfrak{m}_{1,0}, \cdot) = d_H.$$

Pushing the IBL_∞-structure to cohomology

Consider an inclusion of a subcomplex $\iota : (H, d|_H = 0) \hookrightarrow (A, d)$ inducing an isomorphism on cohomology.
(In the de Rham case, H are the harmonic forms.)

$B^{cyc*}H$ carries the canonical dIBL structure $\mathfrak{q}_{1,1,0} = 0$, $\mathfrak{q}_{2,1,0}$, $\mathfrak{q}_{1,2,0}$.

Theorem. There exists an IBL_∞-homotopy equivalence

$$\mathfrak{f} = \{\mathfrak{f}_{k,\ell,g}\} : B^{cyc*}A \to B^{cyc*}H$$

with $\mathfrak{f}_{1,1,0} = \iota^*$.

Construction of the homotopy equivalence $\mathfrak{f}$

We define

$$\mathfrak{f}_{k,\ell,g} := \sum_{\Gamma \in R_{k,\ell,g}} \mathfrak{f}_\Gamma,$$

Γ ribbon graph satisfying:

- the thickened surface Σ_Γ has k (interior) vertices, ℓ boundary components, and genus g;
- each boundary component meets at least one exterior edge.

To define $\mathfrak{f}_\Gamma$, we pick a projection $\Pi : A \to A$ onto B and a Green's operator (propagator) $G : A \to A$ satisfying

$$\begin{aligned} \Pi d &= d\Pi, & \langle \Pi a, b\rangle &= \langle a, \Pi b\rangle, \\ dG + Gd &= \mathbb{1} - \Pi, & \langle Ga, b\rangle &= (-1)^{|a|}\langle a, Gb\rangle. \end{aligned}$$

Let $G^{ij} := \langle Ge^i, e^j\rangle$.

Construction of the homotopy equivalence $\mathfrak{f}$

Suppose we are given $\Gamma \in R_{k,\ell,g}$ as well as

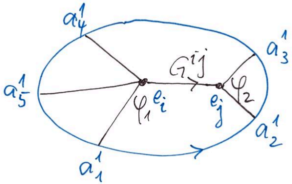
- $\phi_1, \dots, \phi_k \in B^{cyc*}A$;
- $a_j^b \in H$ for $b = 1, \dots, \ell$ and $j = 1, \dots, s_b$, where s_b is the number of exterior edges meeting the b-th boundary component.

Then

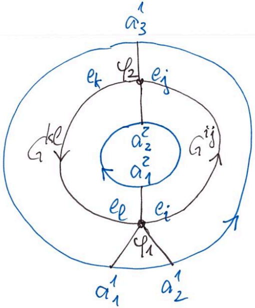
$$\mathfrak{f}_\Gamma(\phi_1, \dots, \phi_k)(a_1^1 \cdots a_{s_1}^1, \dots, a_1^\ell \cdots a_{s_\ell}^\ell)$$

is the sum over all basis elements of the following numbers:

Construction of the homotopy equivalence f

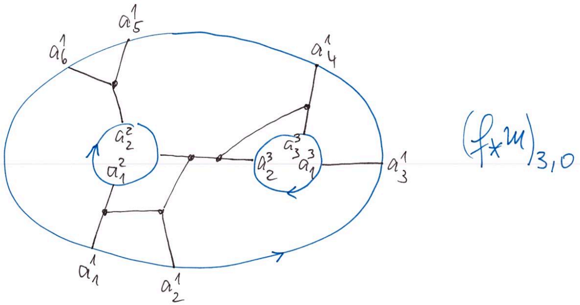


$\Gamma \in \mathcal{R}_{2,1,0}$



$\Gamma \in \mathcal{R}_{2,2,0}$

The terms $(\mathfrak{f}_*\mathfrak{m})_{\ell,g}$ of the push-forward $\mathfrak{f}_*\mathfrak{m}$ of the canonical Maurer-Cartan element to $B^{cyc*}H$ is given by the same expressions, where the graphs Γ are **trivalent** and the operation to each vertex is the triple intersection product $\mathfrak{m}_{1,0}(a_1, a_2, a_3) := \langle a_1 \wedge a_2, a_3\rangle$.



