G-STRUCTURES WITH PRESCRIBED GEOMETRY—EDS TECHNIQUES

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References: For these three lectures (these are the slide for the second), the reader who wants more background information on exterior differential systems might want to consult the brief introduction

http://www.math.duke.edu/~bryant/Introduction_to_EDS.pdf

Many of the examples discussed here and the main variants of Cartan's theory of structure equations can be found in the lecture notes on EDS that can be found here

http://arxiv.org/abs/1405.3116

This latter article contains many references to the literature and further resources.

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An integral element of $\mathcal{I}$ is a subspace $E \subset T_xM$ such that $E^*(\alpha) = 0$ for all $\alpha \in \mathcal{I}$. Let $\mathcal{V}_p(\mathcal{I}) \subset \mathrm{Gr}_p(TM)$ denote the set of p-dimensional integral elements of $\mathcal{I}$.

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Fundamental Problem: Given an $E \in \mathcal{V}_n(\mathcal{I})$, when does there exist an integral manifold $f : N \to M$ and an $x \in N$ such that $f'(T_x N) = E$?

Polar spaces. Fix $E \in \mathcal{V}_p(\mathcal{I})$, with $E \subset T_xM$, and let $e_1, \ldots, e_p$ be a basis of E. The *polar space* of E is the subspace

$$H(E) = \{ v \in T_xM \mid \alpha(v, e_1, \ldots, e_p) = 0 \ \forall \alpha \in \mathcal{I}^{p+1} \ \} \subset T_xM.$$

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Any $E_+ \in \mathcal{V}_{p+1}(\mathcal{I})$ that contains E must lie in $H(E)$ and, conversely, any $E_+ \in \mathrm{Gr}_{p+1}(TM)$ that satisfies $E \subset E_+ \subseteq H(E)$ satisfies $E_+ \in \mathcal{V}_{p+1}(\mathcal{I})$. Set $c(E) = \dim\big(T_xM/H(E)\big)$.

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Cartan's Bound Let $E \in \mathcal{V}_n(I)$ be fixed, and let $F = (E_0, E_1, \ldots, E_{n-1})$ be a flag of subspaces of E, with $\dim E_i = i$. Thus,

$$(0)_x = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E \subset T_xM.$$

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Proposition: There is an open E-neighborhood $U \subset \mathrm{Gr}_n(TM)$ such that $\mathcal{V}_n(\mathcal{I}) \cap U$ is contained in a smooth submanifold of U of codimension

$$c(F) = c(E_0) + c(E_1) + \cdots + c(E_{n-1}).$$

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When equality holds, we say that F is a regular flag and E is ordinary.

Cartan-Kähler Theorem: If $\mathcal{I} \subset \Omega^+(M)$ is a real-analytic EDS and $E \in \mathcal{V}_n(\mathcal{I})$ is ordinary, then there is an $\mathcal{I}$-integral manifold $f : N \to M$ with $E = f'(T_xN)$ for some $x \in N$.

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Generality: The character sequence of the flag $F = (E_0, E_1, \ldots, E_{n-1})$ is

$$s_i(F) = \begin{cases} c(E_0) & i = 0, \\ c(E_i) - c(E_{i-1}) & 1 \leq i < n, \\ \dim H(E_{n-1}) - n & i = n. \end{cases}$$

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Then the 'generic' ordinary integral manifold of $\mathcal{I}$ depends on

$s_0(F)$ constants
$s_1(F)$ functions of 1 variable,
$s_2(F)$ functions of 2 variables,
$\vdots$
$s_n(F)$ functions of n variables.

Example: (Cartan's Third Theorem) Let $C^i_{jk} = -C^i_{kj}$ and F^α_i (with $1 \leq i, j, k \leq n$ and $1 \leq \alpha \leq s$) be functions on $\mathbb{R}^s$. One wants to know whether or not there exist linearly independent 1-forms ω^i on $\mathbb{R}^n$ and a function $a = (a^\alpha) : \mathbb{R}^n \to \mathbb{R}^s$ that satisfy the *Cartan structure equations*

$$\mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}(a)\, \omega^j \wedge \omega^k \qquad \text{and} \qquad \mathrm{d}a^\alpha = F^\alpha_i(a)\, \omega^i.$$

