G-STRUCTURES WITH PRESCRIBED GEOMETRY—EXAMPLES

ROBERT L. BRYANT

DUKE UNIVERSITY

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References: For these three lectures (these are the slide for the third), the reader who wants more background information on exterior differential systems might want to consult the brief introduction

http://www.math.duke.edu/~bryant/Introduction_to_EDS.pdf

Many of the examples discussed here and the main variants of Cartan's theory of structure equations can be found in the lecture notes on EDS that can be found here

http://arxiv.org/abs/1405.3116

This latter article contains many references to the literature and further resources.

I. Riemannian Surfaces with $|\nabla K|^2 = 1$

Consider (M^2, g) whose Gauss curvature satisfies $|\nabla K|^2 = 1$. The structure equations on the orthonormal frame bundle $\pi : B \to M$ have $g = {\omega_1}^2 + {\omega_2}^2$ and are

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_{12} \wedge \omega_2 \\ \mathrm{d}\omega_2 &= \quad \omega_{12} \wedge \omega_1 \\ \mathrm{d}\omega_{12} &= a\, \omega_1 \wedge \omega_2 \\ \mathrm{d}a &= \cos b\, \omega_1 + \sin b\, \omega_2 \end{aligned} \qquad \omega_1 \wedge \omega_2 \wedge \omega_{12} \neq 0,$$

where $K = a$ is the Gauss curvature and b is the free derivative.

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Using these equations, we see that $\mathrm{d}^2\omega_1 = \mathrm{d}^2\omega_2 = \mathrm{d}^2\omega_{12} = 0$ are identities, but, using the structure equations, one finds

$$0 = \mathrm{d}(\mathrm{d}a) = (\mathrm{d}b - \omega_{12}) \wedge (-\sin b\, \omega_1 + \cos b\, \omega_2).$$

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It follows that the hypotheses of the Variant of Cartan's Theorem are satisfied, with the characters of the tableau of free derivatives being $s_1 = 1$, $s_2 = s_3 = 0$. Thus, the general (local) solution depends on one function of one variable.

I. Ricci solitons in dimension 2

Consider (M^2, g) that is a *gradient Ricci soliton*, i.e.,

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$$\mathrm{d}K = -K\,(f_1\, \omega_1 + f_2\, \omega_2) = -K\, \mathrm{d}f.$$

There are now no more ‘free derivatives’, but $\mathrm{d}^2 = 0$ is an identity.

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For an oriented Finsler surface (M^2, F), Cartan showed that the 'tangent indicatrix' $\Sigma \subset TM$ has a canonical coframing $(\omega_1, \omega_2, \omega_3)$, satisfying

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_2 \wedge \omega_3 \\ \mathrm{d}\omega_2 &= -\omega_3 \wedge \omega_1 - I\,\omega_2 \wedge \omega_3 \\ \mathrm{d}\omega_3 &= -K\,\omega_1 \wedge \omega_2 - J\,\omega_2 \wedge \omega_3 \end{aligned} \qquad \omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0,$$

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The functions I, J, and K are the *Finsler structure functions*. K is the Finsler-Gauss (or 'flag') curvature; I is the Cartan scalar, which vanishes iff (M^2, F) is Riemannian; J is the Landsberg scalar.

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Differentiating the above equations yields the Finsler-Bianchi identities

$$\begin{aligned} dI &= J\,\omega_1 + I_2\,\omega_2 + I_3\,\omega_3\,, \\ dJ &= -(K_3 + KI)\,\omega_1 + J_2\,\omega_2 + J_3\,\omega_3\,, \\ dK &= K_1\,\omega_1 + K_2\,\omega_2 + K_3\,\omega_3\,. \end{aligned}$$

for seven new functions, K_i, etc. These are the 'free derivatives', and their tableau has $(s_1, s_2, s_3) = (3, 3, 1)$.

Case 1: $I = 0$ (the Riemannian case), forces $J = 0$, so

$$\begin{aligned} \mathrm{d}\omega_1 &= -\omega_2 \wedge \omega_3 \\ \mathrm{d}\omega_2 &= -\omega_3 \wedge \omega_1 \\ \mathrm{d}\omega_3 &= -K\,\omega_1 \wedge \omega_2 \\ \mathrm{d}K &= K_1\,\omega_1 + K_2\,\omega_2 \end{aligned}$$

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Case 2: $J = 0$ (Landsberg surfaces)

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\mathrm{d}I &= & & + I_2\,\omega_2 & & + I_3\,\omega_3\,, \\
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Tableau has $(s_1, s_2, s_3) = (2, 2, 0)$, so 2 functions of 2 variables.

