Introduction to D-modules in representation theory

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LECTURE I

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References for most of what we will do can be found on Dragan Miličić's homepage

http://www.math.utah.edu/˜milicic

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These generators satisfy the commutation relations

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The nontrivial relations come from the Leibniz rule:

$$\partial_i (x_j P) = \partial_i (x_j) P + x_j \partial_i (P).$$

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A crucial remark is that $\mathbb{D}(1)$ cannot have any finite-dimensional modules.

Namely, if M were a finite-dimensional $\mathbb{D}(1)$-module, then the operator $[\partial, x]$ on M would have trace 0, while the operator 1 would have trace $\dim M$, a contradiction.

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Another way to describe an isomorphic module is as $\mathbb{C}[\partial]$, with

$$\partial \cdot \partial^i = \partial^{i+1}; \qquad x \cdot \partial^i = -i\partial^{i-1}.$$

("Fourier transform" of $\mathbb{C}[x]$.)

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One can generalize this by replacing the x_1-axis by any curve $Y \subset \mathbb{C}^2$, and consider the D-module consisting of regular functions on Y tensored by the "normal derivatives" to Y. Such a module is typically not of the form $M_1 \otimes M_2$ as above. We will define such modules more precisely later.

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This will make dimension theory easier, but on the other hand Bernstein filtration has no analogue on more general varieties, where there is no notion of degree for a regular function.

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Note that while $\operatorname{Gr} D$ is the same for both filtrations, its grading is different, and individual $\operatorname{Gr}_n D$ are different.

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- $F_pM = 0$ for $p << 0$;
- $\bigcup_p F_pM = M$;
- $\exists p_0 \in \mathbb{Z}$ such that for $p \geq p_0$ and for any n,

$$D_nF_pM = F_{n+p}M.$$

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There is k such that for any p,

$$F_p M \subseteq F'_{p+k} M \subseteq F_{p+2k} M.$$

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Proposition. *For $M \neq 0$, there are $d, e \in \mathbb{Z}_+$, independent of the choice of FM, such that for large p,*

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The proposition is proved by passing to the graded setting and using the analogous fact for modules over polynomial rings. The proof of the last fact involves studying Poincaré series and Hilbert polynomials.

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For any nonzero $\mathbb{D}(n)$*-module* M, $d(M) \geq n$.

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So the dimension of $\mathbb{C}[x_1, \ldots, x_n]$ is n. (And the multiplicity is 1.)

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3. Let FM be a good filtration; can assume $F_pM = 0$ for $p < 0$ and $F_0M \neq 0$.

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So $2n \leq 2d(M)$ and we are done.

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Now $T(m) = \lambda m = 0$ for any $m \in F_pM \neq 0$, so $\lambda = 0$, so $T = 0$ and the theorem follows.

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This is proved by choosing compatible good filtrations for M, M' and M''. Then

$$\dim F_p M = \dim F_p M' + \dim F_p M''$$

and the lemma follows easily.

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Namely, if M is not irreducible, then it fits into a nontrivial short exact sequence, with M' and M'' holonomic with strictly smaller multiplicity.

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More generally,

$$\mathbb{C}[x_1, \ldots, x_k, \partial_{k+1}, \ldots, \partial_n]$$

is a holonomic $\mathbb{D}[n]$-module.

Category of holonomic modules

So submodules, quotients and extensions of holonomic modules are holonomic.

One can also show that the localization of a holonomic module with respect to powers of a nonconstant polynomial is holonomic.

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is a holonomic $\mathbb{D}[n]$-module.

Finally, one easily sees that $d(\mathbb{D}(n)) = 2n$, so $\mathbb{D}(n)$ is not a holonomic module over itself.

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Since $\operatorname{Gr} M$ is finitely generated over $\operatorname{Gr} D = \mathbb{C}[x_1, \dots, x_n, \xi_1, \dots, \xi_n]$, we can consider the ideal

$$I = \operatorname{Ann}_{\operatorname{Gr} D} \operatorname{Gr} M$$

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The characteristic variety of M is the zero set of I in $\mathbb{C}^{2n}$:

$$\operatorname{Ch}(M) = V(I).$$

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Here $\operatorname{Supp} M$ is the support of M as a $\mathbb{C}[x_1, \ldots, x_n]$-module:

$$\operatorname{Supp} M = \operatorname{Ann}_{\mathbb{C}[x_1,\ldots,x_n]} M = \{x \in \mathbb{C}^n \mid M_x \neq 0\}.$$

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Bernstein's original proof used a sequence of (weighted) filtrations interpolating between the Bernstein filtration and the filtration by degree of differential operators, and is also quite involved.