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Now $\mathrm{d}^2 = 0$, and this implies that C and F must satisfy compatibility

$$F^\alpha_j \frac{\partial C^i_{kl}}{\partial u^\alpha} + F^\alpha_k \frac{\partial C^i_{lj}}{\partial u^\alpha} + F^\alpha_l \frac{\partial C^i_{jk}}{\partial u^\alpha} = \left(C^i_{mj} C^m_{kl} + C^i_{mk} C^m_{lj} + C^i_{ml} C^m_{jk} \right)$$

and

$$F^\beta_i \frac{\partial F^\alpha_j}{\partial u^\beta} - F^\beta_j \frac{\partial F^\alpha_i}{\partial u^\beta} = C^l_{ij}\, F^\alpha_l\,.$$

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Theorem: If C and F as above satisfy compatibility, and are real-analytic, then for each $u_0 \in \mathbb{R}^s$, there exists a pair (a, ω) on $\mathbb{R}^n$ with $a(0) = u_0$ satisfying the above Cartan structure equations (unique up to $\mathrm{Diff}(\mathbb{R}^n, 0)$).

Proof: Let $M = \mathrm{GL}(n,\mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^s$, and let $p : M \to \mathrm{GL}(n,\mathbb{R})$, $x : M \to \mathbb{R}^n$, and $u : M \to \mathbb{R}^s$ be the projections. Consider the ideal $\mathcal{I}$ generated on M by the n 2-forms

$$\Upsilon^i = \mathrm{d}(p^i_j\,\mathrm{d}x^j) + \tfrac{1}{2}C^i_{jk}(u)(p^j_l\,\mathrm{d}x^l) \wedge (p^k_m\,\mathrm{d}x^m)$$

and the s 1-forms

$$\theta^\alpha = \mathrm{d}u^\alpha - F^\alpha_i(u)\,(p^i_j\,\mathrm{d}x^j).$$

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Note that one can write

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for some 1-forms $\pi^i_j = \mathrm{d}p^i_j + P^i_{jk}\,\mathrm{d}x^k$ for some functions P^i_{jk} on M and that the forms π^i_j, $\mathrm{d}x^k$, and θ^α define a coframing on M, i.e., they are linearly independent everywhere and span the cotangent space everywhere.

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Now, $\mathrm{d}(\mathcal{I}) \subset \mathcal{I}$ if and only if C and F satisfy compatibility. Also, the n-plane field defined by $\pi^i_j = \theta^\alpha = 0$ consists of ordinary integral elements. Now apply the Cartan-Kähler Theorem, obtaining $(a, \omega) = \big(u(x), p^i_j(x)\,\mathrm{d}x^j\big)$. QED.

Involutivity: Let V and W be vector spaces over $\mathbb{R}$ of dimensions n and m respectively, and let $A \subset W \otimes V^*$ be an r-dimensional subspace. We want to understand the set of functions $f : V \to W$ such that $f'(x)$ lies in A for all $x \in V$.

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Set up an EDS as follows: Let $M = V \times W \times A$ and let $u : M \to W$, $x : M \to V$, and $p : M \to A$ denote the projections. Let $\mathcal{I}$ be the ideal generated by the components of the W-valued 1-form $\theta = \mathrm{d}u - p\,\mathrm{d}x$. Thus, $\mathcal{I}$ is generated in degree 1 by $m = \dim W$ 1-forms and in degree 2 by the (at most) m independent 2-forms that are the components of $\mathrm{d}\theta = -\mathrm{d}p{\wedge}\mathrm{d}x$.

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A $E \in \mathcal{V}_n(TM)$ at $(u_0, x_0, p_0) \in M$ on which the components of $\mathrm{d}x$ are independent will be described by equations of the form

$$\mathrm{d}u - p_0\,\mathrm{d}x = \mathrm{d}p - s\,\mathrm{d}x = 0$$

where $s \in A \otimes V^*$ must satisfy $(s\,\mathrm{d}x){\wedge}\mathrm{d}x = 0$, i.e.,

$$s \in A \otimes V^* \cap W \otimes S^2(V^*) = A^{(1)}.$$

Whether $E \in \mathcal{V}_n(I)$ is ordinary or not depends only on $A \subset W \otimes V^*$, and we say that A is *involutive* if E is ordinary. We define the Cartan characters of A to be the characters $s_i(A) = s_i(F)$ of any regular flag F.

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When A is involutive, if one takes the Taylor series of the 'general' solution $f : V \to W$ of the equations forcing $f'(x)$ to lie in A for all x, one gets

$$f(x) = f_0 + f_1(x) + f_2(x) + \cdots + f_k(x) + \cdots ,$$

where f_k is a W-valued homogeneous polynomial of degree k on V and hence lies in the subspace

$$A^{(k-1)} = \bigl(W \otimes S^k(V^*)\bigr) \cap \bigl(A \otimes S^{k-1}(V^*)\bigr).$$

which has dimension

$$\dim A^{(k-1)} = \sum_{j=1}^{n} \binom{j+k-2}{k-1} s_j(A)\,,$$

which is exactly what one would expect if f were to be thought of as being comprised of $s_1(A)$ functions of 1 variable, $s_2(A)$ functions of 2 variables, etc.