Case 3: K-basic (i.e., $K_3 = 0$, so K well-defined on M)

$$\begin{aligned} \mathrm{d}I &= J\,\omega_1 + I_2\,\omega_2 + I_3\,\omega_3\,, \\ \mathrm{d}J &= -KI\,\omega_1 + J_2\,\omega_2 + J_3\,\omega_3\,, \\ \mathrm{d}K &= K_1\,\omega_1 + K_2\,\omega_2\,. \end{aligned}$$

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Consider the problem of the generality of Riemannian 3-manifolds (M^3, g) for which there exists a function f (a 'Ricci potential') such that

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The four functions (f, f_1, f_2, f_3) will play the role of the a^α in the structure equations, and $\mathrm{d}(\mathrm{d}f) = 0$ implies that there exist $f_{ij} = f_{ji}$ so that

$$\mathrm{d}f_i = -\omega_{ij}\, f_j + f_{ij}\, \omega_j$$

The equations $\mathrm{d}(\mathrm{d}\omega_i) = 0$ are identities (because R is symmetric), but the equations $\mathrm{d}(\mathrm{d}\omega_{ij}) = 0$ can be written as

$$\big(2(f_{11} + f_{22} + f_{33}) - H'(f)\big)\,\mathrm{d}f = 0.$$

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Thus, either $\mathrm{d}f = 0$ (in which case, the metric is Einstein) or else

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so that one has

$$\mathrm{d}f_i = -\omega_{ij}f_j + \big(b_{ij} + \tfrac{1}{6}H'(f)\delta_{ij}\big)\,\omega_j\,.$$

where the (new) $b_{ij} = b_{ji}$ are subject to the trace condition $b_{11}+b_{22}+b_{33} = 0$. These b_{ij} will play the role of the 'free derivatives' in the structure equations.

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We can now easily check that these structure equations satisfy the compatibility and involutivity conditions for the Cartan Variant Theorem, with characters

$$(s_1, s_2, s_3, s_4, s_5, s_6) = (3, 2, 0, 0, 0, 0),$$

so, up to diffeomorphism, these structures (M^3, g, f) depend on 2 functions of 2 variables.

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These are $\mathrm{CO}(3)$-structures on 3-manifolds endowed with a compatible connection on $B^7 \to M^3$ that satisfies the structure equations

$$\begin{pmatrix}\mathrm{d}\eta_1\\ \mathrm{d}\eta_2\\ \mathrm{d}\eta_3\end{pmatrix} = -\begin{pmatrix}\theta_0 & \theta_3 & -\theta_2\\ -\theta_3 & \theta_0 & \theta_1\\ \theta_2 & -\theta_1 & \theta_0\end{pmatrix} \wedge \begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}$$

and

$$\begin{pmatrix}\mathrm{d}\theta_0\\ \mathrm{d}\theta_1\\ \mathrm{d}\theta_2\\ \mathrm{d}\theta_3\end{pmatrix} = \begin{pmatrix}0\\ \theta_2{\wedge}\theta_3\\ \theta_3{\wedge}\theta_1\\ \theta_1{\wedge}\theta_2\end{pmatrix} + \begin{pmatrix}2H_1 & 2H_2 & 2H_3\\ H_0 & H_3 & -H_2\\ -H_3 & H_0 & H_1\\ H_2 & -H_1 & H_0\end{pmatrix} \begin{pmatrix}\eta_2{\wedge}\eta_3\\ \eta_3{\wedge}\eta_1\\ \eta_1{\wedge}\eta_2\end{pmatrix}.$$

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and

$$\begin{pmatrix} \mathrm{d}\theta_0 \\ \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \\ \mathrm{d}\theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \theta_2\wedge\theta_3 \\ \theta_3\wedge\theta_1 \\ \theta_1\wedge\theta_2 \end{pmatrix} + \begin{pmatrix} 2H_1 & 2H_2 & 2H_3 \\ H_0 & H_3 & -H_2 \\ -H_3 & H_0 & H_1 \\ H_2 & -H_1 & H_0 \end{pmatrix} \begin{pmatrix} \eta_2\wedge\eta_3 \\ \eta_3\wedge\eta_1 \\ \eta_1\wedge\eta_2 \end{pmatrix}.$$

Differentiating these equations gives relations of the following form ((i,j,k) is an even permutation of (1,2,3)), where, the 9 L_{ij} satisfy $L_{11}+L_{22}+L_{33}=0$.

$$\mathrm{d}H_0 = 2H_0\,\theta_0 + (L_{ij} - L_{ji})\,\eta_k$$
$$\mathrm{d}H_i = 2H_i\,\theta_0 - H_j\theta_k + H_k\,\theta_j + L_{ij}\,\eta_j$$

V. Einstein-Weyl structures in dimension 3

These are $\mathrm{CO}(3)$-structures on 3-manifolds endowed with a compatible connection on $B^7 \to M^3$ that satisfies the structure equations

$$\begin{pmatrix} \mathrm{d}\eta_1 \\ \mathrm{d}\eta_2 \\ \mathrm{d}\eta_3 \end{pmatrix} = - \begin{pmatrix} \theta_0 & \theta_3 & -\theta_2 \\ -\theta_3 & \theta_0 & \theta_1 \\ \theta_2 & -\theta_1 & \theta_0 \end{pmatrix} \wedge \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{d}\theta_0 \\ \mathrm{d}\theta_1 \\ \mathrm{d}\theta_2 \\ \mathrm{d}\theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ \theta_2\wedge\theta_3 \\ \theta_3\wedge\theta_1 \\ \theta_1\wedge\theta_2 \end{pmatrix} + \begin{pmatrix} 2H_1 & 2H_2 & 2H_3 \\ H_0 & H_3 & -H_2 \\ -H_3 & H_0 & H_1 \\ H_2 & -H_1 & H_0 \end{pmatrix} \begin{pmatrix} \eta_2\wedge\eta_3 \\ \eta_3\wedge\eta_1 \\ \eta_1\wedge\eta_2 \end{pmatrix}.$$