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- $\operatorname{Ch} \mathbb{C}[x]_x = (\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$.
- If $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, then the module $M = \mathbb{C}[x]_x x^\alpha$ is irreducible, but $\operatorname{Ch} M$ is still $(\mathbb{C} \times \{0\}) \cup (\{0\} \times \mathbb{C})$.

D-modules on smooth varieties

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A general X can be covered by affine varieties X_i, and we obtain $\mathcal{D}_X$ by glueing the sheaves $\mathcal{D}_{X_i}$ together.

All our varieties will be smooth. This is to ensure that the algebras of differential operators have good properties (like the noetherian property).

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In particular, $[D, a] \in A$, so

$$[[D, a], b] = 0, \qquad a, b \in A.$$

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Conversely, if $[[D, a], b] = 0,\ a, b \in A$, then D is in $A \oplus \mathrm{Der}(A)$.
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Definition. For an affine variety X, the algebra of differential operators on X is $D(X) = \mathrm{Diff}\, O(X)$.

Presheaves

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Let X be a topological space. A presheaf of abelian groups on X is a map (functor) $\mathcal{F}$

$$\text{open } U \subseteq X \quad \longmapsto \quad \mathcal{F}(U), \text{ an abelian group}$$

such that for any $U \subseteq V$ open, there is a map $r_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$, and $U \subseteq V \subseteq W$ implies $r_{V,U} r_{W,V} = r_{W,U}$.

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(Think of $\mathcal{F}(U)$ as functions on U and of $r_{V,U}$ as the restriction. Notation: $r_{V,U}(f) = f\big|_U$.)

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A presheaf $\mathcal{F}$ is a sheaf if $U = \cup U_i$ implies $f \in \mathcal{F}(U)$ is 0 iff $f\big|_{U_i} = 0$ for all i, and if for any family $f_i \in \mathcal{F}(U_i)$ agreeing on intersections, there is $f \in \mathcal{F}(U)$ with $f\big|_{U_i} = f_i$.

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One can analogously define presheaves and sheaves of vector spaces, rings, algebras, modules, etc.

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For example, there are no nonconstant holomorphic functions on the Riemann sphere (Liouville's theorem).

LECTURE II

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- The notion of dimension of certain filtered algebras, including enveloping algebras and also $\mathbb{D}(n)$, is due to Gel'fand-Kirillov.
- There are other algebras with dimension theory similar to $\mathbb{D}(n)$, i.e., satisfying an analogue of Bernstein's theorem $d(M) \geq n$. These include certain quotients of $U(\mathfrak{g})$ for a semisimple Lie algebra. The situation was systematically studied by Bavula.

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$D(X)$ is a filtered algebra with respect to the filtration $D_p(X)$. It is also clearly an $O(X)$-module.

Sheaves of differential operators on affine varieties

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Any $O(X)$-module M on an affine variety X can be localized to a sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules on X, where $\mathcal{O}_X$ is the sheaf of (local) regular functions on X. (The construction of $\mathcal{O}_X$ itself follows the same scheme, which we describe below.)

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(X_f is affine, because if $X \hookrightarrow \mathbb{C}^N$, then X_f can be identified with the graph of $1/f$ which is a closed subset of $\mathbb{C}^{N+1}$.)

On X_f, one simply defines $\mathcal{M}(X_f) = M_f$, the localization of M with respect to powers of f. Since $(M_f)_g = M_{fg}$, one can define restriction maps in a compatible way.

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Following the above procedure, we can localize the $O(X)$-module $D(X)$ and obtain a quasicoherent $\mathcal{O}_X$-module $\mathcal{D}_X$.

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Following the above procedure, we can localize the $O(X)$-module $D(X)$ and obtain a quasicoherent $\mathcal{O}_X$-module $\mathcal{D}_X$.

It remains to see that $\mathcal{D}_X$ is a sheaf of algebras. This follows from the fact $D(X)_f = D(X_f)$ for any principal open set X_f, and the fact that an inverse limit of algebras is an algebra.

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Then $\mathcal{D}_X$ is a sheaf of algebras on X, and an $\mathcal{O}_X$-module.

