G-STRUCTURES WITH PRESCRIBED GEOMETRY—OVERVIEW

ROBERT L. BRYANT

DUKE UNIVERSITY

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References: For these three lectures (these are the slide for the first), the reader who wants more background information on exterior differential systems might want to consult the brief introduction

http://www.math.duke.edu/~bryant/Introduction_to_EDS.pdf

Many of the examples discussed here and the main variants of Cartan's theory of structure equations can be found in the lecture notes on EDS that can be found here

http://arxiv.org/abs/1405.3116

This latter article contains many references to the literature and further resources.

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Dual Formulation: Let $\delta : L^* \to \Lambda^2(L^*)$ be a linear map. If its extension $\delta : \Lambda^*(L^*) \to \Lambda^*(L^*)$ as a graded derivation of degree 1 satisfies $\delta^2 = 0$, then there is a DGA homomorphism $\phi : \big(\Lambda^*(L^*), \delta\big) \to \big(\Omega^*(L), \mathrm{d}\big)$ satisfying

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Basis Formulation: If $C^i_{jk} = -C^i_{kj}$ $(1 \le i, j, k \le n)$ are constants, then there exist linearly independent 1-forms ω^i $(1 \le i \le n)$ on $\mathbb{R}^n$ satisfying the structure equations

$$\mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}\, \omega^j \wedge \omega^k$$

if and only if these formulae imply $\mathrm{d}\big(\mathrm{d}\omega^i\big) = 0$ as a formal consequence.

A geometric problem: Classify those Riemannian surfaces (M^2, g) whose Gauss curvature K satisfies the second order system

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for some functions a and b.

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and the condition to be studied is encoded as

$$\begin{pmatrix} \mathrm{d}K_1 \\ \mathrm{d}K_2 \end{pmatrix} = \begin{pmatrix} -K_2 \\ K_1 \end{pmatrix} \omega_{12} + \begin{pmatrix} a(K) + b(K)\,{K_1}^2 & b(K)\,K_1K_2 \\ b(K)\,K_1K_2 & a(K) + b(K)\,{K_1}^2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

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Applying $\mathrm{d}^2 = 0$ to these two equations yields

$$\big(a'(K) - a(K)b(K) + K\big)\,K_i = 0 \qquad \text{for } i = 1, 2.$$

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$$\big(a'(K) - a(K)b(K) + K\big)\,K_i = 0 \qquad \text{for } i = 1, 2.$$

Thus, unless $a'(K) = a(K)b(K) - K$, such metrics have K constant.

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Answer: A theorem of É. Cartan [1904] implies that a 'solution' (N^3, ω) does indeed exist and is determined uniquely (locally near p, up to diffeomorphism) by the 'value' of (K, K_1, K_2) at p.

Cartan's result: Suppose that $C^i_{jk} = -C^i_{kj}$ and F^α_i (with $1 \leq i, j, k \leq n$ and $1 \leq \alpha \leq s$) are real-analytic functions on $\mathbb{R}^s$ such that the equations

$$\mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}(a)\,\omega^j \wedge \omega^k \qquad \text{and} \qquad \mathrm{d}a^\alpha = F^\alpha_i(a)\,\omega^i$$

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formally satisfy $\mathrm{d}^2 = 0$. Then, for every $b_0 \in \mathbb{R}^s$, there exists an open neighborhood U of $0 \in \mathbb{R}^n$, linearly independent 1-forms η^i on U, and a function $b : U \to \mathbb{R}^s$ satisfying

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Remark 1: Cartan assumed that $F = (F^\alpha_i)$ has constant rank, but it turns out that, for a 'solution' (η, b) with U connected, $F(b) = \big(F^\alpha_i(b)\big)$ always has constant rank anyway.

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Remark 2: Cartan worked in the real-analytic category and used the Cartan-Kähler theorem in his proof, but the above result is now known to be true in the smooth category. (Cf. P. Dazord)

The Hessian equation example:

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Remark: The F-matrix either has rank 0 (when $K_1 = K_2 = a(K) = 0$) or 2 (all other cases). The rank 0 cases have K constant. The rank 2 cases have a 1-dimensional symmetry group and each represents a surface of revolution.