Cartan's Theorem (Variant 1): Let $C^i_{jk} = -C^i_{kj}$ (with $1 \leq i, j, k \leq n$) be functions on $\mathbb{R}^s$ and F^α_i $(1 \leq \alpha \leq s)$ be functions on $\mathbb{R}^{s+r}$ One wants to find linearly independent 1-forms ω^i on $\mathbb{R}^n$ and functions $a = (a^\alpha) : \mathbb{R}^n \to \mathbb{R}^s$ and $b = (b^\rho) : \mathbb{R}^n \to \mathbb{R}^r$ that satisfy the *Cartan structure equations*

$$\mathrm{d}\omega^i = -\tfrac{1}{2} C^i_{jk}(a)\, \omega^j \wedge \omega^k \qquad \text{and} \qquad \mathrm{d}a^\alpha = F^\alpha_i(a,b)\, \omega^i.$$

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Again, $\mathrm{d}^2\omega^i = 0$ implies that C and F must satisfy C-compatibility

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But $\mathrm{d}^2 a^\alpha = 0$ turns out to be equivalent to the existence of G^ρ_j satisfying F-compatibility

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We also need involutivity of the subspaces $A(u, v)$ spanned by the r matrices

$$\left(\frac{\partial F^\alpha_i}{\partial v^\rho}(u, v) \right) \qquad 1 \leq \rho \leq r.$$

Theorem: If C and F as above satisfy the compatibility and involutivity hypotheses and are real-analytic, then for any $(u_0, v_0) \in \mathbb{R}^{s+r}$, there exist (a, b, ω) on an open neighborhood V of $0 \in \mathbb{R}^n$ that satisfy

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The general solution, up to diffeomorphism, depends on $s_q(A)$ functions of q variables where $q \leq n$ is the largest integer for which $s_q(A) > 0$.

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Remark: The proof is similar to the proof of Cartan's Theorem; one defines a differential ideal $\mathcal{I}$ on

$$M = \mathrm{GL}(n, \mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^s \times \mathbb{R}^r$$

that is generated by the n 2-forms

$$\Upsilon^i = \mathrm{d}(p^i_j\, \mathrm{d}x^j) + \tfrac{1}{2} C^i_{jk}(u)(p^j_l\, \mathrm{d}x^l) \wedge (p^k_m\, \mathrm{d}x^m)$$

and the s 1-forms

$$\theta^\alpha = \mathrm{d}u^\alpha - F^\alpha_i(u, v)\, (p^i_j\, \mathrm{d}x^j).$$

and shows that the hypotheses imply that there is a regular flag.

Example: (Torsion-free H-structures) Let $\mathfrak{m}$ be a vector space over $\mathbb{R}$ of dimension m, and let $H \subset \mathrm{GL}(\mathfrak{m})$ be a connected Lie subgroup of dimension r with Lie algebra $\mathfrak{h} \subset \mathfrak{gl}(\mathfrak{m}) = \mathfrak{m} \otimes \mathfrak{m}^*$.

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Problem: Determine the generality, modulo diffeomorphism, of the (local) H-structures that are torsion-free, and, more generally, of torsion-free connections on m-manifolds with holonomy contained in (a conjugate of) H.

Remark: When the first prolongation space of $\mathfrak{h}$ vanishes, i.e., when

$$\mathfrak{h}^{(1)} = (\mathfrak{h} \otimes \mathfrak{m}^*) \cap \big(\mathfrak{m} \otimes S^2(\mathfrak{m}^*)\big) = (0),$$

these two questions are essentially the same, since, in this case, an H-structure that is torsion-free has a (unique) compatible torsion-free connection and conversely. For simplicity, I will assume this holds.

Let $\pi : B \to M^m$ be an H-structure on M^m endowed with a torsion-free compatible connection. Let $\eta : TB \to \mathfrak{m}$ be the canonical $\mathfrak{m}$-valued 1-form on B, then the torsion-free compatible connection defines an $\mathfrak{h}$-valued 1-form $\theta : TB \to \mathfrak{h}$ satisfying the *first structure equation*

$$\mathrm{d}\eta = -\theta \wedge \eta,$$

and having the equivariance $R_h^*(\theta) = \mathrm{Ad}(h^{-1})\big(\theta\big)$ for all $h \in H$.