Moreover, $\mathcal{D}_X$ is a quasicoherent $\mathcal{O}_X$-module, i.e., for an affine cover U_i of X, $\mathcal{D}_X(U_i)$ is obtained from the $O(U_i)$-module $D(U_i)$ by localization.

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This defines a filtration on $\mathcal{D}_X$. The corresponding $\operatorname{Gr} \mathcal{D}_X$ is isomorphic to $\pi_*(\mathcal{O}_{T^*(X)})$, where $\pi : T^*(X) \to X$ is the cotangent bundle, and π_* denotes the O-module direct image functor. $(\pi_*(\mathcal{O}_{T^*(X)})(U) = \mathcal{O}_{T^*(X)}(\pi^{-1}(U))$; more details later.)

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The proofs use symbol calculus: for $T \in \mathcal{D}_p(U)$, $\operatorname{Symb}_p(T) \in \mathcal{O}_{T^*(X)}(\pi^{-1}(U))$ is given by

$$\operatorname{Symb}_p(T)(x, df) = \frac{1}{p!} \underbrace{[\ldots[[T, f], f], \ldots, f]}_{p}(x).$$

$\mathcal{D}_X$-modules

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- $\dim \mathrm{Ch}(\mathcal{V}) \geq \dim X$ (sketch of proof later).

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Then $\bar{f}(\mathcal{G})$ is a presheaf on X, and we let $f^{\cdot}(\mathcal{G})$ be the associated sheaf. Example: if $f : \{y\} \hookrightarrow Y$, then $f^{\cdot}(\mathcal{G}) = \mathcal{G}_y$, the stalk.

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(Namely, $(gf)_{\cdot} = g_{\cdot}f_{\cdot}$ is obvious, and $(gf)^{\cdot} = f^{\cdot}g^{\cdot}$ follows by adjunction.)

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If $\mathcal{V}$ is an $\mathcal{O}_X$-module, then $f.(\mathcal{V})$ is an $f.\mathcal{O}_X$-module, and therefore an $\mathcal{O}_Y$-module via $- \circ f$. We denote this $\mathcal{O}_Y$-module by $f_*(\mathcal{V})$.

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If $\mathcal{W}$ is an $\mathcal{O}_Y$-module, then $f^{\cdot}(\mathcal{W})$ is an $f^{\cdot}\mathcal{O}_Y$-module. By adjunction, $- \circ f$ defines a morphism $f^{\cdot}\mathcal{O}_Y \to \mathcal{O}_X$, which we can use to extend scalars:

$$f^*(\mathcal{W}) = \mathcal{O}_X \otimes_{f^{\cdot}\mathcal{O}_Y} f^{\cdot}(\mathcal{W}).$$

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Moreover, if $X \xrightarrow{f} Y \xrightarrow{g} Z$, then $(gf)^+ = f^+g^+$.

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This functor does not have good properties in general, but it does if X and Y are affine. One can then get the functor we want by glueing the affine pieces via the Čech resolution. To do this, one needs to pass to derived categories.

Derived categories

Objects of the derived category $D(\mathcal{A})$ of an abelian category $\mathcal{A}$ are complexes over $\mathcal{A}$. This includes objects of $\mathcal{A}$, viewed as complexes concentrated in degree 0. One often imposes boundedness conditions on the complexes in $D(\mathcal{A})$.

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If $F : \mathcal{A} \to \mathcal{B}$ is a functor between abelian categories, then the left derived functor $LF : D(\mathcal{A}) \to D(\mathcal{B})$ is computed as $LF(X) = F(P)$, where P, with a quasiisomorphism $P \to X$, is a suitable resolution (e.g. a projective complex, or a flat complex).

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The right derived functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ is computed as $RF(X) = F(I)$, where I, with a quasiisomorphism $X \to I$, is a suitable resolution (e.g. an injective complex).

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Moreover, the functor f_+ has nice properties. Notably, if $X \xrightarrow{f} Y \xrightarrow{g} Z$ then $(gf)_+ = g_+ f_+$.

(There is however no adjunction property between Lf^+ and f_+ in general. Also, f_+ is not a derived functor of any functor on the level of abelian categories.)

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It follows that p^+ is exact, and that $p^+(W) = O(F) \otimes W$ for $W \in \mathcal{M}(D(Y))$.