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$$\{,\} : \Gamma(Y) \times \Gamma(Y) \to \Gamma(Y)$$

and a bundle map $\alpha : Y \to TA$ that induces a Lie algebra homomorphism on sections and satisfies the Leibnitz compatibility condition

$$\{U, fV\} = \alpha(U)(f)\, V + f\,\{U, V\} \qquad \text{for } f \in C^\infty(A) \text{ and } U, V \in \Gamma(Y).$$

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A 'solution' is a $b : B^n \to A$ covered by a bundle map $\eta : TB \to Y$ of rank n that induces a Lie algebra homomorphism on sections.

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Many more examples drawn from classical differential geometry.

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C^i_{jk}, F^a_i, and $A^a_{i\alpha}$ (where $1 \le i, j, k \le n$, $1 \le a \le s$, and $1 \le \alpha \le r$) are specified functions on a domain $X \subset \mathbb{R}^s$.

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In the present case, $W = \mathbb{R}^s$ and $V = \mathbb{R}^n$, while $A(h)$ is spanned by the r matrices (of size s-by-n)

$$\big(A^a_{i1}(h)\big),\ \big(A^a_{i2}(h)\big),\ \ldots,\ \big(A^a_{ir}(h)\big).$$

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(3) $\mathrm{d}^2 = 0$ reduces to equations of the form

$$0 = A^a_{i\alpha}(h)\big(\mathrm{d}p^\alpha + B^\alpha_j(h,p)\,\eta^j\big) \wedge \eta^i$$

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Remark: The proof is a straightforward modification of Cartan's proof in the case $r = 0$ (i.e., when there are no 'free derivatives' p^α).

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2^{nd} Bianchi implies that there are functions a_i and b_i so that

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where $q_{ii} = 0$ and $q_{ij} = -p_{jk}/(c_i{-}c_j)$ when (i,j,k) are distinct.

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This defines an involutive tableau (in the p-variables) of rank $r = 9$ and with characters $s_1 = 6$, $s_2 = 3$, and $s_3 = 0$. Cartan's criteria are satisfied, so the desired metrics depend on three functions of two variables.

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When $c_1 = c_2 = c_3$, the Cartan analysis gives the expected result that the solutions depend on a single constant.

General Holonomy. A torsion-free H-structure on M^n (where $H \subset \mathrm{GL}(\mathfrak{m})$ and $\dim(\mathfrak{m}) = n$) satisfies the first structure equation

$$\mathrm{d}\omega = -\phi \wedge \omega$$

where ϕ takes values in $\mathfrak{h} \subset \mathrm{GL}(\mathfrak{m})$ and the second structure equation

$$\mathrm{d}\phi = -\phi \wedge \phi + R(\omega \wedge \omega)$$

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Ex: For $G_2 \subset \mathrm{GL}(7, \mathbb{R})$, the tableau $K^1(\mathfrak{g}_2)$ has $s_6 = 6 > s_7 = 0$, so the general metric with G_2-holonomy depends on 6 functions of 6 variables.

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4. (B—, 1987) The Ricci-solitons in dimension 3 depend on 2 functions of 2 variables. (More generally, $\mathrm{Ric}(g) = a(f)\,g + b(f)\,\mathrm{d}f^2 + c(f)\ \mathrm{Hess}_g(f)$.)

Other examples.

1. (Cartan, 1926) The metrics in dimension 4 with holonomy SU(2) depend locally on 2 functions of 3 variables.

2. (Cartan-Einstein, 1929-32) Analysis of Einstein's proposed unified field theory via connections with torsion.

3. (Cartan, 1943) The Einstein-Weyl structures in dimension 3 depend on 4 functions of 2 variables.

4. (B—, 1987) The Ricci-solitons in dimension 3 depend on 2 functions of 2 variables. (More generally, $\mathrm{Ric}(g) = a(f)\,g + b(f)\,\mathrm{d}f^2 + c(f)\ \mathrm{Hess}_g(f)$.)

5. (B—, 2008) The solitons for the G_2-flow in dimension 7 depend on 16 functions of 6 variables.