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One then has the *second structure equation*

$$\mathrm{d}\theta = -\theta \wedge \theta + \tfrac{1}{2}\, R(\eta \wedge \eta)$$

for a unique *curvature function* $R : B \to \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*)$. It satisfies the first Bianchi identity,

$$0 = \mathrm{d}(\mathrm{d}\eta) = -\mathrm{d}\theta \wedge \eta + \theta \wedge \mathrm{d}\eta = -(\mathrm{d}\theta + \theta \wedge \theta) \wedge \eta = -\tfrac{1}{2}\, R(\eta \wedge \eta) \wedge \eta = 0.$$

I.e., R takes values in the kernel $K_0(\mathfrak{h}) \subset \mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*)$ of the natural map

$$\mathfrak{h} \otimes \Lambda^2(\mathfrak{m}^*) \subset \mathfrak{m} \otimes \mathfrak{m}^* \otimes \Lambda^2(\mathfrak{m}^*) \to \mathfrak{m} \otimes \Lambda^3(\mathfrak{m}^*).$$

Differentiating the second structure equation gives the second Bianchi identity

$$0 = \mathrm{d}(\mathrm{d}\theta) = \tfrac{1}{2}\big(\mathrm{d}R + \rho_0'(\theta)R\big)(\eta \wedge \eta),$$

where $\rho_0 : H \to \mathrm{GL}\big(K_0(\mathfrak{h})\big)$ is the induced representation of H on $K_0(\mathfrak{h})$, and $\rho_0' : \mathfrak{h} \to \mathfrak{gl}\big(K_0(\mathfrak{h})\big)$ is the induced map on Lie algebras. This means that

$$\mathrm{d}R = -\rho_0'(\theta)R + R'(\eta),$$

where $R' : B \to K_0(\mathfrak{h}) \otimes \mathfrak{m}^*$ takes values in the kernel $K_1(\mathfrak{h}) \subset K_0(\mathfrak{h}) \otimes \mathfrak{m}^*$ of the natural linear mapping defined by skew-symmetrization

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The structure equations with $\omega = (\eta, \theta)$, and $a = R$ while $b = R'$ becomes

$$\mathrm{d}\eta = -\theta \wedge \eta, \quad \mathrm{d}\theta = -\theta \wedge \theta + \tfrac{1}{2}\, a(\eta \wedge \eta), \qquad \mathrm{d}a = -\rho_0'(\theta)a + b(\eta)$$

where a and b take values in $K_0(\mathfrak{h})$ and $K_1(\mathfrak{h})$ respectively. This is exactly of the type treated by the Variant Theorem.

Example: When $m = 4$ and $H = \mathrm{SU}(2) \subset \mathrm{SO}(4)$, this becomes a Riemannian manifold with holonomy $\mathrm{SU}(2)$, i.e, the case of a Ricci-flat Kähler surface.

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$$\begin{pmatrix} \mathrm{d}\eta_0 \\ \mathrm{d}\eta_1 \\ \mathrm{d}\eta_2 \\ \mathrm{d}\eta_3 \end{pmatrix} = - \begin{pmatrix} 0 & \theta_1 & \theta_2 & \theta_3 \\ -\theta_1 & 0 & -\theta_3 & \theta_2 \\ -\theta_2 & \theta_3 & 0 & -\theta_1 \\ -\theta_3 & -\theta_2 & \theta_1 & 0 \end{pmatrix} \wedge \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \\ \mathrm{d}\theta_3 \end{pmatrix} = - \begin{pmatrix} 2\,\theta_2\wedge\theta_3 \\ 2\,\theta_3\wedge\theta_1 \\ 2\,\theta_1\wedge\theta_2 \end{pmatrix} + \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} \begin{pmatrix} \eta_0\wedge\eta_1 - \eta_2\wedge\eta_3 \\ \eta_0\wedge\eta_2 - \eta_3\wedge\eta_1 \\ \eta_0\wedge\eta_3 - \eta_1\wedge\eta_2 \end{pmatrix}$$

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where $R_{ij} = R_{ji}$ and $R_{11} + R_{22} + R_{33} = 0$.

Calculation shows that, in the equation $\mathrm{d}R = -\rho_0'(\theta)R + R'(\eta)$, the tableau of free derivatives is involutive, with characters

$$(s_1, s_2, s_3, s_4) = (5, 5, 2, 0),$$

so the general solution depends on 2 functions of 3 variables.