Example 1 – continued

To calculate the derived functors of p_+, we should resolve $D_{X \to Y} = O(F) \otimes D(Y)$ by projective modules over $D(X) = D(F) \otimes D(Y)$. To do this, we should resolve the $D(F)$-module $O(F)$.

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So $p_+(M)$ and $L_1 p_+(M)$ are the cohomology modules of the complex $0 \to M \xrightarrow{\partial} M \to 0$.

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This is equal to $D(Y) \otimes \Delta(F)$, where $\Delta(F) = \mathbb{C} \otimes_{O(F)} D(F)$ is the space of "normal derivatives" to Y in X. For example, if F is $\mathbb{C}$ or $\mathbb{C}^*$, then $\Delta(F) = \oplus_i \mathbb{C}\partial^i$.

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This is equal to $D(Y) \otimes \Delta(F)$, where $\Delta(F) = \mathbb{C} \otimes_{O(F)} D(F)$ is the space of "normal derivatives" to Y in X. For example, if F is $\mathbb{C}$ or $\mathbb{C}^*$, then $\Delta(F) = \oplus_i \mathbb{C}\partial^i$.

In particular, $D_{Y \to X}$ is free over $D(Y)$, so i_+ is exact, and

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Namely, if $f : X \to Y$ is a morphism, we can consider its graph, which is a closed subvariety of $X \times Y$, and it is isomorphic to X.

In this way we get $i_f : X \hookrightarrow X \times Y$. If $p_Y : X \times Y \to Y$ is the projection, then $f = p_Y \circ i_f$.

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So there is no need to derive the tensor product functor. Moreover, since i is an affine morphism, $i.$ is exact on quasicoherent sheaves, and one need not derive $i.$ either.

Kashiwara's equivalence

Thus $i_+ : \mathcal{M}^R_{qc}(\mathcal{D}_Y) \to \mathcal{M}^R_{qc}(\mathcal{D}_X)$, given by

$$i_+(\mathcal{V}) = i_.(\mathcal{V} \otimes_{\mathcal{D}_Y} \mathcal{D}_{Y \to X})$$

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This functor defines an equivalence of the category $\mathcal{M}^R_{qc}(\mathcal{D}_Y)$ with the category $\mathcal{M}^R_{qc,Y}(\mathcal{D}_X)$ of quasicoherent right $\mathcal{D}_X$-modules supported in Y. The inverse is the functor $i^!$ given by

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In addition, both i_+ and $i^!$ take coherent modules to coherent modules, so they also make the categories $\mathcal{M}^R_{coh}(\mathcal{D}_Y)$ and $\mathcal{M}^R_{coh,Y}(\mathcal{D}_X)$ equivalent.

LECTURE III

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$i : Y \hookrightarrow X$ a closed embedding $\Rightarrow$

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is an equivalence of categories (Kashiwara).

Holonomic defect

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So i_+ preserves the "holonomic defect".

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Since i_+ preserves holonomic defect, and since we know Bernstein's theorem for $\mathbb{C}^N$, the result follows in general.

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For any morphism $f : X \to Y$ of general algebraic varieties, the functors f_+ and Lf^+ preserve holonomicity.

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Connections are also called local systems.

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A typical example: the Lie algebra of upper triangular matrices is a Borel subalgebra of $\mathfrak{sl}(n, \mathbb{C})$.

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This flag variety $\mathcal{B}$ can be described as G/B where B is the stabilizer in G of a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$.

So $\mathcal{B}$ is a smooth algebraic variety. Moreover, $\mathcal{B}$ is a projective variety.

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So the flag variety is a closed subvariety of a projective variety, and hence it is itself projective.

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So the flag variety of $\mathfrak{sl}(2, \mathbb{C})$ is the complex projective space $\mathbb{P}^1$, or the Riemann sphere.

Universal enveloping algebra

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$U(\mathfrak{g})$ is the associative algebra with 1, generated by $\mathfrak{g}$, with relations

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In this way we get a map from $\mathfrak{g}$ into (global) vector fields on $\mathcal{B}$.

This map extends to a map from $U(\mathfrak{g})$ into (global) differential operators on $\mathcal{B}$, $\Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}})$.