The twisted IBL_∞-structure on cohomology

Theorem. $B^{cyc*}H$ with the twisted IBL_∞-structure $\mathfrak{q}^{\mathfrak{f}_*\mathfrak{m}}$ is IBL_∞-homotopy equivalent to $B^{cyc*}A$ with its canonical dIBL-structure $\mathfrak{p}$. In particular, the homology of $(B^{cyc*}H, \mathfrak{q}^{\mathfrak{f}_*\mathfrak{m}})$ equals the cyclic homology of $(B^{cyc*}A, \mathfrak{p})$.

Application to the de Rham complex

Now we apply this to the de Rham complex $(A, \wedge)$ on M with the intersection product

$$\langle a, b \rangle := \int_M a \wedge b.$$

Let H be the space of harmonic forms on M.

Theorem (in progress). Suppose that M is odd-dimensional, simply connected, and its tangent bundle TM is trivializable. Then there exists a twisted IBL_∞-structure $\mathfrak{q}^{\mathfrak{f}_*\mathfrak{m}}$ on $B^{cyc*}H$ such that

- the homology of $(B^{cyc*}H, \mathfrak{q}^{\mathfrak{f}_*\mathfrak{m}})$ equals the homology of $H_*(\Sigma)$ of the string space of M;
- the involutive Lie bialgebra structure on the homology of $(B^{cyc*}H, \mathfrak{q}^{\mathfrak{f}_*\mathfrak{m}})$ agrees with the the structure arising from string topology.

Idea of the proof

The Maurer-Cartan element $\mathfrak{f}_*\mathfrak{m}$ is constructed by sums over trivalent ribbon graphs, where to each graph Γ we associate an integral of the type

$$\int_{M^{3k}} \prod G(x,y) \prod a(x),$$

where we assign

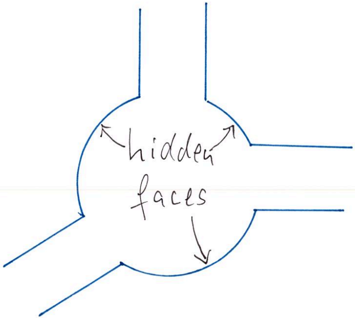
- to each interior vertex an integration variable x;
- to each interior edge the Green kernel $G(x,y)$;
- to each exterior edge a harmonic form $\alpha(x)$.

Difficulties in the proof

(1) The Green kernel $G(x, y)$ is singular at $x = y$.
Solution: Blow up the edge diagonals in the configuration space M^{3k} to obtain a (singular) manifold $\widetilde{M^{3k}}$ with boundary.

(2) The boundary of $\widetilde{M^{3k}}$ has **hidden faces**, which may lead to extra terms in Stokes' theorem and destroy the IBL_∞-relations.
Solution: Construct a specific G, by pulling back a standard form from $\mathbb{R}^n$ via the trivialization, to ensure that the integrals over hidden faces vanish.

Blow-up of configuration space with hidden faces



Relation to perturbative Chern-Simons theory

The Chern-Simons action of a G-connection $A \in \Omega^1(M, \gg)$ on a 3-manifold M is

$$S(A) = \frac{1}{4\pi} \int_M \mathrm{Tr}\Big(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\Big).$$

Perturbative expansion of the partition function

$$Z_k = \int_{\{A\}} DA\, e^{ikS(A)}$$

around the trivial flat connection leads in the case $G = U(1)$ to the same kind of integrals over configuration spaces associated to trivalent graphs.

Literature: Witten, Bar-Natan, Bott and Taubes, ...

Open questions

1. How exactly is the IBL_∞-structure related to perturbative Chern-Simons theory? What is the relation between **anomalies** in both theories?
2. Is there a variant of the IBL_∞-structure $U(1)$ replaced by for $U(N)$?
3. Can one incorporate knots and links into the IBL_∞-structure (as in perturbative Chern-Simons theory)?
4. Does the IBL_∞-structure yield interesting invariants of manifolds (and possibly knots and links)?
5. Can one drop the assumptions of the theorem (odd dimension, TM trivializable)?

Thank you!