Differentiating these equations gives relations of the following form ((i,j,k) is an even permutation of (1,2,3)), where, the 9 L_{ij} satisfy $L_{11} + L_{22} + L_{33} = 0$.

$$\mathrm{d}H_0 = 2H_0\,\theta_0 + (L_{ij} - L_{ji})\,\eta_k$$

$$\mathrm{d}H_i = 2H_i\,\theta_0 - H_j\theta_k + H_k\,\theta_j + L_{ij}\,\eta_j$$

These are involutive with $(s_1, s_2, s_3, \ldots) = (4, 4, 0, \ldots)$.

VI. Nearly-Kähler 6-manifolds

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A nearly-Kähler 6-manifold is a manifold M^6 endowed with a $\mathrm{SU}(3)$-structure $\pi : B \to M$ with torsion, whose first structure equations take the form

$$\mathrm{d}\eta_i = -\theta_{i\bar{l}} \wedge \eta_l + \lambda\, \overline{\eta_j} \wedge \overline{\eta_k}$$

where (i, j, k) is an even perm. of $(1, 2, 3)$, $\overline{\theta_{i\bar{\jmath}}} = -\theta_{j\bar{\imath}}$ and $\theta_{i\bar{\imath}} = 0$, and $\lambda = \bar{\lambda}$.

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The identities $\mathrm{d}(\mathrm{d}\eta_i) = 0$ imply that $\mathrm{d}\lambda = 0$ and

$$\mathrm{d}\theta_{i\bar{\jmath}} = -\theta_{i\bar{k}} \wedge \theta_{k\bar{\jmath}} + \lambda^2 \left(\tfrac{3}{4}\eta_i \wedge \overline{\eta_j} - \tfrac{1}{4}\delta_{i\bar{\jmath}}\, \eta_l \wedge \overline{\eta_l}\right) + K_{i\bar{\jmath}p\bar{q}}\, \eta_q \wedge \overline{\eta_p}\,,$$

where the functions $K_{i\bar{\jmath}p\bar{q}}$ satisfy

$$K_{i\bar{\jmath}p\bar{q}} = K_{p\bar{\jmath}i\bar{q}} = K_{i\bar{q}p\bar{\jmath}} = \overline{K_{j\bar{\imath}q\bar{p}}} \quad \text{and} \quad K_{i\bar{\imath}p\bar{q}} = 0.$$

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This leaves 27 real components in the tensor K, and $\mathrm{d}(\mathrm{d}\theta_{i\bar{j}}) = 0$ yields

$$\mathrm{d}K_{i\bar{j}p\bar{q}} = -K_{\ell\bar{j}p\bar{q}}\, \theta_{i\bar{\ell}} + K_{i\bar{\ell}p\bar{q}}\, \theta_{\ell\bar{j}} - K_{i\bar{j}\ell\bar{q}}\, \theta_{p\bar{\ell}} + K_{i\bar{j}p\bar{\ell}}\, \theta_{\ell\bar{q}} + K_{i\bar{j}p\bar{q}\bar{\ell}}\, \eta_\ell + K_{i\bar{j}p\bar{q}\ell}\, \eta_{\bar{\ell}}\,,$$

where $K_{i\bar{j}p\bar{q}\bar{\ell}} = K_{i\bar{j}p\bar{\ell}\bar{q}}$ and $K_{i\bar{j}p\bar{q}\ell} = K_{i\bar{j}\ell\bar{q}p}$, etc.

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where the functions $K_{i\bar{j}p\bar{q}}$ satisfy

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$$\mathrm{d}K_{i\bar{j}p\bar{q}} = -K_{\ell\bar{j}p\bar{q}}\, \theta_{i\bar{\ell}} + K_{i\bar{\ell}p\bar{q}}\, \theta_{\ell\bar{j}} - K_{i\bar{j}\ell\bar{q}}\, \theta_{p\bar{\ell}} + K_{i\bar{j}p\bar{\ell}}\, \theta_{\ell\bar{q}} + K_{i\bar{j}p\bar{q}\bar{\ell}}\, \eta_\ell + K_{i\bar{j}p\bar{q}\ell}\, \eta_{\bar{\ell}}\,,$$

where $K_{i\bar{j}p\bar{q}\bar{\ell}} = K_{i\bar{j}p\bar{\ell}\bar{q}}$ and $K_{i\bar{j}p\bar{q}\ell} = K_{i\bar{j}\ell\bar{q}p}$, etc.

This is involutive, with characters $(s_1, \ldots, s_5, s_6) = (27, 27, 19, 9, 2, 0)$.