Theorem

The map $U(\mathfrak{g}) \to \Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}})$ *is surjective.*
The kernel is the ideal I_ρ *of* $U(\mathfrak{g})$ *generated by the annihilator in the center of* $U(\mathfrak{g})$ *of the trivial* $\mathfrak{g}$*-module* $\mathbb{C}$*.*

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Denoting $U(\mathfrak{g})/I_\rho$ by U_ρ, we get

$$U_\rho \xrightarrow{\cong} \Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}}).$$

Localization

If $\mathcal{V}$ is a $\mathcal{D}_{\mathcal{B}}$-module, then its global sections $\Gamma(\mathcal{B}, \mathcal{V})$ form a module over $\Gamma(\mathcal{B}, \mathcal{D}_{\mathcal{B}}) \cong U_\rho$.

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$\Delta_\rho : \mathcal{M}(U_\rho) \to \mathcal{M}_{qc}(\mathcal{D}_{\mathcal{B}})$ is called the localization functor.

Theorem (Beilinson-Bernstein equivalence)

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The functors Δ_ρ and Γ are mutually inverse equivalences of categories $\mathcal{M}(U_\rho)$ and $\mathcal{M}_{qc}(\mathcal{D}_\mathcal{B})$.

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So $\Gamma(X, \mathcal{O}_{\mathcal{B}})$ is the trivial $\mathfrak{g}$-module $\mathbb{C}$.

More examples for $\mathfrak{sl}(2, \mathbb{C})$

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We will use the usual basis of $\mathfrak{sl}(2, \mathbb{C})$:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

with commutation relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

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In each case all the h-eigenspaces are one-dimensional, e moves them up by 2, and f moves them down by 2.

All these modules are related to representations of the real Lie group $SU(1, 1)$; $D_{\pm 2}$ to the discrete series representations, and P to the principal series representation.

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To describe sheaves on $\mathcal{B} = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, we cover $\mathcal{B}$ by two copies of $\mathbb{C}$:
$\mathbb{P}^1 \setminus \{\infty\}$ with variable z, and $\mathbb{P}^1 \setminus \{0\}$ with variable $\zeta = 1/z$.

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Any quasicoherent O-module or D-module on $\mathcal{B}$ is determined by its sections on these two copies of $\mathbb{C}$, which have to agree on the intersection $\mathbb{C}^* = \mathbb{P}^1 \setminus \{0, \infty\}$.

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By the chain rule, $\partial_\zeta = -z^2\partial_z$. By a short computation one computes the map $\mathfrak{g} \to \Gamma(\mathcal{B}, \mathcal{D}_\mathcal{B})$:

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More examples for $\mathfrak{sl}(2,\mathbb{C})$

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Finally, P is obtained from the D-module equal to $\mathbb{C}[z, z^{-1}]z^{1/2}$ on $\mathbb{P}^1 \setminus \{\infty\}$, and to $\mathbb{C}[\zeta, \zeta^{-1}]\zeta^{1/2}$ on $\mathbb{P}^1 \setminus \{0\}$.

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$\mathcal{O}(\lambda)$ does not have an action of $\mathcal{D}_\mathcal{B}$, but of a slightly modified sheaf $\mathcal{D}_\lambda$ of differential operators on the line bundle $\mathcal{O}(\lambda)$.

Twisted differential operators

If λ is regular and integral but not dominant, one still has $\mathcal{O}(\lambda)$ and $\mathcal{D}_\lambda$, but now F_λ appears in higher cohomology of $\mathcal{O}(\lambda)$, and there are no global sections (Bott).

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One can again define the localization functor
$\Delta_\lambda : \mathcal{M}(U_\lambda) \to \mathcal{M}_{qc}(\mathcal{D}_\lambda)$.

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This is useful because if $w \in W$, then $U_\lambda = U_{w\lambda}$, but $\mathcal{D}_\lambda \neq \mathcal{D}_{w\lambda}$, and so one gets several possible localizations and can use their interplay (e.g., intertwining functors).

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If λ is singular (i.e., has nontrivial stabilizer in W), then there are more sheaves than modules (recall $\mathcal{O}(\lambda)$). In this case, $\mathcal{M}(U_\lambda)$ is a quotient category of $\mathcal{M}_{qc}(\mathcal{D}_\lambda)$ if λ is dominant; an analogous fact is true for the derived categories if λ is not necessarily dominant.

Equivariant group actions

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1. $K = N$ or $K = B$: highest weight modules;
2. $G_{\mathbb{R}}$ a real form of G, $G_{\mathbb{R}} \cap K$ a maximal compact subgroup of $G_{\mathbb{R}}$. Then $(\mathfrak{g}, K)$-modules correspond to group representations of $G_{\mathbb{R}}$.

Equivariant group actions

Some care is needed to define quasicoherent equivariant sheaves. One can turn a K-action π on V into a dual action of $O(K)$:

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On the sheaf level one considers $p, \mu : K \times \mathcal{B} \to \mathcal{B}$, the projection, respectively the action map, and requires to have an isomorphism $\mu^*(\mathcal{V}) \to p^*(\mathcal{V})$, satisfying a certain "cocycle condition".

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Then every coherent $(\mathcal{D}_\lambda, K)$-module is holonomic.

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This leads to a very nice geometric classification of irreducible $(\mathfrak{g}, K)$-modules.

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Since τ corresponds to a bundle, it is given by a representation W of the stabilizer S of a point in Q. The action of the Lie algebra $\mathfrak{s}$ on W should be given by $\lambda - \rho$, and it should integrate to S (compatibility).

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Any irreducible $(\mathcal{D}_\lambda, K)$-module is $\mathcal{L}(Q, \tau)$ for unique Q and τ.

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We set $\mathcal{B}' = \mathcal{B} \setminus \partial Q$, and factorize $Q \overset{i}{\hookrightarrow} \mathcal{B}$ as $Q \overset{i'}{\hookrightarrow} \mathcal{B}' \overset{j}{\hookrightarrow} \mathcal{B}$.

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So any two irreducible submodules of $\mathcal{I}(Q,\tau)$ have to intersect, and hence they agree.

Proofs

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The support of τ is all of Q by K-equivariance. So τ is a connection on a dense open subset of Q, hence everywhere by K-equivariance.

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The situation is analogous at ∞, with roles of z and $\zeta = 1/z$ reversed, and we get a lowest weight module with lowest weight $\lambda + \rho$.

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In this last case, the irreducible submodule is the sheaf $\mathcal{O}(\lambda)$ corresponding to the finite-dimensional representation, while the quotient is the direct sum of the standard modules corresponding to $\{0\}$ and $\{\infty\}$.

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$U(\mathfrak{g})$ and U_λ are however weak $(\mathfrak{g}, K)$-modules: they have an action π of $\mathfrak{g}$, and an action ν of K, the action π is K-equivariant, but ν and π do not necessarily agree on $\mathfrak{k}$. Then $\omega = \nu - \pi$ is a new action of $\mathfrak{k}$.

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The family i_X should also be K-equivariant, they should commute with the $\mathfrak{g}$-action, and anticommute with each other.

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A typical example of an equivariant complex is the standard (Koszul) complex of $\mathfrak{g}$,

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$$N(\mathfrak{g}) = U(\mathfrak{g}) \otimes \bigwedge(\mathfrak{g}),$$

with the usual Koszul differential.

The map i_X is simply given by wedging by X. This complex has the structure of a differential graded algebra.

One now as usual passes to homotopic category and localizes with respect to quasiisomorphisms to obtain the equivariant derived category.

Equivariant derived categories

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Bernstein and Lunts also proved that for $(\mathfrak{g}, K)$-modules, the ordinary and equivariant derived categories are equivalent.

This makes it possible to localize certain constructions using homological algebra of $(\mathfrak{g}, K)$-modules, like the Zuckerman functors.

Zuckerman functors

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It also has a $(\mathfrak{k}, T)$-action, the left regular action on $O(K)$ tensored by the action on V.

Zuckerman functors

The $(\mathfrak{g}, K)$-action commutes with the $(\mathfrak{k}, T)$-action and therefore descends to

$$\Gamma(V) = \mathrm{Hom}_{(\mathfrak{k}, T)}(\mathbb{C}, O(K) \otimes V).$$

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The derived functors of Γ are given by the corresponding Ext modules.

On the level of equivariant derived categories, one can construct an analogous functor by setting

$$\Gamma^{eq}(V) = \mathrm{Hom}^{\cdot}_{(\mathfrak{k},T,N(\mathfrak{t}))}(N(\mathfrak{k}), O(K) \otimes V)$$

for an equivariant $(\mathfrak{g}, T)$-complex V.

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One shows that Γ^{eq} is a well defined functor from equivariant $(\mathfrak{g}, T)$-complexes to equivariant $(\mathfrak{g}, K)$-complexes, and that it descends to the level of equivariant derived categories.

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It is possible to localize the above construction. Moreover, there is a purely geometric version. This was done by Sarah Kitchen, along with some further results.

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THANK YOU!